Chapter 4: Report of the Task Group on Learning Processes

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CONTENTS

List of Figures 4-v
List of Tables 4-vii
List of Abbreviations 4-ix

Executive Summary 4-xi
I. Introduction 4-1
II. Methodology 4-1
A. Procedures 4-2
1. Literature Search and Study Inclusion 4-2
2. Criteria for Inclusion 4-2
III. Reviews and Findings 4-2
A. General Principles: From Cognitive Processes to Learning Outcomes 4-2
1. Information Processing 4-4
2. Mental Representations 4-7
B. Social, Motivational, and Affective Influences on Learning 4-10
1. Goals and Beliefs About Learning 4-10
2. Intrinsic and Extrinsic Motivation 4-12
3. Attributions 4-13
4. Self-Regulation 4-13
5. Mathematics Anxiety 4-14
6. Vygotsky's Sociocultural Perspective 4-16
7. Conclusions and Recommendations 4-17
C. What Children Bring to School 4-19
1. Roots of Numerical Understanding 4-19
2. Mathematical Understanding in the Preschool Period 4-20
a. Acquisition of Number Words and Counting 4-20
b. Ordering Numbers 4-20
c. Arithmetic 4-20
d. Measurement 4-21
e. Geometric Knowledge 4-21
f. Number Sense 4-22
3. Differences Among Individuals and Groups 4-22
4. Improving Early Mathematical Knowledge 4-23
5. Conclusions and Recommendations 4-24
D. Mathematical Development in Content Areas 4-24
1. Whole Number Arithmetic 4-25
a. Acquisition of Arithmetic Facts 4-26
b. Learning Arithmetical Algorithms 4-30
c. Core Arithmetical Concepts 4-35
d. Conclusions and Recommendations 4-38

2. Fractions, Decimals, and Proportions 4-40
a. Definitions and Interpretations 4-40
b. Extent of the Problem 4-42
c. Paths of Acquisition 4-43
d. Obstacles to Mastery 4-50
e. Conclusions and Recommendations 4-54
3. Estimation 4-56
a. Understanding the Goals of Estimation 4-57
b. Development of Estimation Skills 4-58
c. Individual Differences in Estimation 4-62
d. Improving Children's Estimation 4-62
e. Conclusions and Recommendations 4-64
4. Geometry 4-66
a. Geometry Performance of U.S. Students on International Mathematics Assessments 4-66
b. Importance of Geometry for Learning Algebra 4-67
c. Limitations of the Relevant Research-Based Literature 4-68
d. Paths of Acquisition 4-69
e. Obstacles to Mastery 4-72
f. Conclusions and Recommendations 4-73
5. Algebra 4-74
a. Algebraic Equations 4-75
b. Word Problems 4-79
c. Conclusions and Recommendations 4-87
E. Differences Among Individuals and Groups 4-90
1. Sex Differences 4-90
2. Race and Ethnicity 4-91
a. The Achievement Gap 4-91
b. Potential Sources of the Achievement Gap 4-92
c. Potential Cognitive and Social Influences 4-94
3. Learning Disabilities 4-107
4. Gifted Students 4-109
5. Conclusions and Recommendations 4-110
F. Brain Sciences and Mathematics Learning 4-111
1. Conclusions and Recommendations 4-112

BIBLIOGRAPHY 4-113

APPENDIX A: Literature Search Guidelines 4-153
APPENDIX B: Search Terms 4-155
APPENDIX C: Sex Differences 4-157
APPENDIX D: Racial/Ethnic Differences 4-181
APPENDIX E: Socioeconomic Differences (SES) 4-203

Figures

Figure 1: Multiplication Algorithms 4-32
Figure 2: Processing Linear Equations 4-76
Figure 3: Algebraic Bugs 4-79
Figure C-1: Average NAEP Scale Scores by Gender, Age 9: Intermittent Years From 1978–2004 4-161
Figure C-2: Percent at NAEP Performance Levels by Gender, 9-Year-Olds: 1978, 1999, and 2004 4-162
Figure C-3: Average NAEP Scale Scores by Gender, Age 13: Intermittent Years From 1978–2004 4-163
Figure C-4: Percent at NAEP Performance Levels by Gender, 13-Year-Olds: 1978, 1999, and 2004 4-164
Figure C-5: Average NAEP Scale Scores by Gender, Age 17: Intermittent Years From 1978–2004 4-166
Figure C-6: Percent at NAEP Performance Levels by Gender, 17-Year-Olds: 1978, 1999, and 2004 4-167
Figure C-7: Average TIMSS Mathematical Scale Scores of U.S. 4th- and 8th-Graders, by Sex: Various Years From 1995–2003 4-171
Figure C-8: Average TIMSS Mathematical Scale Scores of U.S. 4th-Graders, by Sex, by Content Area: 2003 4-172
Figure C-9: Average TIMSS Mathematical Scale Scores of U.S. 8th-Graders, by Sex, by Content Area: 2003 4-172
Figure C-10: IRT—Estimated Average Math Score (10th-Grade), by Sex (HS&B:80, NELS:88, ELS:2002) 4-174
Figure C-11: Average Mathematics Literacy Scores of U.S. 15-Year-Olds, by Gender: 2003 PISA 4-176
Figure C-12: Average Quantitative Literacy Scores of Adults, by Sex: NALS 1992 and NAAL 2003 4-179
Figure D-1: Average NAEP Scale Scores by Race/Ethnicity, Age 9: Intermittent Years From 1978–2004 4-184
Figure D-2: Average NAEP Scale Scores by Race/Ethnicity, Age 13: Intermittent Years From 1978–2004 4-185
Figure D-3: Average NAEP Scale Scores by Race/Ethnicity, Age 17: Intermittent Years From 1978–2004 4-187
Figure D-4: Percent at NAEP Performance Levels by Race/Ethnicity, Age 9: 1978, 1999, and 2004 4-188
Figure D-5: Percent at NAEP Performance Levels by Race/Ethnicity, Age 13: 1978, 1999, and 2004 4-189
Figure D-6: Percent at NAEP Performance Levels by Race/Ethnicity, Age 17: 1978, 1999, and 2004 4-191
Figure D-7: Average TIMSS Mathematical Scale Scores of U.S. 4th- and 8th-Graders, by Race/Ethnicity: Various Years From 1995–2003 4-194
Figure D-8: IRT—Estimated Average Math Score (10th-Grade), by Race/Ethnicity (HS&B:80, NELS:88, ELS:2002) 4-196

