ORS Working Paper Series

Number 48

On the Existence of Pareto-Superior Reversals of Dynamically Inefficient Social Security Programs

Dean R. Leimer*

Division of Economic Research

June 1991

Social Security Administration
Office of Research and Statistics

* Division of Economic Research, Social Security Administration, 4301 Connecticut Ave., N.W., Washington, D.C. 20008

The author wishes to thank Benjamin Bridges, Jr. and Selig Lesnoy for helpful comments on an earlier draft.

I. Introduction.

Some proponents of the privatization of the Social Security program in the United States have suggested that, because privately available rates of return exceed the internal rate of return implicit in that program, it may be possible to find Pareto-superior privatization schemes.[1] In a similar vein, Townley [1981] argues that, so long as the government can incur debt, a Pareto-superior scheme can always be found to convert a dynamically inefficient pay-as-you-go social security program to a fully funded basis. This note uses Townley's own model to demonstrate analytically that Pareto-superior schemes to reverse a dynamically inefficient pay-as-you-go social security program do not exist, either through privatization or through conversion of the program to a fully funded basis.

Townley assumes a model in which a pay-as-you-go social security program has been imposed on an economy in which the exogenously-determined market interest rate always exceeds the exogenously-determined aggregate economic growth rate. A mature pay-as-you-go social security program is dynamically inefficient in such an economy, because each cohort[2] is forced to contribute to a social security program whose implicit rate of return (equal to the aggregate economic growth rate) is less than the market interest rate that could be earned under

---

[1]For example, see Buchanan [1979] and Ferrara [1985].

[2]The term "cohort" is used throughout this note to refer to the group of individuals born within a given period of time, such as a year.

a fully funded social security program.[3] If a fully funded program could be substituted for the pay-as-you-go program, then future cohorts would benefit from the higher returns.

Townley argues that, as long as the government is able to incur debt, a number of Pareto-superior methods are available for transforming the pay-as-you-go program to a fully funded basis; if true, this implies that future cohorts could benefit from the higher rate of return associated with the funded program without disadvantaging any present or intermediate cohorts. The general idea behind the transformations suggested by Townley can be described as follows. First, taxes under the existing social security program are abolished, and a new, fully funded, social security program is instituted with annual tax payments identical to those under the pay-as-you-go program. Second, government debt is created to effectively "buy out" the retirement benefit rights already earned by present retirees and workers net of their prospective tax liability under the pay-as-you-go program. The particular plan considered by Townley accomplishes this gradually over time by issuing government debt as the benefit payments come due; i.e., total retirement benefits are guaranteed not to decline under the new program, with any difference between what benefits would have been under the pay-as-you-go program and what they are under the fully funded program being made up through public borrowing.

The key to the Townley proposal is that benefits under the new fully funded plan are only paid at a rate of return equal to the rate of aggregate economic growth, rather than at the higher market interest rate. The excess interest earnings of the fund are then used to retire the debt

---

[3]This well-known result was developed by Aaron [1966] in a model with similar assumptions to those adopted here. In the real world, this definition of dynamic efficiency requires that the market interest rate used for comparison with the implicit rate of return to social security is appropriately defined, so that both rates reflect the same degree of risk and liquidity. See Leimer [1991] for a discussion of this issue.

created to buy out net benefit rights under the pay-as-you-go program. Because this rate of return paid to workers under the new funded program is equal to the implicit rate that they would have earned on their tax contributions under the pay-as-you-go program, no present or future workers would be disadvantaged by this scheme. On the basis of computer simulations, Townley asserts that the excess interest earnings of the new program will be sufficient to eventually retire the entire buyout debt, at which point the new program can begin paying benefits at the market interest rate, increasing the returns to future cohorts. Under this transition scheme, then, Townley asserts that future cohorts can be made better off without disadvantaging any present or intermediate cohorts.

