Latitude-Dependent Sensitivity to Stationary Perturbations in Simple Climate Models

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ABSTRACT

The steady-state zonally averaged climate is perturbed by adding a latitude-dependent heat source to an energy balance equation of the simplified Budyko-Sellers type. The latitude of the ice edge, which is attached to an isotherm, becomes dependent on the strength of the perturbation. This dependence is given in terms of the well-known iceline-solar constant relation, and the latitude dependence of the perturbed temperature field is then uniquely determined. The exact analytical solution is linearized and expressed in terms of a superposition of line sources at various latitudes. The main features are: 1) The total temperature response is a sum of the direct effect of the perturbation and an indirect ice-albedo effect proportional to the solar ice-edge sensitivity; and 2) the indirect feedback effect produces an enhanced response in polar latitudes.

1. Introduction

Most sensitivity studies with energy balance models have been concerned with the sensitivity of the model climate to solar constant changes. A small fractional change in the solar constant dQ/Q induces a change in the steady-state global average temperature dT_0 for which one may define the sensitivity parameter

$$\beta_0 = \frac{dT^0}{dQ/Q}, \tag{1.1}$$

which depends on the various feedback mechanisms employed by the model. This provides a measure of

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sensitivity to any global-scale perturbation. However, in many instances we wish to know the sensitivity to more localized perturbations such as anthropogenic waste heating rates (Washington, 1972). Moreover, we may want information on the magnitude of the response at different latitudes. For example, several studies have shown a pronounced enhancement in polar latitudes of the steady-state thermal response to changes in CO_2 (e.g., Manabe and Wetherald, 1975). It is clear that the spatial distribution of sensitivity can be as important as the overall magnitude.

In this note we extend the analytical theory given by Cahalan and North (1979, hereafter referred to as CN) to the question of the climatic response to localized perturbations. The models considered are of the Budyko-Sellers type (Budyko, 1968, 1969; Sellers, 1969). The various assumptions and limitations of these models have been much discussed (see, e.g., Warren and Schneider, 1979). Although many effects are poorly represented in such simple models, their tractability allows causal relationships to be determined explicitly, and these provide guides for experimentation in more realistic models (Schneider and Dickinson, 1974). The single spatial variable retained here is the sine of the latitude x and hemispheric symmetry is assumed so that it is sufficient to consider Northern Hemisphere values, $0 \leqslant x \leqslant 1$. Further details of our notation and method of solution are given in CN and are summarized here in the Appendix.

An important feature of Budyko-Sellers models is the enhancement of β_0 associated with ice-albedo feedback. An increase in the solar constant, $S_0 = 4Q$, not only increases the incident energy, but also shrinks the icecap, and both effects contribute to an increase in T_0. Models in which the edge of the icecap is attached to a given isotherm may be solved analytically (North, 1975a,b). The sine of the ice-edge latitude is found to be a multivalued function of the solar constant, $x_s = x_s(Q)$. Its slope, the ice-edge sensitivity

$$\beta_s = \frac{dx_s}{dQ/Q}, \tag{1.2}$$

determines the temperature sensitivity β_0, as well as the stability of the steady-state climate (CN). We shall see that it also plays an important role in determining the spatial distribution of climate sensitivity in these models.

In the remainder of this section we describe the type of perturbation considered, preview our main results, and describe the spatial distribution of sensitivity in general terms. Section 2 contains the detailed derivations. Section 3 concludes with a brief discussion.

Since our emphasis here is on spatially distributed perturbations, the solar constant will be held fixed at its present value $Q = Q_0$. A positive distribution of heat satisfying

$$u(x) \geqslant 0, \tag{1.3a}$$

and

$$\int_0^1 dx u(x) = 1, \tag{1.3b}$$

will be added to the energy balance with strength g (equally in both hemispheres). This steady-state problem is soluble by the same methods as the unperturbed case ($g = 0$), and the ice edge is now a multivalued function of the strength of heating, $x_s = x_s(Q_0; g)$, with the stability determined by the slope dx_s/dg. The ice edge may of course be adjusted to the same latitude either by tuning Q with g fixed, as in the unperturbed case, or by tuning g with Q fixed, as here. Thus, the functions $x_s(Q_0; g)$ may be expressed in terms of previously tabulated values of $x_s(Q; 0)$. The slope dx_s/dg is found to be directly proportional to β_s, and the constant of proportionality is largest when $u(x)$ is most concentrated near the ice edge. This confirms our intuitive notion that a model sensitive to changes in Q will also be sensitive to other perturbations.

