Finance and Economics Discussion Series
Divisions of Research & Statistics and Monetary Affairs
Federal Reserve Board, Washington, D.C.

Diagnosing and Treating Bifurcations in Perturbation Analysis of Dynamic Macro Models

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2007-14

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March 5, 2007

Abstract

In perturbation analysis of nonlinear dynamic systems, the presence of a bifurcation implies that the first-order behavior of the economy cannot be characterized solely in terms of the first-order derivatives of the model equations. In this paper, we use two simple examples to illustrate how to detect the existence of a bifurcation. Following the general approach of Judd (1998), we then show how to apply l'Hospital's rule to characterize the solution of each model in terms of its higher-order derivatives. We also show that in some cases the bifurcation can be eliminated through renormalization of model variables; furthermore, renormalization may yield a more accurate first-order solution than applying l'Hospital's rule to the original formulation.

- JEL Classification: C63; C61; E52.
- Keywords: bifurcation; perturbation; relative price distortion; optimal monetary policy.

1 Introduction

In recent analysis of nonlinear dynamic macroeconomic models, the characterization of their first-order dynamics has been an important step for understanding

*We have benefited from discussion with Gary Anderson, Jean Boivin, Chris Sims, and participants at the Canadian Macroeconomics Study Group meeting in Montreal. The views in this paper are solely the responsibility of the authors and should not be interpreted as reflecting the views of the Board of Governors of the Federal Reserve System or any other person associated with the Federal Reserve System.

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theoretical implications and evaluating empirical success. However, the presence of a bifurcation in perturbation analysis of nonlinear dynamic systems implies that the first-order behavior of the economy cannot be characterized solely in terms of the first-order derivatives of the model equations.

In this paper, we use two simple macroeconomic models to address several issues regarding bifurcations. In particular, the bifurcation problem would emerge in conjunction with the price dispersion generated by staggered price setting in the part of firms. We then show how to apply l'Hospital's rule to characterize the solution of each model in terms of its higher-order derivatives. We also show that in some cases the bifurcation can be eliminated through renormalization of model variables; furthermore, renormalization may yield a more accurate first-order solution than applying l'Hospital's rule to the original formulation.

Before presenting our results, it is noteworthy that our definition of bifurcation is distinct from the one analyzed in Benhabib and Nishimura (1979). In particular, their analysis on bifurcation is associated with time evolution of dynamic systems. However, our concern with bifurcation arises in the process of approximating nonlinear equations, as discussed in Judd (1998).

We proceed as follows. Section 2 describes the two examples and illustrates how to detect the existence of a bifurcation problem. Section 3 follows the general approach of Judd (1998) and applies l'Hospital's rule to characterize the first-order behavior of each model. Section 4 shows how the bifurcation can be eliminated through renormalization of model variables. Section 5 concludes.

2 Diagnosis of Bifurcations

This section discusses how we can detect the existence of bifurcation in two simple economies. In both models, Calvo-style price setting behavior of firms can be summarized by the following law of motion for the relative price distortion:

$$\Delta_t = (1-\alpha)\left(\frac{1-\alpha\Pi_t^{\epsilon-1}}{1-\alpha}\right)^{\frac{\epsilon}{\epsilon-1}} + \alpha\Delta_{t-1}\Pi_t^{\epsilon}, \tag{2.1}$$

where the distortion index is defined as

$$\Delta_t = \int_0^1 \left(\frac{P_t(z)}{P_t}\right)^{-\epsilon} dz.$$

The parameters α and ϵ represent the percentage of firms that cannot change their price in each period and the elasticity of substitution across goods $z \in (0,1)$, respectively. The variable Π_t $(= P_t/P_{t-1})$ is the gross inflation rate of the price index aggregated over firms.

2.1 A Single-Equation Setting

To discuss the issue of bifurcation, we have to close the model with another equation. In the first example, we simply assume that inflation follows an exogenous stochastic process,

$$\Pi_t = U_t,$$

where the logarithm of U_t follows a mean zero process. We can rationalize this process in terms of monetary policy by a version of strict inflation targeting around the exogenous process or a version of strict output-gap targeting in a model with cost-push shocks.