Although Townley does not supply an analytical proof, the apparent intuition behind the scheme is that the buyout debt is finite and arises from only those cohorts participating in the program as taxpayers or beneficiaries at the time that the new program is put into effect. In contrast, the number of future cohorts potentially benefiting from the transition to a fully funded program can be treated as infinite. Consequently, it might not seem unreasonable that the buyout debt could eventually be retired without disadvantaging any present or intermediate cohorts. This note, however, will demonstrate analytically the fallacy of this conclusion. In addition, this note indicates how computer simulations of such proposals could support the incorrect conclusion.

II. A Mathematical Proof.

To facilitate comparison, this paper uses similar notation to that used by Townley. The basic model assumes an exogenous and constant interest rate, r, which is larger than the exogenous and constant population growth rate, g. For simplicity, population growth is assumed to be the only source of aggregate economic growth, although this is not critical to the basic conclusion. A multi-period life cycle is assumed, during which each individual enters employment at age W, pays a constant annual social security tax of Y during the working years, retires at age R, receives a constant annual social security benefit of Z during the retirement years, and finally dies at age T.

The relative sizes of successive birth cohorts and age groups can be determined from the constant growth rate assumption. Let $N_{a,t}$ denote the population of age a at time t. This implies that

(1) $$N_{o,t} = N_{o,o}(1+g)^{t} \, , \quad \text{and}$$

(2) $$N_{a,t} = N_{o,t-a} = N_{o,o}(1+g)^{t-a} \, .$$

The population of working age, then, can be derived as

$$P_t^W = \sum_{a=W}^{R-1} N_{a,t} = \sum_{a=W}^{R-1} N_{o,o}(1+g)^{t-a} = N_{o,o}(1+g)^{t-W}\left[\frac{1-(1+g)^{-(R-W)}}{1-(1+g)^{-1}}\right] \Rightarrow$$

(3) $$P_t^W = \frac{N_{o,o}(1+g)^{t-W+1}}{g}\left[1-(1+g)^{-(R-W)}\right] .$$

By analogy, the retired population can be represented as

(4) $$P_t^R = \sum_{a=R}^{T-1} N_{a,t} = \frac{N_{o,o}(1+g)^{t-R+1}}{g}\left[1-(1+g)^{-(T-R)}\right] .$$

The relationship between the annual individual social security tax payment and benefit under the pay-as-you-go program can be derived from the equality of aggregate annual taxes and benefits; i.e.,

$$Y P_t^W = Z P_t^R \;\Rightarrow\; Z = Y \left(\frac{P_t^W}{P_t^R} \right) \;\Rightarrow$$

$$\text{(5)} \quad Z = Y(1+g)^{R-W} \left[\frac{1-(1+g)^{-(R-W)}}{1-(1+g)^{-(T-R)}} \right] ,$$

from equations (3) and (4).

We can simplify analysis of the Townley proposal by identifying an alternative, but equivalent, funding scheme and breaking it into parts. Under this alternative scheme, the social security program is abolished, and government debt is created immediately to buy out the unfunded liability[4] of the program at that time; i.e., to buy out the net benefit rights already earned by cohorts participating in the pay-as-you-go program at the time that it is abolished. At the same time, a fully funded program is initiated, with tax contributions equal to those under the pay-as-you-go program, but paying the full market rate of return on those contributions. Taxes are imposed on future cohorts entering employment after the pay-as-you-go program is abolished, however, equal to the lifetime gains that these cohorts experience by virtue of the higher rate of return realized under the new fully funded program relative to the lower rate of return that they would have earned under the old pay-as-you-go program. These tax proceeds are then applied to retire the accumulating buyout debt that was created to leave unchanged the lifetime incomes of those cohorts participating in the pay-as-you-go program at the time it was abolished.

---

[4]A "closed group" definition of unfunded liability is appropriate here. Under our assumptions, the unfunded liability equals the present value of expected lifetime benefits less taxes for all present participants in the social security program, evaluated at the market interest rate.

As under the Townley proposal, then, no cohort is disadvantaged by this alternative scheme. In fact, the schemes are functionally equivalent, both from the perspective of individuals and the government. The main differences between the two schemes are the timing of the buyout debt creation and the timing and form of the recapture from future cohorts of the potential gains that they experience from the substitution of a fully funded program for the dynamically inefficient pay-as-you-go program.