Increases in g increase the heating both directly in the $u(x)$ source, and indirectly in the solar absorption due to the ice edge shift. Due to the nonlinear feedback the temperature response is generally nonlinear is g, and is also generally not given by the sum effect of each latitude strip alone. However, for small g the ice edge shift is also small, and it has the effect of an additional heat source localized in the region of the ice edge and superimposed on the direct effect of g. This localized feedback effect gives a sensitivity or "response function" which peaks at the ice edge, even when the $u(x)$ perturbation itself is uniform.

For readers not wishing to indulge in the detailed derivation of the following section, the main features of the temperature response may be summarized schematically:

Total response per unit heat added to a given latitude	=	Direct response in the absence of ice-albedo feedback
	+	Indirect response due to albedo change multiplied by β_s. (1.4)

The direct response peaks at the latitude of the source, and dies off smoothly over some characteristic distance. Physically this falloff is due to the fact that a significant fraction of heat is lost by infrared radiation during the time it takes to transport it, so that heat sources produce a response over some limited effective range. Since albedo changes occur primarily where snow and ice melt, the indirect response peaks in that region. When the source is smoothly distributed, the peak in the direct response is also smoothed, but the peak in the indirect response is enhanced since heat from each latitude increases the albedo change in the ice-edge region.

2. Perturbed solutions and latitude-dependent sensitivity

The class of models which we consider are defined by the energy-balance equation

$$L[T](x) + A(x) = QS(x)a(x,x_s) + gu(x), \quad (2.1)$$

or its equivalent integral form

$$T(x) = \int_0^1 dyG_0(x,y) \times [QS(y)a(y,x_s) + gu(y) - A(y)]. \quad (2.2)$$

Here L is a homogeneous linear operator associated with horizontal and vertical (infrared radiation) heat transport and G_0 is its inverse, satisfying

$$LG_0(x,y) = \delta(x - y). \quad (2.3)$$

The boundary conditions are zero horizontal heat flux at the equator ($x = 0$) and pole ($x = 1$). The function $A(x)$ represents the inhomogeneous part of the heat transport; Q is the solar constant divided by four; $S(x)$ is the normalized mean annual distribution of incident solar flux; $a(x,x_s)$ is the coalbedo, which is temperature-dependent due to the ice-edge condition,

$$T(x_s) = T_s, \quad (2.4)$$

and $gu(x)$ is the perturbation, satisfying (1.3a) and (1.3b). A particularly simple example of this type of model is that of North (1975a,b). That model has horizontal diffusion with a constant coefficient, Budyko's linear infrared rule, and a coalbedo having one constant value over ice-covered regions and a higher constant value over ice-free regions, changing discontinuously at $x = x_s$. We first solve the general problem, linearize and then illustrate results for the North model.

Evaluating (2.2) at $x = x_s$ and using (2.4), we obtain

$$T_s = Q(Sa)_s + g(u)_s - (A)_s, \quad (2.5)$$

where the notation

$$(f)_s = \int_0^1 dyG_0(x_s,y)f(y) \quad (2.6)$$

has been used. In principle (2.5) determines the ice edge as a function of Q and g, i.e., $x_s = x_s(Q; g)$. The "present climate" is defined as that for which $g = 0$ and $Q = Q_0$, and the model must be tuned to the present ice edge, so that $x_s(Q_0; 0) = x_0$. In previous studies Q has been varied with $g = 0$, in which case (2.5) can be solved for $Q(x_s)$. Its inverse $x_s(Q; 0)$ has a well-known multivalued form with icecaps larger than a certain critical size being unstable, as shown by negative values of the slope, β_s. In the present case we hold $Q = Q_0$ and vary g, so that (2.5) can be solved for $g(x_s)$. The $g = 0$ problem and the $Q = Q_0$ problem can be related by equating the corresponding expressions for T_s. When this is done the $(A)_s$ terms cancel, so that

$$Q(Sa)_s = Q_0(Sa)_s + g(u)_s. \quad (2.7)$$

This expresses the fact that a given ice-edge latitude may be obtained either by tuning Q with $g = 0$, or by tuning g with $Q = Q_0$. Although the two resulting temperature fields are equal at the ice edge, they are generally unequal at other latitudes.