By combining the two equations, we now have a single-equation model:

$$\Delta_t = (1-\alpha)\left(\frac{1-\alpha U_t^{\epsilon-1}}{1-\alpha}\right)^{\frac{\epsilon}{\epsilon-1}} + \alpha \Delta_{t-1} U_t^{\epsilon}. \tag{2.2}$$

Since this equation is backward looking, this exact nonlinear form can be used for any dynamic analysis. However, we suppose that we have to rely on approximation methods to analyze this model as would be the case when there are forward-looking equations.

Woodford (2003) and Benigno and Woodford (2005) pointed out that, when deviations of the (net) inflation rate from its zero steady state are of first order in terms of exogenous variations, deviations of the distortion index from one is of second order. Based on this observation, one can naturally approximate the system with respect to the square root of the logarithm of the relative price distortion index. Note that this distortion index is unity at the steady state with zero inflation rate. We follow the convention of using lower cases for log deviations,

$$\begin{aligned} u_t &= \log U_t, \\ \delta_t &= \log \Delta_t, \\ \gamma_t &= \sqrt{\log \Delta_t}. \end{aligned}$$

Specifically, γ_t corresponds to the approximation in Woodford (2003) and Benigno and Woodford (2005). It will be also shown in Section 4 that δ_t can be used as the basis of an alternative approximation.

Under the choice of γ_t as the approximation variable, (2.2) can be rewritten as follows:

$$f\left(\gamma_t;\gamma_{t-1},u_t\right) \equiv \exp\left(\gamma_t^2\right) - (1-\alpha)\left[\frac{1-\alpha\left(\exp u_t\right)^{\epsilon-1}}{1-\alpha}\right]^{\frac{\epsilon}{\epsilon-1}} - \alpha\exp\left(\gamma_{t-1}^2\right)\left(\exp u_t\right)^{\epsilon} = 0. \tag{2.3}$$

Now let's see what happens if we try a Taylor approximation of this system with respect to γ_t and u_t. It is easy to see that the derivative with respect to the endogenous variable $(\partial f/\partial\gamma_t)$ would be zero at the steady state. Based on this zero derivative, we can diagnose the bifurcation problem in this case. Put in an alternative way, the implicit function theorem cannot be applied when the derivative with respect to the endogenous variable is zero.

It is noteworthy to find out what would happen if we feed this case into computer codes commonly available for dynamic macroeconomic analysis. The Dynare package (version 3.05) produces an message saying '`Warning: Matrix is singular to working precision`', and AIM (developed by Gary Anderson and George Moore, and widely used at the Federal Reserve Board) returns a code indicating '`Aim: too many exact shiftrights`'. The routine developed by Christopher Sims (`gensys.m`) ends without any output or error message.

2.2 A Multi-Dimensional Setting

The second example is a case with multiple equations. Our example is a prototypical Calvo-style sticky-price model, and the optimal policy problem is to maximize the household welfare subject to the following four constraints: the law of motion for relative price distortions, the social resource constraint, the firms' profit maximization condition, and the present-value budget constraint of the household. However, it is shown in Yun (2005) that the optimal policy problem can be reduced to minimizing the index for relative price distortion (2.1). At the optimum, we have the following relationship:

$$\Pi_t = \frac{\Delta_t}{\Delta_{t-1}}. \tag{2.4}$$

Therefore, the solution to the optimal policy problem can be represented with the following bivariate nonlinear system:

$$\begin{bmatrix} \Delta_t - (1-\alpha)\left(\frac{1-\alpha\Pi_t^{\epsilon-1}}{1-\alpha}\right)^{\frac{\epsilon}{\epsilon-1}} - \alpha\Delta_{t-1}\Pi_t^{\epsilon} \\ \Delta_t - \Delta_{t-1}\Pi_t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2.5}$$