Figure D-9: Average Quantitative Literacy Scores of Adults, by Race/Ethnicity: NALS 1992 and NAAL 2003 4-199
Figure D-10: Average Mathematic Literacy Scores of U.S. 15-Year-Olds, by Race/Ethnicity: 2003 PISA 4-202
Figure E-1: Average NAEP Scale Scores, by Parents’ Highest Level of Education, Age 13: Intermittent Years From 1978–2004 4-206
Figure E-2: Average NAEP Scale Scores, by Parents’ Highest Level of Education, Age 17: Intermittent Years From 1978–2004 4-207
Figure E-3: Percent at NAEP Performance Levels, by Parents’ Highest Level of Education, Age 13: 1978, 1999, and 2004 4-209
Figure E-4: Percent at NAEP Performance Levels, by Parents’ Highest Level of Education, Age 17: 1978, 1999, and 2004 4-210
Figure E-5: Average TIMSS Mathematical Scale Scores of U.S. 4th- and 8th-Graders, by School Poverty Level: 1999 and 2003 4-214
Figure E-6: IRT—Estimated Average Math Score (10th Grade), by SES (HS&B:80, NELS:88, ELS:2002) 4-216
Figure E-7: Average Quantitative Literacy Scores of Adults, by Household Income: NAAL 2003 4-218
Figure E-8: Average Quantitative Literacy Scores of Adults, by Highest Educational Attainment: NALS 1992 and NAAL 2003 4-219
Figure E-9: Average Mathematics Literacy Scores of U.S. 15-Year-Olds, by Quarters on the International Socioeconomic Index: 2003 PISA 4-221

Tables

Table C-1: Number of Students in Each NAEP Reporting Group, by Age, Gender, Race/Ethnicity, and Parents' Level of Education: 1978, 1999, and 2004 4-160
Table C-2: Probability of 10th-Grade Proficiency in Mathematics, by Gender 4-174
Table C-3: Percentage of U.S. 15-Year-Old Students Scoring at Each Proficiency Level, by Gender: 2003 PISA 4-177
Table C-4: Comparison of U.S. and Organisation for Economic Co-operation and Development (OECD) Countries' Average Scores on 2003 PISA Math Literacy 4-177
Table C-5: Percentage of Adults in Each Quantitative Literacy Level, by Gender: NALS 1992 and NAAL 2003 4-179
Table D-1: Probability of 10th-Grade Proficiency in Mathematics by Race/Ethnicity 4-197
Table D-2: Percentage of Adults in Each Quantitative Literacy Level, by Race/Ethnicity: NALS 1992 and NAAL 2003 4-200
Table E-1: Probability of 10th-Grade Proficiency in Mathematics, by SES 4-216
Table E-2: Percentage of Adults in Each Quantitative Literacy Level, by Household Income: NAAL 2003 4-218
Table E-3: Percentage of Adults in Each Quantitative Literacy Level, by Highest Education Attainment: NALS 1992 and NAAL 2003 4-219

Abbreviations

CTBS	Comprehensive Test of Basic Skills
ECLS	Early Childhood Longitudinal Study
ELS	Education Longitudinal Study
GPA	Grade Point Average
GRE	Graduate Record Exam
HS&B	High School and Beyond
IRT	Item Response Theory
LD	Learning Disability
MLD	Mathematics Learning Disability
NALS	National Adult Literacy Survey
NAAL	National Assessment of Adult Literacy
NAEP	National Assessment of Education Progress
NCES	National Center for Education Statistics
NCTM	National Council of Teachers of Mathematics
NELS	National Education Longitudinal Study
OECD	Organisation for Economic Co-operation and Development
PISA	Programme for International Student Assessment
SAT	Scholastic Aptitude Test
SES	Socioeconomic Status
TEMA	Test of Early Mathematics Ability
TIMSS	Trends in International Mathematics and Science Study

Executive Summary

The charge of the Learning Processes Task Group is to address what is known about how children learn and understand areas of mathematics related to algebra and preparation for algebra. This summary provides brief overviews of and recommendations from the corresponding in-depth reviews provided later in the report. The reviews cover the areas of 1) General Principles of Cognition and Learning; 2) Social, Motivational, and Affective Influences on Learning; 3) What Children Bring to School; 4) Mathematical Development in Content Areas of a) Whole Number Arithmetic; b) Fractions; c) Estimation; d) Geometry; and e) Algebra; 5) Differences Among Individuals and Groups; and 6) The Brain Sciences and Mathematics Learning. For the mathematical content areas included in the reviews, the recommendations are organized around classroom practices, training of teachers and researchers, curriculum, and future research efforts.

General Principles: From Cognitive Processes to Learning Outcomes

Cognitive science is the study of the processes that underlie learning and cognition and is a foundational component of scientifically informed educational practice. There is a large body of high-quality research on learning mechanisms that can be directly applied to the classroom to improve student learning and achievement; however, this research at present is not being optimally used.

The two main classes of cognitive mechanism that control learning are information processing operations and mental representations. Students also engage in metacognitive processing, which controls information-processing operations such as selecting strategies for effective problem solving.

Information processing begins when a student encounters information and lasts until that information is acted upon and a response is made. The process starts with *attention*, without which information is lost. Information that is the focus of attention becomes available to learners' *working memory*, and with practice the information can be transferred to long-term memory. Deficiencies or superiorities in working memory capacities are major contributors to learning disabilities or accelerated learning, respectively. Improving the effectiveness of working memory can be assisted by achieving automaticity.

Mental representations are represented in different ways in the brain, including *declarative* knowledge, *procedural* knowledge, and *conceptual* knowledge.

The number line is a core tool in modern mathematics and is used in many contexts. One important cognitive mechanism in mathematics learning is the so-called *mental* number line.

Memories occur in either *verbatim* or *gist* form. Verbatim recall of math knowledge is an essential feature of math education, and it requires a great deal of time, effort, and practice. Gist memory is the form of memory that is typically relied on in reasoning. A combination of gist knowledge and verbatim knowledge is critical for success in math.

Social, Motivational, and Affective Influences on Learning

Children's goals and beliefs about learning are related to their mathematics performance. Mastery-oriented students are focused on learning the material and show better long-term academic development in mathematics and the pursuit of difficult academic tasks. Performance-oriented students are focused on grades and show less persistence on complex tasks. When students are told that beliefs about effort and ability can be changed, they are shown to undergo a significant rebound in their mathematics grades.

Young children's intrinsic motivation to learn is positively correlated with academic outcomes in mathematics and other domains and is related to mastery goals. Extrinsic motivation is related to performance goals.

Students' attributions or beliefs about the causes of their success and failure have been repeatedly linked to their engaging and persisting in learning activities. Students' self-regulation improves math learning.

Anxiety is an emotional reaction that is related to low math achievement, failure to enroll in advanced mathematics courses, and poor scores on standardized tests of math achievement. Math anxiety creates a focus of limited working memory on managing anxiety reaction rather than on solving the math problem, but it can be reduced by therapeutic interventions.

Vygotsky's characterization of the learning process as one of social induction may be applicable to the sharing of informal mathematics knowledge when it is embedded in everyday practices.

Recommendations

The Task Group recommends extension of experimental studies that have demonstrated that: 1) children's beliefs about the relative importance of effort and ability can be changed; 2) increased emphasis on the importance of effort is related to greater engagement and persistence on mathematics tasks; and 3) improved mathematics grades result from these changed beliefs.