From (2.7) the function $g(x_s)$ can be determined from previously tabulated values of $Q(x_s)$ for any chosen perturbation $u(x)$. Its inverse, $x_s(Q_0; g)$, determines the possible ice-edge latitudes corresponding to a given g, which in turn specifies the coalbedo on the right side of (2.2), so that the perturbed solution is given formally by

$$T^g(x) = \int_0^1 dyG_0(x,y)[Q_0S(y)a(y,x_s(Q_0; g)) + gu(y) - A(y)]. \quad (2.8)$$

It can be shown that this solution is stable if and only if

$$\frac{dx_s}{dg} \geq 0. \quad (2.9)$$

For stable solutions, then, an increase in g shrinks the icecap and adds to the warming produced by the u distribution.

Because of the nonlinear ice-albedo feedback, the effect of a sum of perturbations, say, $u = u_1 + u_2$, is not generally given by adding the effect of each one along, $T^g \neq T_1 + T_2$. If g is sufficiently small, however, then (2.8) can be linearized. In that case the temperature field responds independently to the heat added in each latitude belt. This approximation should hold for anthropogenic heat, presently about 10^{-4} times the total absorbed solar energy. As discussed below, the importance of the nonlinear terms also depends on the proximity of the perturbation to the ice edge, so that the linear approximation also requires that heat sources not be located too near to x_s. We set

$$x_s = x_0 + \delta x_s, \quad (2.10)$$

$$T^g = T^{(0)} + \delta T, \quad (2.11)$$

where δx_s and δT are the deviation from the "present climate" ($g = 0$) values given by x_0 and $T^{(0)}$. We may determine δT to first order in g by expanding (2.7) and (2.8) to first order in δx_s and combining the resulting expressions. Using the notation of (1.2) for the iceline sensitivity evaluated at x_0, we have $Q - Q_0 = (Q_0/\beta_s)\delta x_s$. The terms in parentheses in (2.7) can be evaluated at x_0, so that

$$\delta x_s = \frac{u}{Q_0}\frac{\beta_s}{(Sa)_0}g + O(g^2). \quad (2.12)$$

If this expression is inserted into the first-order term in the expansion of the coalbedo in (2.8), and the integration variable in $(u)_0$ is interchanged with y in that term, the result can be written in the form

$$\delta T(x) = g\int_0^1 dyG(x,y)u(y) + O(g^2), \quad (2.13)$$

where G is given by

$$G(x,y) = G_0(x,y) + \left[\int_0^1 dzG_0(x,z)S(z)\frac{\partial a}{\partial x_0}(z,x_0)\right] \times \frac{\beta_s}{(Sa)_0}G_0(x_0,y). \quad (2.14)$$

By setting $u(y) = \delta(y - x_1)$ in (2.13) it can be seen that $G(x,x_1)$ represents the temperature response at x due to a unit of heat added at x_1. The integration in (2.13) just adds up the independent responses due to the heat in each latitude belt. Incidently, Eq. (2.12) proves that dx_s/dg has the same sign as β_s near $g = 0$, and hence the stability Corollary (2.9). The argument is easily extended to finite g.

Eq. (2.14) is the result described schematically in (1.4). G_0 is the direct response due to transport alone. The indirect response, the second term in (2.14), may be interpreted beginning at the extreme right as follows: heat from y is transported to x_0, shifting the ice edge by an amount proportional to $\beta_s G_0(x_0,y)$, so that more heat is added at all latitudes where the coalbedo changes, and this heat is transported to x, producing additional responses $G_0(x,z)$ for each unit of heat added at each z. Superposition of the direct and indirect responses gives the total response.