The Townley scheme creates the initial buyout debt gradually as benefit claims under the old pay-as-you-go program come due. In contrast, the scheme considered here creates the initial buyout debt immediately at the time that the pay-as-you-go program is abolished. The two approaches are functionally equivalent, however, since benefit promises under the Townley proposal are guaranteed not to fall below those under the pay-as-you-go program. Hence, the initial buyout debt created under our alternative proposal can be interpreted as the amount that would have to be invested at the market interest rate at the time that the Townley proposal is instituted in order to fund all future benefits claims less tax payments under that proposal for each cohort of employment age or older at the time that the proposal is instituted.

Similarly, the Townley proposal recaptures the potential gains to future cohorts under the funded program by only returning benefits under that program at the rate of growth in aggregate output; the potential gains are captured gradually, then, over the lifetimes of these cohorts. Our alternative proposal effects the same end result by paying these cohorts the market interest rate in the funded program, but imposing a one-time tax, payable at the time of employment entry, equal to the present value of the lifetime gains experienced by these cohorts under the funded program. As under the Townley proposal, this tax can be dropped should the buyout debt be retired at some future point. It should be clear, then, that, while the Townley proposal and the alternative considered here differ in detail, they have the same effects on the lifetime incomes of all affected cohorts, the same aggregate effects in terms of government finances, and the same implications for establishing the existence or non-existence of Pareto-superior reversals of dynamically inefficient social security programs.

This alternative proposal can be broken into parts to facilitate analysis. First, we compute the present value of the lifetime gains from the new program that are potentially realized by each future cohort member entering employment after the start-up of the funded program. This present value, denoted as L in the equations below, is constant across cohorts under our simplifying assumption of zero productivity growth and represents the present value of the lifetime taxes that can be imposed on each future cohort member under the funded program without decreasing net lifetime income below what would have been experienced under the pay-as-you-go program. Second, we compute the unfunded liability of the pay-as-you-go program at the time of the start-up of the funded program. Denoted as D_o in the equations below, this represents the present value of the net benefit rights already earned under the pay-as-you-go program that must be bought out under the new program in order to leave unchanged the lifetime income of all cohorts working or retired at the time of the new program start-up. Finally, the status of the buyout debt over time under this alternative funding scheme can be identified by accumulating the initial buyout debt, D_o, less the lifetime tax payments, L, imposed on members of successive future cohorts and applied to the retirement of the buyout debt. Although Townley claims that this buyout debt eventually will be retired completely, we shall find that it continues to grow exponentially over time at the rate of growth in aggregate output.

The value of L can be identified as the excess of the present value of taxes less benefits for each future cohort member under the old pay-as-you-go program; i.e., this excess represents the present value of the lifetime gains from the new program that can be taxed away without decreasing their net lifetime income below what would have been experienced under the pay-as-you-go program. Specifically,

$$L = \sum_{a=W}^{R-1} \frac{Y}{(1+r)^{a-W}} - \sum_{a=R}^{T-1} \frac{Z}{(1+r)^{a-W}} \Rightarrow$$

$$L = Y\left[\frac{1-(1+r)^{-(R-W)}}{1-(1+r)^{-1}}\right] - \frac{Z}{(1+r)^{R-W}}\left[\frac{1-(1+r)^{-(T-R)}}{1-(1+r)^{-1}}\right] \Rightarrow$$

$$L = \frac{Y(1+r)}{r}\left[1-(1+r)^{-(R-W)}\right] - \frac{Z(1+r)}{r(1+r)^{R-W}}\left[1-(1+r)^{-(T-R)}\right] \Rightarrow$$

$$(6) \quad L = \left(\frac{1+r}{r}\right)\left[Y - \frac{Y}{(1+r)^{R-W}} - \frac{Z}{(1+r)^{R-W}} + \frac{Z}{(1+r)^{T-W}}\right].$$

As expected, L is constant across cohorts. By virtue of the assumption that $r > g$, L must also be positive.[5]