The Task Group recommends studies that experimentally assess the implications of the relation between intrinsic motivation and mathematics learning.

The Task Group recommends experimental and longitudinal studies that assess the relative contributions of self-efficacy (i.e., the belief that one has the specific skills needed to be successful, which differs from self-esteem) factors to mathematics learning.

Although self-regulation (i.e., making goals, planning, monitoring, and self evaluating progress) appears promising, research is needed to establish the causal relation between these processes and the ability to learn a wider range of mathematics knowledge and skills.

The Task Group recommends research that assesses the potential risk factors of anxiety; it also recommends development of promising interventions for reducing debilitating mathematics anxiety.

A shortage of controlled experiments makes the usefulness of Vygotsky's approach for improving mathematics learning difficult to evaluate, and thus its utility in mathematics classrooms and mathematics curricula needs to be scientifically tested.

What Children Bring to School

Mathematical learning begins at birth and continues through the time children first arrive at school. The amount of mathematical knowledge students bring to school has important consequences for their long-term learning, as children who start kindergarten behind their peers tend to stay behind throughout their schooling.

Mathematical development begins in the first months of infancy, as people posses an innate nonverbal sense of number that provides a foundation for learning the verbal number system.

While 3- and 4-year old children may be able to count from 1 to 10, many have only mastered the superficial form of counting without understanding counting's purpose. By kindergarten, most children begin to understand the magnitudes of the numbers from 1 to 10.

By the start of kindergarten, most children also can retrieve from memory answers to a few basic addition and subtraction facts, know a variety of other procedures for solving simple addition and subtraction problems, and show some understanding of basic arithmetic concepts. Children of this age also choose effectively among strategies; use measurement strategies that reflect basic understanding of *more than*, *less than*, and *equal to*; and show basic geometrical knowledge of simple shapes.

Mathematical knowledge during preschool and kindergarten is predictive of mathematical knowledge in third, fifth, and eighth grade. Students who are at risk for low mathematics achievement tend to come from single-parent families with low-parental education levels, families where English is not the primary language, and families living in poverty. African-American and Hispanic children are more likely than other children to enter kindergarten with poor mathematical knowledge.

Effective instructional programs designed to improve mathematical knowledge of preschool children focus on forming mental representations of numbers, such as the mental number line, the language of numbers, and tools found through computer software programs.

Recommendations

Research that scales up interventions to improve the mathematical knowledge of preschoolers and kindergartners, especially those from at-risk backgrounds, and research that evaluates the utility of these interventions in classroom settings are urgently needed.

Mathematical Development in Content Areas

Whole Number Arithmetic

The mastery of whole number arithmetic is a critical step in children's mathematical education. The road to mastery involves learning arithmetic facts, algorithms, and concepts.

The quick and efficient solving of simple arithmetic problems is achieved when children retrieve answers from long-term memory or retrieve related information that allows them to quickly reconstruct the answer. Retention of these facts requires repeated practice.

Research indicates that learning of addition and multiplication facts is easier to achieve than learning of subtraction and division facts, due to the commuted relation within addition and multiplication pairs. Children and many adults in the United States have not reached the point of fast and efficient recall of simple arithmetic problems.

Algorithms range in complexity from counting as a way to solve simple addition problems to the lengthy sequence of steps involved in solving division problems. Learning of complex algorithms is highly dependent on working memory resources and requires repeated use of the algorithm extended over time. Mastery of standard algorithms is dependent on committing these problem-solving steps to long-term procedural memory, at which point the algorithm can be executed automatically with little demand on working memory resources. Algorithms that are mastered are less prone to disruption due to anxiety or in contexts such as high-stakes testing.

The core concepts that children should understand and use when solving arithmetic problems include mathematical equality, the commutative and associative properties of addition and multiplication, the distributive property of multiplication, identity elements for addition and multiplication, the composition of numbers, connections between arithmetic and counting, and the inverse relation between addition and subtraction and between multiplication and division.

Conceptual understanding is critical for children's ability to identify and correct errors, for appropriately transferring algorithms to solve novel problems, and for understanding novel problems in general.

Recommendations

Training

For teachers to take full advantage of formative assessments, they must have a better understanding of children's learning and the sources of children's conceptual and procedural errors in the content areas they are teaching. The development of courses in mathematical cognition for inclusion in teacher training programs will be necessary to address this goal.

Programs that support cross-disciplinary pre-doctoral and post-doctoral training in cognition, education, and mathematics are needed to ensure that a sufficient number of researchers study children's mathematical learning, and have the background needed to bridge the gap between laboratory studies and classroom practice.

Curricula

Although definitive conclusions cannot be drawn at this time due to lack of relevant, large-scale experimental studies, the research that has been conducted suggest that effective practice should: 1) present more difficult problems more frequently than less difficult problems, 2) highlight the relations among problems, 3) order practice problems in ways that reinforce core concepts, and 4) include key problems that support formative assessments.

Research

Although much is known about some areas of children's arithmetical cognition and learning, further research is needed in the areas of children's learning of complex algorithms; the relation between conceptual knowledge and procedural learning; and on the learning of core concepts, including the base-10 number system, the distributive property of multiplication, and identity elements, among others.

Studies that focus on the translation of cognitive measures of children's learning into formative assessments that are easily understood by teachers and easily used in the classroom are needed.

Funding priorities that target areas of deficit in children's arithmetical cognition and learning are recommended, along with priorities that encourage projects that bridge the gap between basic research and classroom practice.

Fractions

Fractions, decimals, and proportions are introduced into the mathematics curriculum as early as elementary school, and yet solving problems with these quantities remains difficult for many adults. Understanding and manipulating fractions is crucial for further progress in mathematics and for tasks of everyday life.

A fraction is defined as a point on the number line, based on the concept of a part-whole relation, with the unit segment [0,1] (the segment from 0 to 1) serving as a whole. From this mathematical definition of a fraction, other definitions can be derived, such as the division interpretation.

Difficulties with fractions extend beyond those with learning disabilities in mathematics. The failure to attain basic facility with fractions constitutes an obstacle to progress to more advanced topics in mathematics, including algebra and, presumably, to career paths that require mathematical proficiency, as well as interfering with potential life-and-death aspects of daily functioning (e.g., understanding and adhering to medical regimen).

To accurately assess competence, it is important to separate children's understanding of formal fractional notation from their intuitive ability to understand fractional relations and perform calculations using fractional quantities. Young children reveal a nascent ability to understand ratios, and preschool children's experiences with and understanding of part-whole relations among sets of physical objects may contribute to an early understanding of simple ratios.

Similarly, the ability to manipulate fractions is also present early. Research shows that sharing forms the basis for preschool age children's ability to partition a quantity into roughly equal parts through a process of distributive counting. This does not mean that they understand the inverse relation among quantities, but with a few lessons they are able to appreciate and generalize the inverse relation.