As in the single-equation case, we start with a normalization according to which γ_t and π_t are endogenous variables and γ_{t-1} is exogenous:

$$\begin{bmatrix} \exp\left(\gamma_t^2\right) - (1-\alpha)\left[\frac{1-\alpha(\exp\pi_t)^{\epsilon-1}}{1-\alpha}\right]^{\frac{\epsilon}{\epsilon-1}} - \alpha\exp\left(\gamma_{t-1}^2\right)(\exp\pi_t)^{\epsilon} \\ \exp\left(\gamma_t^2\right) - \exp\left(\gamma_{t-1}^2\right)(\exp\pi_t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where $\pi_t\,(=\log\Pi_t)$ is the net inflation rate. When there are multiple equations in the system, the assumption of the implicit function theorem involves the non-singularity of the Jacobian. Computing the determinant for the Jacobian, we have

$$\left|J_{(\gamma_t,\pi_t;\gamma_{t-1})}\right| = \begin{vmatrix} 0 & 0 \\ 0 & -1 \end{vmatrix} = 0.$$

Since the Jacobian is singular, the implicit function theorem cannot be applied and the regular perturbation method does not work. We need to rely on the bifurcation method.

3 Resolution of Bifurcations

As explained in Judd (1998) and Judd and Guu (2001), the bifurcation problem can be resolved by using l'Hospital's rule.

3.1 A Single-Equation Setting

To understand the approximated behavior of γ_t in the singe-equation example, we need to compute $\partial\gamma_t\left(\gamma_{t-1},u_t\right)/\partial\gamma_{t-1}$ and $\partial\gamma_t\left(\gamma_{t-1},u_t\right)/\partial u_t$ where $\gamma_t\left(\gamma_{t-1},u_t\right)$ is defined as an implicit function as follows:

$$f\left(\gamma_t\left(\gamma_{t-1},u_t\right);\gamma_{t-1},u_t\right)=0.$$

In cases for which regular perturbation analysis could be applied, the first-order approximation of $\gamma_t\left(\gamma_{t-1},u_t\right)$ would come from the implicit function theorem as follows:

$$\gamma_t^{(1)}\left(\gamma_{t-1},u_t\right) = -\frac{\partial f/\partial\gamma_{t-1}}{\partial f/\partial\gamma_t}\gamma_{t-1} - \frac{\partial f/\partial u_t}{\partial f/\partial\gamma_t}u_t.$$

The number in the parenthesis indicates the order of approximation. However, the assumption of the implicit function theorem does not hold in our case since $\partial f/\partial\gamma_t = 0$. We need to adopt an advanced asymptotic method—the bifurcation method in this case.

Noting that the derivatives in the numerators are also zero at the steady state, we apply l'Hospital's rule to the two ratios in the form of 0/0 and obtain the following first-order approximation:[1]

$$\gamma_t^{(1)}\left(\gamma_{t-1}, u_t\right) = \sqrt{\alpha}\gamma_{t-1} + \sqrt{\frac{\alpha\epsilon}{2\left(1-\alpha\right)}}u_t. \tag{3.6}$$

This is an example of the transcritical bifurcation.[2]

In this single-equation model, it is easy to avoid the bifurcation problem when we consider the following equation that is equivalent to (2.3),

$$\tilde{f}\left(\gamma_t; \gamma_{t-1}, u_t\right) \equiv \gamma_t - \sqrt{\log\left\{\left(1-\alpha\right)\left[\frac{1-\alpha\left(\exp u_t\right)^{\epsilon-1}}{1-\alpha}\right]^{\frac{\epsilon}{\epsilon-1}} + \alpha\exp\left(\gamma_{t-1}^2\right)\left(\exp u_t\right)^{\epsilon}\right\}} = 0.$$

The derivative $\partial\tilde{f}/\partial\gamma_t$ becomes nonzero, so the assumption of the implicit function theorem is satisfied. However, we still have to use l'Hospital's rule in computing the derivative with respect to the exogenous variables: $\partial\tilde{f}/\partial\gamma_{t-1}$ and $\partial\tilde{f}/\partial u_t$.