Denote as period zero ($t=0$) the time when the pay-as-you-go program is abolished and the new funded program is initiated. As indicated above, the value of the initial buyout debt, D_o, can be identified simply as the present value of future benefits less taxes expected under the pay-as-you-go program by workers and retirees at that time. This represents the present value

[5]See the Appendix in Leimer [1991] for further explanation of this result.

of the net benefit rights already established under the pay-as-you-go program that must be paid off in order to leave unchanged the lifetime income of these cohorts; i.e.,

$$(7)\quad D_o = \sum_{a=R}^{T-1} N_{a,o} \sum_{i=a}^{T-1} \frac{Z}{(1+r)^{i-a}} + \sum_{a=W}^{R-1} N_{a,o} \left[\sum_{i=R}^{T-1} \frac{Z}{(1+r)^{i-a}} - \sum_{i=a}^{R-1} \frac{Y}{(1+r)^{i-a}} \right] .$$

Although the derivation is tedious and therefore relegated to the Appendix, equation (7) simplifies to

$$(8)\quad D_o = \frac{N_{o,o} L (1+g)^{1-W}}{(r-g)} .$$

As noted above, the buyout debt under the Townley proposal can now be identified by accumulating over time the initial buyout debt, D_o, less the lifetime tax payments, L, imposed on members of each future cohort and applied to the retirement of the buyout debt. The buyout debt at time zero, of course, is given by equation (8). The buyout debt in the following period equals the initial buyout debt, accumulated at rate r, less the lifetime gain tax payment levied against the cohort entering employment at that time; i.e.,

$$D_1 = D_o(1+r) - N_{W,1} L = D_o(1+r) - N_{o,o} L (1+g)^{1-W} .$$

Similarly, the accumulated buyout debt in the second period is given by

$$D_2 = D_1(1+r) - N_{W,2} L = D_o(1+r)^2 - N_{o,o} L (1+g)^{1-W} (1+r) - N_{o,o} L (1+g)^{2-W} .$$

The corresponding expression for the third period is given by

$$\begin{aligned} D_3 &= D_2(1+r) - N_{W,3} L \\ &= D_o(1+r)^3 - N_{o,o} L (1+g)^{1-W} (1+r)^2 - N_{o,o} L (1+g)^{2-W} (1+r) - N_{o,o} L (1+g)^{3-W} . \end{aligned}$$

In general, then, it is readily apparent that the accumulated buyout debt for any future period t is given by the expression

$$D_t = D_o(1+r)^t - N_{o,o}L\sum_{s=1}^{t}(1+g)^{s-W}(1+r)^{t-s}$$

$$= D_o(1+r)^t - N_{o,o}L(1+g)^{-W}(1+r)^t\sum_{s=1}^{t}\left[\frac{1+g}{1+r}\right]^s$$

$$= (1+r)^t\left\{D_o - N_{o,o}L(1+g)^{-W}\left[\frac{1+g}{1+r}\right]\left[\frac{1-\left[\frac{1+g}{1+r}\right]^t}{1-\left[\frac{1+g}{1+r}\right]}\right]\right\} \Rightarrow$$

(9) $$D_t = (1+r)^t\left\{D_o - N_{o,o}L(1+g)^{-W}\left[\frac{1+g}{r-g}\right]\left[1-\left[\frac{1+g}{1+r}\right]^t\right]\right\} .$$

Substituting from equation (8) simplifies this expression to

(10) $$D_t = \frac{N_{o,o}L(1+g)^{1-W}}{r-g}(1+g)^t = D_o\,(1+g)^t > 0 ,$$

which identifies the buyout debt remaining at any time $t \geq 0$.