Studies of elementary and middle school-aged children have focused on the acquisition of conceptual knowledge, computational skills, and the ability to use both of these abilities in conjunction with reading comprehension to solve word problems involving fractional quantities. Scores on items assessing conceptual knowledge have consistently been shown to explain unique variance (beyond general intellectual and reading abilities) in performance on computational fraction problems, word problems that include fractions, and estimation tasks with fractional quantities.

Many errors on fraction computation problems could be classified as involving a faulty procedure. Children's accuracy at recognizing formal procedural rules for fractions and automatic retrieval of basic arithmetic facts predicts computational skills, above and beyond the influence of intelligence, reading skills, and conceptual knowledge. Research also shows that on-task time influences performance through its effect on conceptual knowledge.

Motivation also has positive effects on fraction learning. Learning goals rather than performance goals may produce higher self-efficacy, skill, and other achievement outcomes in students. Performance goals with self-evaluation components may be more effective than without. Early levels of basic arithmetic skills may predict those children who will later have difficulty with fractions, and building such skills may enhance performance on fraction computation problems.

Proportional reasoning involves the coordination of two ratio quantities, and early, informal competence can be detected if children are able to use perceptual cues to judge relative numerosity.

A fraction's lack of fit with properties of counting adds to the relative difficulty of learning the concept. Because of the property of infinite divisibility, fractions, unlike counting numbers, do not form a sequence in which each number has a fixed successor. Therefore, it has been argued that the one-to-one and stable-order principles that are important to counting are misleading when children attempt to generalize from whole numbers to fractions.

Pictorial representations, without sufficient emphasis on the nature of wholes in part-whole relations and the importance of equal-sized parts, also may be an obstacle to learning fractions. Number line representation may be more effective. Words also seem to influence the mental representations that children form concerning fractions, particularly when language demarcates parts and wholes in fraction names.

Research on working-memory demands of different tasks shows that different fraction interpretations entail different information-processing demands. Quotient interpretations of fractions are more demanding of memory resources than part-whole interpretations because they involve a more complex series of mappings.

Individual differences in working memory have been associated with performance on fraction tasks; and effects of working memory were independent of effects of conceptual knowledge. While conceptual knowledge carries the greatest weight in predicting performance on all three outcome measures (computation, estimation, and word problems), working memory affected only word problems and only indirectly affected computation through knowledge of basic arithmetic facts.

Recommendations

Classroom

Children should begin fraction instruction with the ability to quickly and easily retrieve basic arithmetic facts. Instruction focusing on conceptual knowledge of fractions is likely to have the broadest and largest impact on problem-solving performance. Procedural knowledge is also essential, however, and although it must be learned separately, it is likely to enhance conceptual knowledge and vice versa.

Successful interventions should include the use of fraction names that demarcate parts and wholes, the use of pictorial representations that are mapped onto the number line, and composite representations of fractions that are linked to representations of the number line. Conceptual and procedural knowledge about fractions less than one do not necessarily transfer to fractions greater than one, and must be taught separately. Appropriate intuitions about sharing, part-whole relations, and proportional relations can be built on in classrooms to support acquisition of conceptual and procedural knowledge of fractions.

Training

Training of teachers should include sufficient coverage of the scientific method so that teachers are able to critically evaluate the evidence for proposed pedagogical approaches and to be informed consumers of the scientific literature. Teachers should be aware of common conceptions and misconceptions involving fractions and of effective interventions involving fractions.

New funding should be provided to train future researchers, to begin new interdisciplinary degree programs with rigorous quantitative training, and to establish support mechanisms for career shifts that encourage rigorous researchers in related fields to focus on education.

Curriculum

The curriculum should allow for sufficient time on task to ensure acquisition of conceptual and procedural knowledge of fractions and of proportional reasoning, with the goal for students being one of learning rather than performance. However, there should be ample opportunity in the curriculum for accurate self-evaluation. The curriculum should include representational supports that have been shown to be effective and tap the full range of conceptual and procedural knowledge.

Research

An area for future study is the relation between the rudimentary understanding of very simple fractional relations and the learning of formal mathematical fractional concepts and procedures. In addition, research is needed to uncover the mechanisms that contribute to the emergence of formal competencies. Research on understanding and learning of fractions should be integrated with what is known and with emerging knowledge in other areas of basic research, such as neuroscience, cognition, motivation, and social psychology. The absence of a coherent and empirically supported theory of fraction tasks is a major stumbling block to developing practical interventions to improve performance in this crucial domain of mathematics.

Classroom-relevant research need not be conducted physically in classrooms, and constraints on funding that require that relevant research be performed in classrooms should be removed. Conversely, many interventions demonstrated to be effective in experiments should be scaled up and evaluated in classrooms.

Estimation

Estimation may be used more often in everyday life than any other quantification process. It is also quite strongly related to other aspects of mathematical ability, such as arithmetic skill and conceptual understanding of computational procedures, and to overall math achievement test scores. It usually requires going beyond rote application of procedures and applying mathematical knowledge in flexible ways.

The Task Group focuses on *numerical estimation*, the process of translating between alternative quantitative representations, at least one of which is inexact and at least one of which is numerical. This category includes many prototypic forms of estimation, including computational, number line, and numerosity.

Many children have highly distorted impressions of the goals of estimation, especially the goals of computational estimation. Accurate computational estimation requires understanding of the simplification principle and the proximity principle. Research shows that students understand the principle of simplification, but they show little if any understanding of the importance of generating an estimate close in magnitude to the correct answer.

Development of computational estimation skills begins surprisingly late and proceeds slowly but does improve considerably with age and experience. From early in the development of computational estimation, individual children use a variety of strategies including rounding, truncating, prior compensation, post-compensation, decomposition, translation, and guessing. Rounding is the most common approach and compensation tends to be among the least common, although it is among the most useful.

Both children and adults adapt their strategy choices to problem characteristics. The range and appropriateness of computational estimation strategies increase with age and mathematical experience. The sophistication of strategies used also changes, and in particular, compensation shows especially substantial growth with age and experience.

The number line task has proved highly informative, not only for improving understanding of estimation but also for providing useful information about children's understanding of the decimal number system more generally.

Children use two primary mental representations of numerical magnitude on number line estimation tasks, including linear representation and logarithmic representation. With age and experience, children progress from using the less accurate logarithmic representation to the more accurate linear one on the number line task.

Both children and adults show substantial individual differences in skill at computational estimation that are associated with broader individual differences in mathematical understanding and general mathematical ability.

Playing board games with linearly arranged, consecutively numbered, equal-size spaces leads children to shift from logarithmic to linear representations of numerical magnitude. These games are particularly effective in improving low-income preschoolers' numerical knowledge and reducing disparities in the numerical knowledge brought to school by children from low-income homes and those from middle-income homes.

Another procedure that is effective for improving elementary school children's number line estimation is to provide students with feedback on their estimates.