3.2 A Multi-Dimensional Setting

To illustrate how we can invoke the bifurcation method in the multi-dimensional example, we substitute the second equation in (2.5) into the first to obtain:

$$\begin{aligned} 0 &= F\left(\gamma_t; \gamma_{t-1}\right) \\ &\equiv \exp\left(\gamma_t^2\right) - \left(1-\alpha\right)\left[\frac{1-\alpha\left(\frac{\exp\left(\gamma_t^2\right)}{\exp\left(\gamma_{t-1}^2\right)}\right)^{\epsilon-1}}{1-\alpha}\right]^{\frac{\epsilon}{\epsilon-1}} - \alpha\exp\left(\gamma_{t-1}^2\right)\left(\frac{\exp\left(\gamma_t^2\right)}{\exp\left(\gamma_{t-1}^2\right)}\right)^{\epsilon}. \end{aligned}$$

Were the assumptions of the bifurcation theorem to hold, then differentiation of the implicit expression $F\left(\gamma_t\left(\gamma_{t-1}\right); \gamma_{t-1}\right) = 0$ with respect to γ_{t-1} would produce the equation

$$\frac{\partial\gamma_t\left(\gamma_{t-1}\right)}{\partial\gamma_{t-1}} = -\left(\frac{\partial F}{\partial\gamma_t}\right)^{-1}\frac{\partial F}{\partial\gamma_{t-1}}.$$

However, since both derivatives on the right-hand side are zero at the steady state, we need to apply l'Hospital's rule to compute $\partial\gamma_t/\partial\gamma_{t-1}$. The first-order solution for γ_t is

$$\gamma_t^{(1)}\left(\gamma_{t-1}\right) = \sqrt{\alpha}\gamma_{t-1},$$

[1] A detailed derivation of this first-order approximation is available upon request.

[2] See Judd (1998, Ch. 15) and Judd and Guu (2001) for details on the bifurcation method.

and the second-order accurate expression for inflation is

$$\pi_t^{(2)}\left(\gamma_{t-1}\right) = -\left(1-\alpha\right)\gamma_{t-1}^2. \tag{3.7}$$

Note that the dependence of π_t on γ_{t-1} is purely quadratic (i.e. the zero coefficient for the linear term) around the steady state with zero inflation rate.

4 Renormalization of Model Variables

The presence of bifurcations is not only related to the economic model in hand, but also to the choice of the variable with respect to which the Taylor approximation is applied. This section shows that the bifurcation can be eliminated through renormalization of model variables; furthermore, renormalization may yield a more accurate first-order solution than applying l'Hospital's rule to the original formulation.

4.1 A Single-Equation Setting

In the single-equation setting, if we can approximate the model with respect to δ_t and δ_{t-1} instead of γ_t and γ_{t-1}, then the bifurcation problem would not emerge.[3] To see this, rewrite (2.2) as follows:

$$g\left(\delta_t;\delta_{t-1},u_t\right) \equiv \exp\delta_t - \left(1-\alpha\right)\left[\frac{1-\alpha\left(\exp u_t\right)^{\epsilon-1}}{1-\alpha}\right]^{\frac{\epsilon}{\epsilon-1}} - \alpha\left(\exp\delta_{t-1}\right)\left(\exp u_t\right)^{\epsilon} = 0.$$

With this renormalization, the second-order Taylor approximation of δ_t yields the second-order solution for the endogenous variable:

$$\delta_t^{(2)}\left(\delta_{t-1},u_t\right) = \alpha\delta_{t-1} + \frac{\alpha}{2}\delta_{t-1}^2 + \alpha\epsilon\delta_{t-1}u_t + \frac{\alpha\epsilon}{2\left(1-\alpha\right)}u_t^2.$$

This choice of expansion variable implies that, when the initial relative price distortion is of first—rather than purely second—order, the current relative price distortion is also of first order. That is, the relative price distortion is of the same order of magnitude as the shocks. This equation differs from what we would obtain by squaring both sides of (3.6) because the renormalization leads to the presence of the $\left(\alpha\delta_{t-1}^2/2 + \alpha\epsilon\delta_{t-1}u_t\right)$ term.