Now we specialize (2.14) to the North model. When the coalbedo is a simple stepfunction, a small shift in the ice edge changes the absorption only at $z = x_0$, so that

$$\frac{\partial a}{\partial x_0}(z,x_0) = \Delta a\delta(z - x_0),$$

where Δa is the discontinuity at the ice edge. Now the integral in (2.14) is easily done. If we also use Budyko's infrared rule, it can be shown that

$$S(x_0)\Delta a\frac{\beta_s}{(Sa)_0} = \left(\beta_0 - \frac{A + BT_0}{B}\right)\frac{B^2}{A + BT_s}.$$

We set $A = 203.3$ W m^{-2}, $B = 2.09$ W m^{-2} (North and Coakley, 1979) and choose $T_0 = 15°C$ and $T_s = -10°C$. Then for the North model (2.14) simplifies to

$$G^D(x,y) = G_0^D(x,y) + G_0^D(x,x_0)\left(\frac{\beta_0 - 112°C}{42°C}\right) \times G_0^D(x_0,y). \quad (2.15)$$

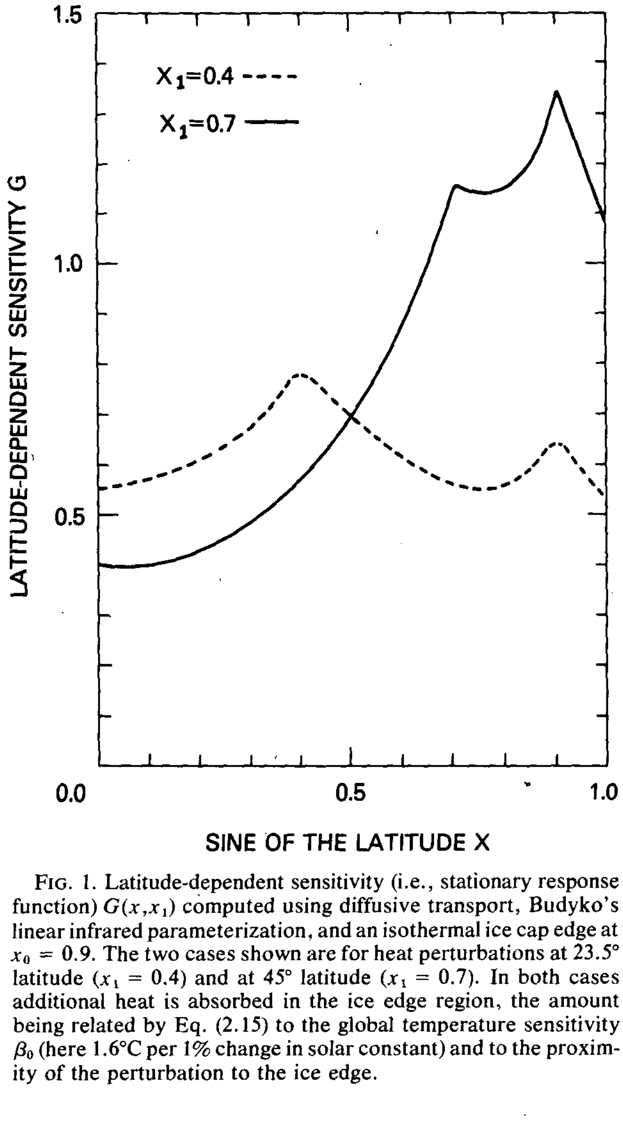

FIG. 1. Latitude-dependent sensitivity (i.e., stationary response function) $G(x,x_1)$ computed using diffusive transport, Budyko's linear infrared parameterization, and an isothermal ice cap edge at $x_0 = 0.9$. The two cases shown are for heat perturbations at 23.5° latitude ($x_1 = 0.4$) and at 45° latitude ($x_1 = 0.7$). In both cases additional heat is absorbed in the ice edge region, the amount being related by Eq. (2.15) to the global temperature sensitivity β_0 (here 1.6°C per 1% change in solar constant) and to the proximity of the perturbation to the ice edge.