Recommendations

Classroom

Teachers should broaden instruction in computational estimation beyond rounding. They should insure that students understand that the purpose of estimation is to approximate the correct value and that rounding is only one of several means for accomplishing this goal.

Teachers should provide examples of alternative procedures for compensating for the distortions introduced by rounding, emphasize that there are many reasonable procedures for estimating rather than just a single correct one, and discuss reasons why some procedures are reasonable and others are not.

Teachers in facilities serving preschoolers from low-income backgrounds should be made aware of the usefulness of numerical board games for improving the children's knowledge of numbers and of the importance of such early knowledge for long-term educational success.

Teachers should not assume that children understand the magnitudes represented by fractions even if the children can perform arithmetic operations with them. Examining children's ability to perform novel estimation tasks, such as estimating the positions of fractions on number lines, can provide a useful tool for assessing children's knowledge of fractions. Providing feedback on such number line estimates can improve children's knowledge of the fractions' magnitudes.

Training

Teachers in preservice and in-service programs should be informed of the tendency of elementary school students to not fully understand the magnitude of large whole numbers, and they should be taught how to assess individual students' understanding and research-based techniques for improving the children's understanding.

Teachers should be made aware of the inadequate understanding by elementary school, middle school, and high school students of the magnitudes of fractions. Teachers also should be familiarized with the usefulness of feedback on number line estimates of the magnitudes of fractions for overcoming these difficulties.

Curriculum

Textbooks need to explicitly explain that the purpose of estimation is to produce accurate approximations. Illustrating multiple useful estimation procedures for a single problem and explaining how each procedure achieves the goal of accurate estimation are useful means for achieving this goal. Contrasting these procedures with others that produce less accurate estimates and explaining why the one set of procedures produces more accurate estimates than the other are also likely to be helpful.

Research

Research is needed regarding simple instruments that teachers can use in the classroom for assessing children's estimation skills, and regarding instruction that can efficiently improve children's estimation.

Research is needed on how the inadequate representations of whole number numerical magnitudes that have been identified by studies of estimation influence learning of other mathematical skills, such as arithmetic.

Research is needed on how children can be taught to accurately estimate the magnitudes of fractions and on how learning to estimate those magnitudes influences acquisition of other numerical skills involving fractions, such as arithmetic and algebra.

Research is needed on how estimation is used by students (e.g., to solve complex problems) and by adults in everyday life and in professional tasks (e.g., to rule out implausible answers).

Geometry

Geometry is the branch of mathematics concerned with properties of space, and of figures and shapes in space. Euclidean geometry is the domain typically covered in mathematics curricula in the United States, although a separate year-long course is not usually taught until high school. Units on geometry as well as measurement are frequently included in middle school mathematics classes, whereas only the latter tends to be emphasized in the elementary grades.

The Conceptual Knowledge and Skills Task Group found that the single aspect of geometry that is most directly relevant for early learning of algebra is that of similar triangles. NCTM's *Focal Points* and some state frameworks also underscore the importance of this aspect of geometry.

To understand the mathematics underlying the proof that the slope of a straight line is independent of the choice of the points selected, students must successfully develop a conceptual understanding of the following: points, lines, length, angle, right triangle, correspondence, ratio, proportion, translation, reflection, rotation, dilation, congruence, and similarity.

One of the earliest and most influential theories of the development of spatial and geometric concepts was put forth by Piaget and Inhelder, who proposed that young children initially conceptualize space and spatial relations topologically as characterized by the following properties: proximity, order, separation, and enclosure. With development, children subsequently begin to represent space in relation to different points of view, and then sometime between middle and late childhood the Euclidean conceptual system emerges permitting preservation of metric relationships such as proportion and distance. The consensus of research is that evidence supporting this developmental model is comparatively weak.

The van Hiele model (1986) has been the dominant theory of geometric reasoning in mathematics education for the past several decades. According to this model the learner moves sequentially through five levels of understanding: Level 0: Visualization/Recognition, Level 1: Description/Analysis, Level 2: Informal Deduction or Ordering, Level 3: Formal Deduction, and Level 4: Rigor. The majority of high school geometry courses are taught at Level 3.

Research shows that the van Hiele theory provides a generally valid description of the development of students' geometric reasoning, yet this area of research is still in its infancy.

A common misconception that impedes learning includes the belief that shapes with the same perimeter must have the same area. Initial formal instruction may inadvertently promote this misconception as a consequence of students being presented with the concepts of perimeter and area pertaining to the same shapes and kinds of problems.

In addition, there is a common misconception that the linear (or proportional) model can pertain to situations where it is, in fact, not applicable. Research has found that only a long-term classroom intervention can produce a positive effect in overcoming the illusion of linearity.

Recommendations

Classroom

Teachers should recognize that from early childhood through the elementary school years, the spatial visualization skills needed for learning geometry have already begun to develop. Proper instruction is needed to ensure that children adequately build upon and make explicit this core knowledge for subsequent learning of formal geometry.

Training

Teachers need to learn more about the latest research concerning the development of children's spatial abilities, in general, and their geometric conceptions and misconceptions, in particular, to capitalize on their strengths and aid them in overcoming their weaknesses.

Researchers investigating geometry learning need to have a firm grounding in cognitive development and spatial information processing, in addition to having a background in mathematics education.

Curriculum

Early exposure to common shapes, their names, and so forth appears to be beneficial for developing young children's basic geometric knowledge and skills. While reliance on manipulatives may enhance the initial acquisition of some concepts under specified conditions, students must eventually transition from concrete or visual representations to internalized abstract representations. The crucial steps in making such transitions are not clearly understood at present.

Research

Longitudinal studies are needed to assess more directly how developmental changes in spatial cognition can inform the design of instructional units in geometry. Studies are needed to demonstrate whether and to what extent knowledge about similar triangles enhances the understanding that the slope of a straight line is the same regardless of the two points chosen, thus leading to a more thorough understanding of linearity.

More research is needed that specifically links cognitive, theory-driven research to classroom contexts. At the same time, cognitive theorizing pertaining to geometry learning needs to take into account more facets of classroom settings if it is to eventually have a large impact on the design of instructional approaches.

Algebra

Because it is not known if the early algebra achievement of elementary school children reflects an actual implicit understanding of aspects of algebra, the Task Group focuses on explicit algebra content typically encountered in middle school to high school algebra courses.

Studies of skilled adults and high school students who have taken several mathematics courses reveal that the processing of algebraic expressions is guided by an underlying syntax or system of implicit rules that guides the parsing and processing of the expressions.

Research shows that skilled problem solvers scan and process basic subexpressions in these equations in a fraction of a second, or have automaticity. There are substantial benefits to cumulative practice, which results in better short-term and long-term retention of individual rules and a better ability to apply rules to solve problems that involve the integration of multiple rules and to discriminate between rules that might otherwise be used inappropriately.