[3]Examples include Schmitt-Grohe and Uribe (2006) and Levin et al. (2006).

Under this renormalization, the expression for the relative price distortion is richer—and more accurate—than (3.6) derived using l'Hospital's rule. Another renormalization that produces a solution similar to (3.6) is to approximate with respect to γ_{t-1} (instead of δ_{t-1}). This alternative way is based on the interpretation that the initial relative price distortion is of second order. Specifically, we rewrite the model as

$$h\left(\delta_t; \gamma_{t-1}, u_t\right) \equiv \exp \delta_t - (1-\alpha)\left[\frac{1-\alpha\left(\exp u_t\right)^{\epsilon-1}}{1-\alpha}\right]^{\frac{\epsilon}{\epsilon-1}} - \alpha\left(\exp \gamma_{t-1}^2\right)\left(\exp u_t\right)^{\epsilon} = 0,$$

and the second-order behavior of the endogenous variable becomes purely quadratic,

$$\delta_t^{(2)}\left(\gamma_{t-1}, u_t\right) = \alpha\gamma_{t-1}^2 + \frac{\alpha\epsilon}{2\left(1-\alpha\right)}u_t^2.$$

Since this expression is purely second order, it is consistent with the results under the timeless perspective—a la Woodford (2003) and Benigno and Woodford (2005)—that the relative price distortions are zero when we focus solely on the first-order approximation.[4]

4.2 A Multi-Dimensional Setting

In the multi-dimensional case, the two ways of renormalization would correspond to

$$\begin{bmatrix} \exp \delta_t - (1-\alpha)\left[\frac{1-\alpha(\exp \pi_t)^{\epsilon-1}}{1-\alpha}\right]^{\frac{\epsilon}{\epsilon-1}} - \alpha\left(\exp \delta_{t-1}\right)\left(\exp \pi_t\right)^{\epsilon} \\ \exp \delta_t - \left(\exp \delta_{t-1}\right)\left(\exp \pi_t\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} \exp \delta_t - (1-\alpha)\left[\frac{1-\alpha(\exp \pi_t)^{\epsilon-1}}{1-\alpha}\right]^{\frac{\epsilon}{\epsilon-1}} - \alpha\left(\exp \gamma_{t-1}^2\right)\left(\exp \pi_t\right)^{\epsilon} \\ \exp \delta_t - \left(\exp \gamma_{t-1}^2\right)\left(\exp \pi_t\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Either way, the determinant of the Jacobian is nonzero,

$$\left|J_{(\delta_t, \pi_t)}\right| = \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} = -1,$$

and the implicit function theorem can be applied. The computer codes written for the regular perturbation methods would work.

[4]In perturbation analysis, the timeless perspective has implications for the choice of normalization variables. This point can be equivalently addressed in a linear-quadratic analysis. The timeless perspective is also different from the Ramsey optimal policy in terms of how to deal with the lagged Lagrange multiplier for the behavior of models and their welfare implications.

According to the first renormalization, the second-order approximation of (2.1) is

$$\delta_t^{(2)} = \alpha\delta_{t-1} + \frac{\alpha}{2}\delta_{t-1}^2 + \alpha\epsilon\delta_{t-1}\pi_t + \frac{\alpha\epsilon}{2(1-\alpha)}\pi_t^2, \tag{4.8}$$

and the logarithmic transformation of (2.4) is

$$\pi_t = \delta_t - \delta_{t-1}.$$

Therefore, the second-order solution of this problem would be

$$\begin{aligned}\delta_t^{(2)}(\delta_{t-1}) &= \alpha\delta_{t-1} + \frac{\alpha(1-\epsilon+\alpha\epsilon)}{2}\delta_{t-1}^2, \\ \pi_t^{(2)}(\delta_{t-1}) &= -(1-\alpha)\delta_{t-1} + \frac{\alpha(1-\epsilon+\alpha\epsilon)}{2}\delta_{t-1}^2.\end{aligned}$$

It is noteworthy to point out that, according to this renormalization, the first-order relationship between inflation and relative price distortions ($\pi_t^{(1)}(\delta_{t-1}) = \delta_t^{(1)}(\delta_{t-1}) - \delta_{t-1}$) replicates the exact nonlinear relationship (2.4).