Here $G_0^D(x,y)$ is the diffusive response function, and as discussed in CN, it is positive and peaks where the heat is added, at $x = y$. These two physically reasonable properties are not special to diffusive transport, but hold much more generally. The total response given by (2.15) is plotted in Fig. 1 for the case $\beta_0 = 1.6°C$ per 1% dQ/Q and $x_0 = 0.9$, with heat added at 23.5° ($x_1 = 0.4$) and at 45° ($x_1 = 0.7$). Like G_0^D, G^D also peaks where the heat is added, but it has a second peak at the ice edge, $x = x_0$, due to the additional absorption there. Note that tropical heating produces a relatively smaller

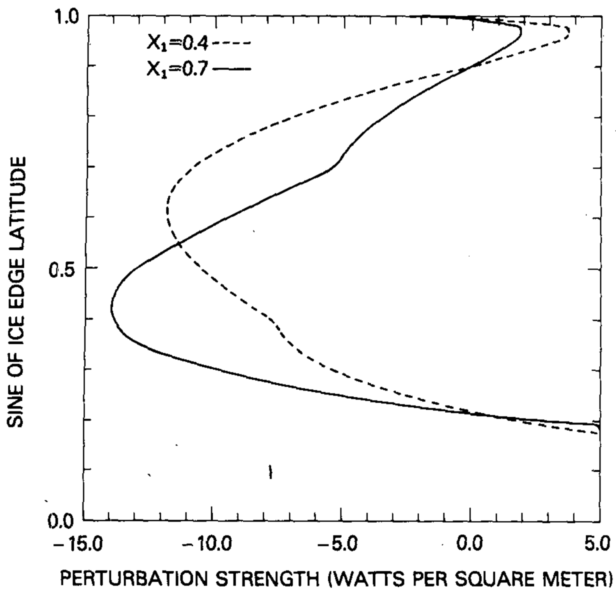

FIG. 2. Latitude of the ice cap edge x_s as a function of the perturbation strength g, with the solar constant fixed at its present value. As in Fig. 1, results are shown both for a tropical perturbation (x_1 = 0.4) and a midlatitude perturbation (x_1 = 0.7). In both cases the unperturbed climate is assumed to have $x_s = x_0 = 0.9$. The tangent at x_0 is a good approximation for x_s when g is small, but for larger g the response is nonlinear. Solutions with $dx_s/dg < 0$ are unstable.

response over the ice, while middle latitude heating produces an amplified effect over the ice. In general, the temperature response given by (2.13) is most concentrated in the neighborhood of the ice edge when the perturbation u is also concentrated there. However, even for a uniform perturbation, having $u = 1$, while the diffusive transport alone produces a uniform response, given by $g \int dy G_0^D = g/B$, the total response is given by the integral of (2.15):

$$\delta T(x)\big|_{u=1} = \frac{g}{B}\left[G_0^D(x,x_0)\,\frac{\beta_0 - 112^\circ\mathrm{C}}{42^\circ\mathrm{C}} + 1 \right]. \qquad (2.16)$$

This is still largest at the ice edge. Here we have an example of a purely global perturbation having a localized response due to the localized character of the feedback.

For large values of g the linear response given by (2.13) becomes invalid, and the temperature change must be found from (2.8), using the ice-edge function $x_s(Q_0; g)$ determined from (2.7). In Fig. 2 we show this function for the same parameters and perturbations as in Fig. 1. Note that the slope at $g = 0$, which is proportional to the ice-edge sensitivity of the present climate [Eq. (2.12)], is larger for the midlatitude perturbation (x_1 = 0.7) than for the tropical one (x_1 = 0.4). Mathematically this is because the δx_s associated with a source at x_1 is proportional to $(u)_0 = G_0(x_0,x_1)$, which is largest when x_1 is closest to x_0. Physically the ice-edge shift is smaller for a more distant source due to the increased fraction of heat lost to space during the time it takes to transport it. Also, note from Fig. 2 that the range of g for which the linear approximation remains accurate is smaller when the source is nearer to the ice edge. This is also related to the larger δx_s, which causes the nonlinear terms in (2.7) to become important at smaller values of g. (See also Salmun, 1979.)