Students who are first learning algebra and adults who are not skilled in mathematics do not have long-term memory representations of basic forms of linear equations, but this does not prevent the solving of linear equations as long as they understand the general arithmetical and algebraic concepts and rules. Research shows that diagnostic tests in which individual problems varied systematically in terms of the knowledge needed for correct solution can identify sources of common errors, such as those that reflect a poor conceptual understanding of the syntax of algebraic expressions.

A poor understanding of the concept of mathematical equality and the meaning of the "=" is common for elementary school children in the United States, and continues for many children into the learning of algebra.

Errors in the solving of algebraic equations are sometimes classified as procedural bugs. These errors can occur due to overgeneralized use of procedures that are correct for some problems or from a misunderstanding of the procedure itself. Preliminary studies suggest that remediation that focuses on these specific bugs can reduce their frequency.

Research shows that the solution of algebraic word problems requires two general sets of processes: problem translation and problem solution. Problem translation requires an understanding of the meaning and implications of the text within which the problem is embedded. The same potential sources of error described for solving of linear equations can occur during the problem solution stage of word problems.

An analysis of word problems presented in algebra textbooks found that most problems included four types of statements: assignment statements, relational statements, questions, and relevant facts. Problem translation involves taking each of these forms of information and using them to develop corresponding algebraic equations. Translation errors most frequently occur during the processing of relational statements, which specify a single relationship between two variables. Errors also occur in problems where the statement could be *directly* translated into an equation, but the direct translation is incorrect for the problem as a whole (e.g., "z is equal to the sum of 3 and y"). In addition, relational information can sometimes aid problem solving if the information is consistent with students' previous out-of-classroom experiences and if these experiences can be used to create non-algebraic solution strategies.

Abstract problems are more difficult to solve than concrete problems, but the largest effect on students' problem-solving skill is their familiarity with solving the class of word-problem (e.g., interest, rate).

Successful translation of algebraic word problems, as well as the solution of algebraic equations and many other problem types, is guided by schemas including the syntax of equations. Research on children's conceptual knowledge, which was inferred based on how they sorted word problems into categories, shows that the ability to categorize word problems based on the underlying concept and the corresponding reduction in problem solving errors is consistent with development of category-specific schemas.

Researchers have demonstrated that one way in which schema development can occur is the use of worked examples. These provide students with a sequence of steps that can be used to solve problems. The students then solve a series of related problems that are in the same category and involve a very similar series of problem-solving steps. Worked examples are more effective than simply providing students with the procedural steps, as they may promote the automatization and transfer of procedures used across classes of problems.

The best predictors of the ability to solve word problems are computational skills and knowledge of mathematical concepts, as well as intelligence, reading ability, and vocabulary. Students who struggle with algebraic equations also make factoring errors and use algebraic procedures incorrectly. At a cognitive level, problem-solving errors and learning the syntax of algebraic expressions and algebraic schemas are influenced by working memory. Accuracy at solving various forms of mathematics word problems is also related to spatial abilities. It is also very likely that other factors, including motivation, self-efficacy, and anxiety, contribute to skill development in algebra.

Research on learning in general indicates a benefit for practice that is distributed across time, as contrasted with the same amount of practice massed in a single session. Algebraic skills decline steadily over time, and the best predictor of long-term retention of competencies in algebra is the number of mathematics courses taken beyond Algebra I.

Recommendations

Classroom

Teachers should not assume that all students understand even basic concepts, such as equality. Many students will not have a sufficient understanding of the commutative and distributive properties, exponents, and so forth to take full advantage of instruction in algebra.

Many students will likely need extensive practice at basic transformations of algebraic equations and explanation as to why the transformations are done the way they are. The combination of explanation of problem-solving steps combined with associated concepts is critically important for students to effectively solve word problems. For both equations and word problems, it is important that students correctly solve problems before given seatwork or homework.

Training

Teachers should understand how students learn to solve equations and word problems, and causes of common errors and conceptual misunderstandings. This training will better prepare them for dealing with the deficiencies students bring to the classroom, and for anticipating and responding to procedural and conceptual errors during instruction.

The next generation of researchers to study algebra learning will need multi-disciplinary training in mathematics, experimental cognitive psychology, and education. This can be achieved through interdisciplinary doctoral programs or, at a federal level, postdoctoral fellowships that involve work across these disciplines.

Curriculum

There are aspects of many current textbook series in the United States that contribute to the poor preparation and background of algebra students. Presenting operations on both sides of the equation; and showing worked-out examples that include conceptual explanation, procedural steps, and multiple examples are ways in which textbooks can be improved.

Distributed practice should naturally occur as students progress to more complex topics. However, if basic skills are not well learned and understood, the natural progression to complex topics is impeded.

Research

The development of assessment measures that teachers can use to identify core deficiencies in arithmetic (whole number, fractions, and decimals) and likely sources of procedural and conceptual errors in algebra are needed.

Research that explicitly explores the relation between conceptual understanding and procedural skills in solving algebraic equations is needed. Research on how students solve linear equations, and where and why they make mistakes needs to be extended to more complex equations and other key topic areas of Algebra identified by the Conceptual Knowledge and Skills Task Group.

The issue of transfer needs considerable attention, particularly determining the parameters that impede or facilitate transfer. Research on instructional methods that will reduce the working memory demands associated with learning algebra is needed. Longitudinal research is needed to identify the early predictors of later success in algebra.

A mechanism is needed for fostering translation of basic research findings into potential classroom practices and for scientifically assessing their effectiveness in the classroom. Equally important, mechanisms need to be developed for reducing the lag time between basic findings and assessment in classroom settings.

Differences Among Individuals and Groups

For large, nationally representative samples, the average mathematics scores of boys and girls are very similar; when differences are found they are small and typically favor boys.

From preschool to college, there is a mathematics performance gap between black and Hispanic students to their white and Asian counterparts. It is often proposed that socioeconomic status differences account for these disparities, but the research indicates that this is not a sufficient explanation. Other factors include attitudes, beliefs, motivation, and school-based factors such as features of teaching and learning contexts.

Stereotype threat, cognitive load, and strategy use are all potential mechanisms contributing to existing differences, and work in those areas holds promise as a means to improve the mathematics performance of black and Hispanic students. There is not, however, sufficient research to fully evaluate this promise.

There is strong support for a relation between motivational and attitudinal factors, especially task engagement and self-efficacy, and the mathematics outcomes for black and Hispanic students. Recent research also documents that social and intellectual support from peers and teachers is associated with higher mathematics performance for all students and that such support is especially important for black and Hispanic students.

At least 5% of students will experience a significant learning disability in mathematics before completing high school, and many more children will show learning difficulties in specific mathematical content areas.

There are only a few cognitive studies of the sources of the accelerated learning of mathematically gifted students, but those that have been conducted suggest an enhanced ability to remember and process numerical and spatial information. Quasi-experimental and longitudinal studies consistently reveal that accelerated and demanding instruction is needed for these students to reach their full potential in mathematics.