The alternative renormalization consistent with the timeless perspective is to adopt γ_{t-1} (instead of δ_{t-1}) as an exogenous variable. Based on this choice of an expansion parameter, Woodford (2003) concluded that the optimal inflation rate is zero to the first order in the absence of cost-push shocks. Under this normalization, the two model equations are approximated as follows:

$$\begin{aligned}\delta_t^{(2)} &= \alpha\gamma_{t-1}^2 + \frac{\alpha\epsilon}{2(1-\alpha)}\pi_t^2, \\ \pi_t &= \delta_t - \gamma_{t-1}^2.\end{aligned}$$

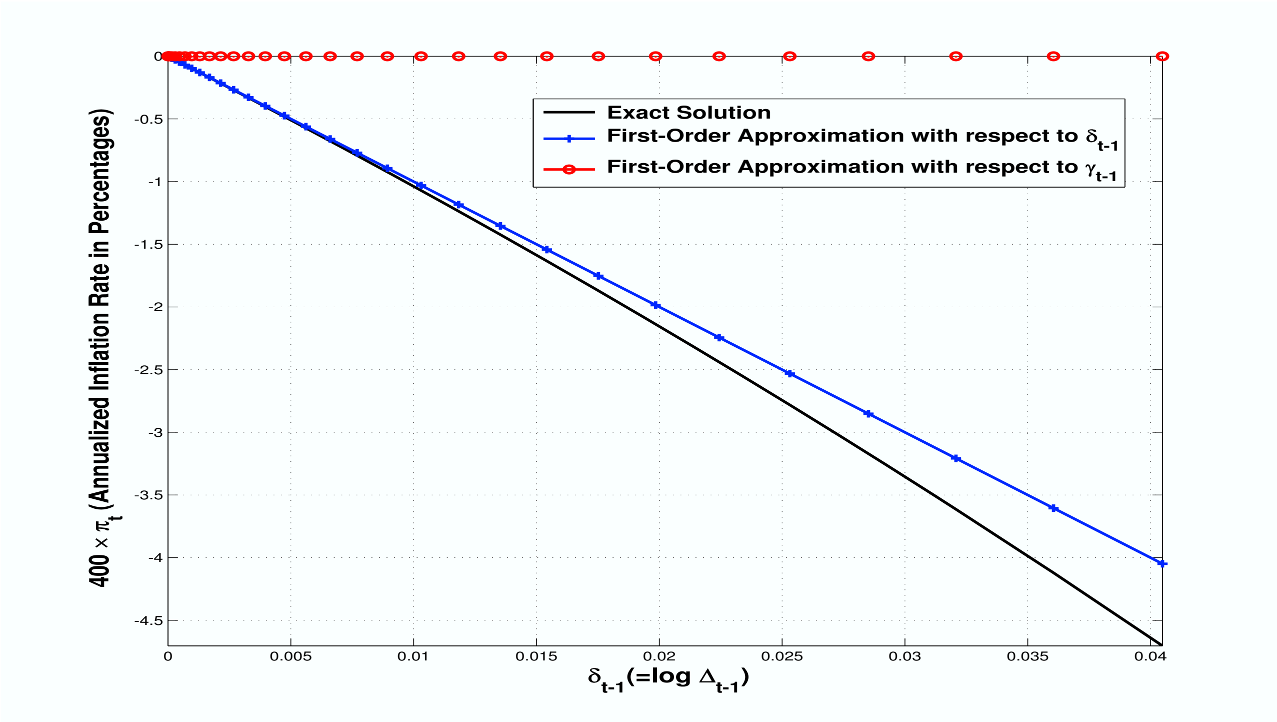
The second-order solution to this system of equations would be purely quadratic[5]

$$\begin{aligned}\delta_t^{(2)}(\gamma_{t-1}) &= \alpha\gamma_{t-1}^2, \\ \pi_t^{(2)}(\gamma_{t-1}) &= -(1-\alpha)\gamma_{t-1}^2.\end{aligned}$$

The first-order approximation of this solution is consistent with the optimality of zero inflation, as derived in the linear-quadratic approximation by Woodford (2003), Benigno and Woodford (2005), and Levine, Pearlman and Pierse (2006). Furthermore, the second-order solution for inflation is equivalent to the one via the bifurcation method, (3.7).