III. Implications.

In contrast to Townley's claim that the buyout debt will eventually be completely retired, it is immediately apparent from equation (10) that the buyout debt remains positive for all time, growing exponentially at the rate of growth in aggregate output.[6] As shown in the Appendix to this note, the unfunded liability of a pay-as-you-go social security program also grows over

[6]Breyer [1989] reaches the same conclusion based on a simpler two-period life cycle model.

time at the rate of growth in aggregate output. Consequently, the buyout debt under the full funding scheme considered above remains equal in all periods to what the unfunded liability of the pay-as-you-go program would have been had that program been continued; i.e., the initial buyout debt is equal to the unfunded liability of the pay-as-you-go program at time zero by definition, and both the buyout debt and unfunded liability grow at the rate of growth in aggregate output thereafter.

This result illustrates the Pareto efficiency of existing pay-as-you-go social security programs, even when such programs are dynamically inefficient in the sense that they generate internal rates of return below the market interest rate. Even if all future cohorts would benefit if a fully funded program could be substituted for the existing pay-as-you-go program, there is no way to make the transition to a fully funded program without reducing the lifetime income and consumption of at least some cohorts or without increasing the total indebtedness of the government, where total indebtedness is defined to include both explicit debt and the implicit unfunded liability of the pay-as-you-go social security program. Any privatization or full funding scheme that increases the lifetime income and consumption of one cohort must necessarily do so at the expense of other cohorts, then, either directly through current transfers or indirectly through changes in national saving.[7]

An inspection of equation (9) above provides some insight into how a computer simulation of the buyout debt D_t could lead one to the wrong conclusion. Limited precision is a well-known deficiency of typical computer implementations of floating point numbers. Because such

---

[7]This conclusion can be extended to models with endogenous determination of factor returns and is also likely to hold in the real world, even when such complications as administrative costs and variable labor supply are introduced. See Leimer [1991] for a discussion of these issues.

numbers are represented internally in the computer with a limited number of significant digits, an expression that can be shown to be zero analytically may actually be calculated by the computer program as a very small non-zero value. In the present application, where values are accumulated over a possibly large number of periods, even very small errors in initial calculations can lead to erroneous conclusions. In equation (9), for example, if the internal expression

$$D_o - N_{o,o} L (1+g)^{-w} \left[\frac{1+g}{r-g} \right]$$

is not calculated precisely as zero, the entire expression for D_t will converge to the value

$$D_t = (1+r)^t \left[D_o - N_{o,o} L (1+g)^{-w} \left[\frac{1+g}{r-g} \right] \right]$$

for large t. Consequently, after some point, the computer simulation is likely to imply that the buyout debt begins to grow at the market interest rate, being either positive or negative in value, depending on the nature of the imprecision of the initial calculations.[8] If negative in value, this result is consistent with the erroneous conclusion that the buyout debt would eventually be completely retired, allowing subsequent cohorts to obtain the market interest rate on their pension contributions. This may explain the misleading simulation results obtained by Townley.

---

[8]This possibility was confirmed by a number of computer simulations of the Townley proposal. In general, the remaining buyout debt in these simulations would grow initially at the rate of growth in output but eventually converge to an ultimate growth rate equal to the assumed market interest rate. In some cases, the buyout debt would be retired but, in other cases, remain positive for all time. These conflicting outcomes could be induced by simply changing the initial values of certain variables, such as population, that should have no effect on the qualitative outcome of whether the buyout debt could be retired. In short, these computer simulations generated erroneous outcomes that depended in seemingly random fashion on the initial values selected for various variables, suggesting the imprecision of initial calculations as the culprit.

References

Aaron, Henry, "The Social Insurance Paradox," *Canadian Journal of Economics and Political Science*, August 1966, *32*, 371-4.

Breyer, Friedrich, "On the Intergenerational Pareto Efficiency of Pay-as-you-go Financed Pension Systems," *Journal of Institutional and Theoretical Economics*, December 1989, *145*, 643-58.

Buchanan, James M., "Commentaries," Comment on Edgar K. Browning, "The Politics of Social Security Reform," in Colin D. Campbell, ed., *Financing Social Security*, Washington, D.C.: American Enterprise Institute for Public Policy Research, 1979, 208-12.

Ferrara, Peter J., "Social Security Reform: Some Theoretical Considerations," in Peter J. Ferrara, ed., *Social Security: Prospects for Real Reform*, Washington, D.C.: Cato Institute, 1985, 173-89.