Recommendations

Research efforts are needed in areas that assess the effectiveness of interventions designed to: 1) reduce the vulnerability of black and Hispanic students to negative stereotypes about their academic abilities, 2) functionally improve working memory capacity, and 3) provide explicit instruction on how to use strategies for effective and efficient problem solving.

More experimental work is needed to specify the underlying processes that link task engagement and self-efficacy, and the mathematics outcomes for black and Hispanic students. Urgently needed are a scaling-up and experimental evaluation of the interventions that have been found to be effective in enhancing engagement and self-efficacy for black and Hispanic students.

Intervention studies of students with a mathematics learning disability (MLD) are in the early stages and should be a focus of future research efforts. Further research also is needed to identify the sources of MLD and learning difficulties in the areas of fractions, geometry, and algebra.

Brain Sciences and Mathematics Learning

Brain sciences research has the potential to contribute to knowledge of mathematical learning and eventually educational practices, yet attempts to make these connections to the classroom are premature. Instructional programs in mathematics that claim to be based on brain sciences research remain to be validated. Yet, promising research emerging from the field of cognitive neuroscience is permitting investigators to begin forging links between neurobiological functions and mathematical cognition.

Most research making use of brain imaging and related techniques has focused on basic mental representations of number and quantity, with a few studies exploring problem solving in arithmetic and simple algebra. In most of these studies, researchers have contrasted, mapped, and differentiated the brain regions active during mathematical activities. It has been repeatedly found that comparisons of number magnitudes, quantitative estimation, use of a mental number line, and problem solving in arithmetic and algebra activate several areas of the parietal cortex. The intraparietal sulcus is also active when nonhuman animals engage in numerical activities, and it has been proposed that a segment of this sulcus, particularly in the left hemisphere, may support an inherent number representational system.

Research also shows that the hippocampus, which supports the formation of declarative memories, is active when involved in the learning of basic arithmetic facts. Other studies suggest the parietal cortex in the adolescent brain may be more responsive than the same regions in the adult brain when individuals are learning to solve simple algebraic equations. Another study suggests differences in the brain regions that contribute to success at solving algebraic word problems and algebraic equations. In addition, research shows there may be differences in the network of posterior brain regions engaged during the learning of different arithmetical operations.

In coming years, brain imaging and related methodologies will almost certainly help answer core questions associated with mathematical learning, such as the sources of learning disabilities and the effects of different forms of instruction on the acquisition of declarative, conceptual, and procedural competencies.

Recommendations

Brain sciences research has a unique potential for contributing to knowledge of mathematical learning and cognition and eventually educational practices. Nevertheless, attempts to connect research in the brain sciences to classroom teaching and student learning in mathematics should not be made until instructional programs in mathematics based on brain sciences research are created and validated.

I. Introduction

This report reflects the work of the Task Group on Learning Processes and addresses what is known about how children learn mathematical concepts and skills. The discussion begins with an introduction to the basic principles of learning and cognition, as well as the social and motivational factors that are relevant to educational practices, and to skill development in the focal mathematical domains addressed. The focus then moves to a review of the mathematical competencies that many children bring to school, followed by reviews of research on conceptual and procedural learning in the core content areas of whole number arithmetic, fractions, estimation, geometry, and algebra. These reviews summarize the scientific literature on what is known about learning within each of these areas and identify areas in which future study is needed before definitive conclusions can be made. Next, the report addresses individual and group differences in achievement in these core domains or in mathematics achievement up to and including algebra; the Task Group addresses mathematics achievement as related to race and ethnicity, gender, learning disabilities, and giftedness. The Task Group report closes with a discussion of future directions, specifically the implications of recent advances in the brain sciences for understanding mathematical learning.

II. Methodology

For all areas and to the extent that high-quality literature was available, the reviews and conclusions of the Task Group are based primarily on studies that test explicit hypotheses about the mechanisms promoting the learning of declarative knowledge (arithmetic facts), procedural knowledge, and conceptual knowledge. The evidence regarded as strongest for this purpose is that which shows convergent results across procedures and study types. When the evidence is not as strong, conclusions are qualified and suggestions are provided for research that will strengthen the ability to draw conclusions.

The multiple approaches, procedures, and study types reviewed and assessed with regard to convergent results include the following:

- Verbal report (e.g., of problem solving approaches).
- Reaction time and error patterns.
- Priming and implicit measures.
- Experimental manipulation of process mechanisms (e.g., random assignment to dual task, or practice conditions).
- Computer simulations of learning and cognition.
- Studies using brain imaging and related technologies.
- Large-scale longitudinal studies.
- International comparisons of math achievement.
- Process-oriented intervention studies.

A. Procedures

1. Literature Search and Study Inclusion

Literature searches were based on key terms linking mathematical content, and learning and cognitive processes (Appendix B). The first search focused on core peer-reviewed learning, cognition, and developmental journals (see Appendix A). A second search supplemented the first and included other empirical journals indexed in PsychInfo and the Web of Science.

2. Criteria for Inclusion

- Published in English.
- Participants are age 3 years to young adult.
- Published in a peer-reviewed empirical journal, or a review of empirical research in books or annual reviews.
- Experimental, quasi-experimental, or correlational methods.

III. Reviews and Findings

A. General Principles: From Cognitive Processes to Learning Outcomes

Cognitive science is the basic discipline that underlies studies of human learning, including learning of academic material, just as biology is the basic discipline that underlies medical practice and physics is the basic discipline that underlies engineering. In all three cases, the basic science identifies the causal pathways to successful outcomes. The next few pages describe the key cognitive processes that control learning: information processing operations (attention, working memory, retrieval, transfer, and retention; Section 1), and mental representations (declarative, procedural, and conceptual knowledge; verbatim and gist memories; Section 2). Students also engage in metacognitive processes, which are processes that control cognitive operations, such as explicitly selecting and monitoring strategies for effective problem solving (Section 1). Students' ability to orchestrate these various cognitive and metacognitive operations depends on the maturity of their prefrontal cortex, which controls attention and working memory, as well as on specific brain regions engaged in the representation of concepts or procedures. Examples of how these cognitive and metacognitive processes affect mathematics learning are presented, as are research-based methods of enhancing each process and thereby potentially improving mathematics learning. These examples and others in the sections that follow illustrate the utility of cognitive research for understanding learning, and suggest that teachers, superintendents, policy makers, curriculum developers, and anyone else whose goal is to increase student achievement, would advance that goal by having at least a rudimentary knowledge of the basic science of cognition.

There is a great deal of scientific knowledge that could be applied today to improve learning and student achievement. Much of that knowledge is currently not used in the nation's classrooms.

The concepts, principles, and processes of cognition presented here are supported by high-quality scientific research. This research provides insights, and sometimes immediate applications, to how student learning can be improved. There is much scientific knowledge that could be applied today to improve learning and student achievement (e.g., Cepeda et al., 2006). However, much of that knowledge is not currently being applied in the nation's classrooms. The following sections provide a review of scientific evidence about topics ranging from simple information processing to complex problem solving. Even creativity has been studied scientifically; excellent work on this topic was conducted in the 1950s and continues to the present day (e.g., Holyoak & Thagard, 1995; Sternberg, 1999). Therefore, this report proceeds through each of the cognitive building blocks to student achievement, to informed citizenship, and to career development in fields that require mathematical proficiency.