[5] To use this solution recursively, we have to make the set of equations complete in the dynamic context by adding an equation ($\delta_t - \gamma_t^2 = 0$) that relates the contemporaneous value of δ_t and γ_t. There are two ways to express the solution of this complete system. One way is to replace γ_{t-1}^2 with δ_{t-1}. Alternatively, we can derive the linear dynamics of γ_t as $\gamma_t^{(1)}(\gamma_{t-1}) = \sqrt{\alpha}\gamma_{t-1}$.

Figure 1: Renormalizations and Optimal Inflation Rates



4.3 Accuracy Comparison

After presenting two different renormalizations, it is natural to compare approximation errors for these two methods.[6] For this purpose, we use as a reference point the closed-form solution to the optimal policy problem (2.5). Specifically, as shown in Yun (2005), the exact nonlinear solution for the optimal inflation rate is

$$\Pi_t\left(\Delta_{t-1}\right)=\left[\alpha+(1-\alpha)\,\Delta_{t-1}^{\epsilon-1}\right]^{\frac{-1}{\epsilon-1}}. \tag{4.9}$$

It is noteworthy that this closed-form solution is feasible only when the relative price distortion is the only distortion—due to the assumption that there is an optimal subsidy and there are no cost-push shocks. The optimal rate of inflation is less than zero ($\Pi_t < 1$) as long as there are initial price distortions ($\Delta_{t-1} > 1$).

The difference between the two methods is that the expansion parameter of the first renormalization is δ_{t-1}, while that of the second is γ_{t-1}. Figure 1 compares

[6]Benigno and Woodford (2005) also comment on the difference between the two methods in Footnotes 30 and 52.

the accuracy of the two normalizations based on the first-order solution under each normalization.[7] The black solid line represents the exact closed-form solution for annualized inflation ($400 \times \pi_t$) in terms of initial relative distortion (δ_{t-1}). The blue line with crosses is the linear approximation of this nonlinear solution. This corresponds to the first-order approximation of π_t when the expansion parameter is δ_{t-1}—that is, $\pi_t^{(1)}(\delta_{t-1})$. It is evident that this approximation is more accurate than $\pi_t^{(1)}(\gamma_{t-1})$: the first-order approximation with γ_{t-1} as the expansion parameter, depicted by the red circles.

We can provide an intuitive understanding about the improved accuracy of the approximation with respect to δ_{t-1} as follows. Since δ_{t-1} is the square of γ_{t-1}, the first-order approximation with respect to δ_{t-1} is equivalent to the second-order approximation with respect to γ_{t-1}:

$$\pi_t^{(1)}(\delta_{t-1}) = \pi_t^{(2)}(\gamma_{t-1}).$$

Note that the equality holds because no linear terms are included in $\pi_t^{(2)}(\gamma_{t-1})$ with zero steady-state inflation rate.

5 Conclusion

We have illustrated how to detect the existence of a bifurcation and demonstrated how to apply l'Hospital's rule to characterize the solution. We have also shown that the bifurcation can be eliminated through renormalization of model variables; furthermore, renormalization may yield a more accurate first-order solution than applying l'Hospital's rule to the original formulation. This paper has focused on the consequences of renormalization on the treatment of bifurcations. However, the renormalization is also associated with the welfare evaluation of different policies as in Benigno and Woodford (2005).

[7] Other ways to measure accuracy include checking Euler equation errors and calculating welfare costs caused by approximation errors.

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