Leimer, Dean R., "The Pareto Optimality of Pay-As-You-Go Social Security Programs," ORS Working Paper, Office of Research and Statistics, Social Security Administration, forthcoming 1991.

Townley, Peter G.C., "Public Choice and the Social Insurance Paradox: A Note," *Canadian Journal of Economics*, November 1981, *14*, 712-7.

Appendix. Derivation of the Pay-As-You-Go Unfunded Liability

Under the assumptions of the model developed in the text, the unfunded liability associated with the strict pay-as-you-go social security program can be derived as

$$F_t = \sum_{a=R}^{T-1} N_{a,t} \sum_{i=a}^{T-1} \frac{Z}{(1+r)^{i-a}} + \sum_{a=W}^{R-1} N_{a,t} \left[\sum_{i=R}^{T-1} \frac{Z}{(1+r)^{i-a}} - \sum_{i=a}^{R-1} \frac{Y}{(1+r)^{i-a}} \right]$$

for any arbitrary point in time t; i.e., the unfunded liability is simply the present value of future benefits less taxes expected under the program by workers and beneficiaries participating in the program at that time. Substituting from equation (2) of the text for the population of each age at that time, this expression can be expanded as

$$F_t = \sum_{a=R}^{T-1} N_{o,o} (1+g)^{t-a} Z \left[\frac{1-(1+r)^{-(T-a)}}{1-(1+r)^{-1}} \right]$$
$$+ \sum_{a=W}^{R-1} N_{o,o} (1+g)^{t-a} \left(\frac{Z}{(1+r)^{R-a}} \left[\frac{1-(1+r)^{-(T-R)}}{1-(1+r)^{-1}} \right] - Y \left[\frac{1-(1+r)^{-(R-a)}}{1-(1+r)^{-1}} \right] \right) \rightarrow$$

$$F_t = \frac{N_{o,o} Z (1+r)(1+g)^t}{r} \left[\sum_{a=R}^{T-1} (1+g)^{-a} - (1+r)^{-T} \sum_{a=R}^{T-1} \left(\frac{1+g}{1+r} \right)^{-a} \right]$$
$$+ \frac{N_{o,o}(1+r)(1+g)^t}{r} \left(\frac{Z}{(1+r)^R} [1-(1+r)^{-(T-R)}] \sum_{a=W}^{R-1} \left(\frac{1+g}{1+r} \right)^{-a} \right.$$
$$\left. - Y \left[\sum_{a=W}^{R-1} (1+g)^{-a} - (1+r)^{-R} \sum_{a=W}^{R-1} \left(\frac{1+g}{1+r} \right)^{-a} \right] \right) \rightarrow$$

$$F_t = \frac{N_{o,o}\,Z\,(1+r)\,(1+g)^t}{r}\left((1+g)^{-R}\left[\frac{1-(1+g)^{-(T-R)}}{1-(1+g)^{-1}}\right] - (1+r)^{-T}\left(\frac{1+g}{1+r}\right)^{-R}\left[\frac{1-\left(\frac{1+g}{1+r}\right)^{-(T-R)}}{1-\left(\frac{1+g}{1+r}\right)^{-1}}\right]\right)$$

$$+\ \frac{N_{o,o}\,(1+r)\,(1+g)^t}{r}\left(\frac{Z}{(1+r)^R}\left[1-(1+r)^{-(T-R)}\right]\left(\frac{1+g}{1+r}\right)^{-W}\left[\frac{1-\left(\frac{1+g}{1+r}\right)^{-(R-W)}}{1-\left(\frac{1+g}{1+r}\right)^{-1}}\right]\right.$$

$$\left.-\ \frac{Y}{(1+g)^W}\left[\frac{1-(1+g)^{-(R-W)}}{1-(1+g)^{-1}}\right] + \frac{Y}{(1+r)^R}\left(\frac{1+g}{1+r}\right)^{-W}\left[\frac{1-\left(\frac{1+g}{1+r}\right)^{-(R-W)}}{1-\left(\frac{1+g}{1+r}\right)^{-1}}\right]\right) \Rightarrow$$