We now consider the sensitivity of the main features of Figs. 1 and 2 to the particular model chosen. First, most models give values of β_0 from 0.5 to 2.5°C per 1% dQ/Q. Our choice is about average. The uncertainty in β_0 contributes an uncertainty of perhaps a factor of 2 to the magnitude of the response at high latitudes. Second, long-term mean meridional heat transport behaves diffusively only for the largest spatial scales. Presumably smaller scale variations would not qualitatively change the falloff in the response to a steady local change in forcing. In this case the inclusion of other transport mechanisms would not eliminate the high-latitude enhancement. Finally, the response function has been greatly simplified by the unrealistic step-function albedo. Actually, the step has been somewhat smoothed out because G_0 has been computed using a finite mode expansion. As more modes are included the step sharpens and the peaks shown in Fig. 1 evolve into sharp cusps. We have also tested the effect of linearly smoothing the albedo around x_s. When the smoothing width becomes comparable to the effective range of the heat transport (~0.1 for the parameters used in Fig. 1) the cusp at the ice edge again becomes a smooth peak. Smoothing the albedo sufficiently also removes the negative slope region near $x_s = 1$ in Fig. 2, so that small icecaps become stable (CN). Despite its strong dependence on the albedo parameterization, the sensitivity appears simply as a multiplying factor of the indirect response, and can be fixed independently of the change in albedo with the ice edge.

Despite the unrealistic simplicity of the particular model chosen here, it exhibits interesting behavior in the heat sink (negative g) region in Fig. 2. The ice edge is lowered to $x_s = 0.7$ if a heat sink having a global average strength of −5 W m^{-2} is placed at x_1 = 0.7, while the same result requires more than twice the strength if the sink is placed at x_1 = 0.4. Once the ice expands below $x_s = 0.7$, however, it becomes much more sensitive to the low-latitude sink. In fact, for a heat sink of −12 W m^{-2}, no stable steady state (other than $x_s = 0$) is found with the low-latitude sink, while the same strength sink in midlatitudes produces a stable climate with nearly 50% ice coverage. These numbers certainly are not intended to be taken seriously, but they show how misleading it can be to extrapolate present climate sensitivities to much different climates. The resistance to total glaciation is not simply related to the ice-edge sensitivity even in our naive example.

3. Discussion

We have determined the steady-state solutions to climate models of the Budyko-Sellers type, generalized to include a latitude-dependent perturbation. The latitude of the ice-cap edge is a nonlinear multivalued function of the perturbation strength, whose slope, the ice edge sensitivity, becomes negative whenever the icecap is unstable. For small perturbations, the temperature relative to the present climate is given in terms of a response function, a generalization of the sensitivity parameter having spatial dependence.

Our response function for the simplest icecap models increases poleward with a peak at the icecap edge. This finding generally agrees with the Manabe and Wetherald (1975) result which also shows a small peak at ~70° latitude—their icecap edge. This is a pleasant surprise since the Manabe-Wetherald model employs many other feedback mechanisms as well as ice feedback.

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APPENDIX

List of Symbols[3]

- A — that part of the heat transport which is inhomogeneous in temperature
- a — coalbedo of the earth-atmosphere system
- β_0, β_s — global temperature sensitivity—steady-state change in T_0 due to a small fractional change in solar constant; β_s refers to a change in x_s rather than T_0
- G, G_0 — local temperature sensitivity—steady-state change in T due to a small localized change in heating: G is also referred to as the stationary temperature response function; G_0 is the temperature response in the absence of ice-albedo feedback
- G^D, $G_0{}^D$ — particular forms of G and G_0 for the diffusive model of North (1975a,b)
- g — global average of the heat perturbation
- L — homogeneous linear operator associated with infrared radiation and horizontal heat transport
- Q, Q_0 — solar constant divided by four; Q_0 is the present value
- S — distribution of incident solar flux, normalized to have unit global average
- T_0 — global average temperature
- T, T_s — local temperature distribution; T_s is the temperature at $x = x_s$
- T^g, δT — perturbed temperature distribution; δT is the deviation from the unperturbed distribution, $\delta T = T^g - T^{(0)}$
- u — distribution of added heat, normalized to have unit global average
- x, x_s, x_0 — sine of the latitude; at the latitude of the ice edge $x = x_s$, and x_0 is the present value of x_s.

[3] Annual, vertical, and zonal averages are assumed.

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