$$F_t = \frac{N_{o,o}\,Z\,(1+r)\,(1+g)^t}{r}\left(\frac{(1+g)^{1-R}}{g}\left[1-(1+g)^{-(T-R)}\right] - (1+r)^{-T}\left(\frac{1+r}{1+g}\right)^{R}\left(\frac{1+g}{g-r}\right)\left[1-\left(\frac{1+g}{1+r}\right)^{-(T-R)}\right]\right)$$

$$+\ \frac{N_{o,o}\,(1+r)\,(1+g)^t}{r}\left(\frac{Z}{(1+r)^R}\left[1-(1+r)^{-(T-R)}\right]\left(\frac{1+r}{1+g}\right)^{W}\left(\frac{1+g}{g-r}\right)\left[1-\left(\frac{1+g}{1+r}\right)^{-(R-W)}\right]\right.$$

$$\left.-\ \frac{Y(1+g)^{1-W}}{g}\left[1-(1+g)^{-(R-W)}\right] + \frac{Y}{(1+r)^R}\left(\frac{1+r}{1+g}\right)^{W}\left(\frac{1+g}{g-r}\right)\left[1-\left(\frac{1+g}{1+r}\right)^{-(R-W)}\right]\right) \Rightarrow$$

$$F_t = \frac{N_{o,o}\,(1+r)\,(1+g)^t}{r}\left(\frac{Z(1+g)^{1-R}}{g} - \frac{Z(1+g)^{1-T}}{g} - \frac{Z(1+r)^{R-T}(1+g)^{1-R}}{r-g}\left[\left(\frac{1+r}{1+g}\right)^{T-R} - 1\right]\right.$$

$$+\ \frac{Z(1+r)^{W-R}(1+g)^{1-W}}{r-g}\left[1-(1+r)^{R-T}\right]\left[\left(\frac{1+r}{1+g}\right)^{R-W} - 1\right]$$

$$\left.-\ \frac{Y(1+g)^{1-W}}{g} + \frac{Y(1+g)^{1-R}}{g} + \frac{Y(1+r)^{W-R}(1+g)^{1-W}}{r-g}\left[\left(\frac{1+r}{1+g}\right)^{R-W} - 1\right]\right) \Rightarrow$$

$$F_t = \frac{N_{o,o}\,(1+r)\,(1+g)^t}{r}\left(\frac{Z(1+g)^{1-R}}{g} - \frac{Z(1+g)^{1-T}}{g} - \frac{Z(1+g)^{1-T}}{r-g} + \frac{Z(1+r)^{R-T}(1+g)^{1-R}}{r-g}\right.$$

$$+\ \left[\frac{Z(1+g)^{1-R}}{r-g} - \frac{Z(1+r)^{W-R}(1+g)^{1-W}}{r-g}\right]\left[1-(1+r)^{R-T}\right]$$

$$\left.-\ \frac{Y(1+g)^{1-W}}{g} + \frac{Y(1+g)^{1-R}}{g} + \frac{Y(1+g)^{1-R}}{r-g} - \frac{Y(1+r)^{W-R}(1+g)^{1-W}}{r-g}\right) \Rightarrow$$

$$F_t = \frac{N_{o,o}(1+r)(1+g)^{t+1}}{r(r-g)}\left(\frac{Z(r-g)}{g(1+g)^R} - \frac{Z(r-g)}{g(1+g)^T} - \frac{Z}{(1+g)^T} + \frac{Z}{(1+r)^{T-R}(1+g)^R}\right.$$

$$+ \frac{Z}{(1+g)^R} - \frac{Z}{(1+r)^{R-W}(1+g)^W} - \frac{Z}{(1+r)^{T-R}(1+g)^R} + \frac{Z}{(1+r)^{T-W}(1+g)^W}$$

$$\left. - \frac{Y(r-g)}{g(1+g)^W} + \frac{Y(r-g)}{g(1+g)^R} + \frac{Y}{(1+g)^R} - \frac{Y}{(1+r)^{R-W}(1+g)^W}\right) \Rightarrow$$

$$F_t = \frac{N_{o,o}(1+r)(1+g)^{t-W+1}}{r(r-g)}\left(\frac{Z(r-g)}{g(1+g)^{R-W}} - \frac{Z(r-g)}{g(1+g)^{T-W}} - \frac{Z}{(1+g)^{T-W}} + \frac{Z}{(1+g)^{R-W}} - \frac{Z}{(1+r)^{R-W}}\right.$$

$$\left. + \frac{Z}{(1+r)^{T-W}} - \frac{Y(r-g)}{g} + \frac{Y(r-g)}{g(1+g)^{R-W}} + \frac{Y}{(1+g)^{R-W}} - \frac{Y}{(1+r)^{R-W}}\right) \Rightarrow$$

$$F_t = \frac{N_{o,o}(1+r)(1+g)^{t-W+1}}{r(r-g)}\left(Y - \frac{Yr}{g} - \frac{Y}{(1+r)^{R-W}} - \frac{Z}{(1+r)^{R-W}} + \frac{Z}{(1+r)^{T-W}} + \frac{Z(r-g)(1+g)^{W-R}}{g}\right.$$

$$\left. - \frac{Z(r-g)(1+g)^{W-T}}{g} - Z(1+g)^{W-T} + Z(1+g)^{W-R} + Y(1+g)^{W-R}\left[\frac{r}{g}\right]\right) \Rightarrow$$

$$F_t = \frac{N_{o,o}(1+r)(1+g)^{t-W+1}}{r(r-g)}\left(Y - \frac{Y}{(1+r)^{R-W}} - \frac{Z}{(1+r)^{R-W}} + \frac{Z}{(1+r)^{T-W}}\right.$$

$$+ \frac{Zr(1+g)^{W-R}}{g} - Z(1+g)^{W-R} - \frac{Zr(1+g)^{W-T}}{g} + Z(1+g)^{W-T}$$

$$\left. - Z(1+g)^{W-T} + Z(1+g)^{W-R} - \frac{Yr}{g} + \frac{Yr(1+g)^{W-R}}{g}\right) \Rightarrow$$

$$F_t = \frac{N_{o,o}L(1+g)^{t-W+1}}{(r-g)} + \frac{N_{o,o}(1+r)(1+g)^{t-W+1}}{r(r-g)}\left(\frac{Zr}{g}\left[(1+g)^{W-R} - (1+g)^{W-T}\right] - \frac{Yr}{g}\left[1 - (1+g)^{W-R}\right]\right),$$

from equation (6) in the text. Substituting from equation (5) in the text and further simplifying,

$$F_t = \frac{N_{o,o}L(1+g)^{t-W+1}}{(r-g)}$$

$$+ \frac{N_{o,o}(1+r)(1+g)^{t-W+1}}{g(r-g)}\left(Z\left[(1+g)^{W-R} - (1+g)^{W-T}\right] - Z(1+g)^{W-R}\left[\frac{1-(1+g)^{-(T-R)}}{1-(1+g)^{-(R-W)}}\right]\left[1-(1+g)^{-(R-W)}\right]\right) \Rightarrow$$

$$F_t = \frac{N_{o,o}L(1+g)^{t-W+1}}{(r-g)} + \frac{N_{o,o}(1+r)(1+g)^{t-W+1}Z}{g(r-g)}\left[(1+g)^{W-R} - (1+g)^{W-T} - (1+g)^{W-R} + (1+g)^{W-T}\right] \Rightarrow$$

$$F_t = \frac{N_{o,o} L (1+g)^{1-W}}{(r-g)} (1+g)^t .$$

Note that the unfunded liability is always positive and grows over time at the rate of growth in aggregate output. The initial unfunded liability at time zero ($t=0$), when the pay-as-you-go program is abolished and the funded program is initiated under the Townley proposal, is given by

$$D_o = F_o = \frac{N_{o,o} L (1+g)^{1-W}}{(r-g)} ,$$

as expressed in equation (8) of the text.