Basic research in cognitive science, especially research on the factors that promote learning, provides an essential grounding for the development and evaluation of effective educational practices.

What is cognition? Cognition encompasses attention, learning, memory, conceptual understanding, and problem solving, among other "higher" mental processes. General principles of cognition underlie learning and achievement in mathematics, and other academic domains. Test performance in mathematics, for example, is the end product of cognitive processes that include encoding and storing what has been taught, and retrieving it in response to test questions. Because achievement outcomes are critically dependent on the proper sequencing and execution of multiple cognitive operations, obtaining appropriate outcomes requires instruction to be based on a sound scientific foundation. The analogy to medicine is direct: Understanding the causal pathways that produce healthy outcomes (or go awry and result in disease) allows medical researchers to fashion drugs and therapies to achieve better outcomes. Understanding causal pathways in education works the same way as in medicine as it identifies the steps in the learning process that lead to successful outcomes, as well as missteps in the process and how these can be fixed. Just as in medicine, however, interventions derived from basic science about causal pathways must be tested for practical efficacy in educational settings (much like Phase III clinical trials in medicine).

Cognitive factors are not the only causal factors that have been linked to achievement outcomes. Nevertheless, all factors eventually have their effect via cognition.

Cognitive factors are not the only causal factors that have been linked to achievement outcomes; motivation, anxiety, nutrition, stereotypes, brain functioning, and tangible resources, such as availability of quality teachers and textbooks, are among other factors also relevant to achievement (e.g., Ashcraft, 2002; Cadinu et al., 2005; see following sections in this report). Nevertheless, these factors influence learning outcomes by virtue of their effects on cognitive processing. As an illustration, individuals who are anxious about mathematics perform worse on mathematics tests and on other mathematics tasks than their less anxious peers. The finding that interventions, such as cognitive behavioral therapy, can substantially improve the mathematical performance of many of these individuals indicates that their initial deficit is not related to the ability to learn mathematics (Hembree, 1990). Cognitive studies have identified working memory as one source of the lower mathematics achievement of individuals with mathematics anxiety; while performing math tasks, these anxious individuals have thoughts related to their competence intrude into working memory (described below), which disrupts their problem solving (Ashcraft & Kirk, 2001; Ashcraft, & Krause, 2007). Beilock, Kulp, Holt, & Carr (2004) demonstrated that similar intrusions into working memory can disrupt arithmetical problem solving in high-pressure testing situations but only when the procedures are not well learned; the execution of procedures committed to long-term memory was not disrupted by high-pressure testing. Hence, anxiety disrupts performance by affecting cognitive processing (i.e., by overloading working memory) and interventions to reduce that disruption have been shown to be effective. A goal of the present report is to identify such relevant findings and principles that have emerged from cognitive research and to suggest how they could be used to improve educational practice.

1. Information Processing

Attention is the gateway to the mind and, thus, to learning.

Information processing begins when the student first encounters information and extends until that information is operated on (or transformed) and a response is made, such as when a solution to a problem is produced. The first step in information processing is attention (e.g., Cowan, 1995; Pashler, 1999). Attention is a limited capacity faculty, often described as a bottleneck in information processing. Thus, only a portion of information in the environment can be attended to at any one time. Attention is crucial to learning; information that is unattended is lost to the learner. Distractions, such as noise, further limit the ability to pay attention. In addition, attention changes developmentally: Younger children are less attentive than older children (and adults), and distractions are more costly to younger children.

The ability to pay attention should not be confused with the motivation or desire to pay attention. No matter how much younger children may wish to pay attention, their ability to do so is lower than that of older children (Cowan, Saults, & Elliott, 2002). However, specific practices and environmental supports can enhance younger children's ability to attend (described below).

Because attention is the first step in information processing on which all subsequent steps depend, deficits in attention necessarily influence learning. Educational practices and environmental accommodations can improve children's ability to pay attention, such as by limiting irrelevant distractions (especially in the early phases of learning). For example, guiding children's attention to where the 0 is in comparing .03 to .30 has been shown to be effective in improving performance on judgments of relative magnitude (Rittle-Johnson et al., 2001). Recent evidence also suggests that self-regulation—intentional efforts to control attention and behavior—can be improved with practice (Baumeister, 2005; Gailliot, Plant, Butz, & Baumeister, 2007; Muraven, Baumeister, & Tice, 1999).

Working-memory capacity limits mathematical performance, but practice can overcome this limitation by achieving automaticity.

Once information is attended to, it can be encoded into working memory. Working memory is the ability to hold a mental representation of information in mind while simultaneously engaging in other mental processes. Working memory is composed of a central executive that is expressed as attention-driven control of information represented in one of three content-specific systems (Baddeley, 1986, 2000; Engle, Conway, Tuholski, & Shisler, 1995). These systems are a language-based phonetic buffer, a visuospatial sketch pad, and an episodic buffer (i.e., memories of personal experiences). The workings of these systems can be illustrated in a simple arithmetic context. Students initially solve simple addition problems, such as 3 + 4, by means of counting fingers or manipulatives. The child's ability to control the counting process is influenced by the central executive; if the counting process is not well controlled by the central executive, the child may skip a finger or manipulative, or count a single object twice. The representation of the spoken numbers is in the phonetic buffer; if the phonetic buffer is insufficient, the child may need to repeat a number that has already been stated or may skip a number. The visuospatial sketch pad would come into play if the child were counting imagined objects; insufficiencies here might lead to too few or too many objects being imagined, and therefore to inaccurate counts.

With practice, the addends and answers on problems that have been solved are transferred from working memory into more permanent long-term memory. As illustrated in later sections, deficient working memory is a major contributor to the learning problems encountered by children with mathematical learning disabilities and superior working memory is a major contributor to the accelerated learning shown by gifted children.

Working memory capacity increases as children grow older, due to improvements in their ability to control attention and to increases in the fundamental capacity of the content-specific systems (Cowan et al., 2002). At all ages, there are several ways to improve the functional capacity of working memory. The most central of these is the achievement of automaticity, that is, the fast, implicit, and automatic retrieval of a fact or a procedure from long-term memory (Schneider & Shiffrin, 1977; Shiffrin & Schneider, 1977). Some types of information, such as facial features, are processed automatically and without the need for any type of instruction (Schyns, Bonnar, & Gosselin, 2002). For other types of information, including much of mathematics that is taught in school, automaticity is achieved only with specific types of experiences, including practice that is distributed across time (e.g., Cooper & Sweller, 1987).

For example, repeated practice with addition facts, such as 3 + 4 = 7, eventually transforms addition from a conscious resource-demanding process (e.g., counting on one's fingers) to an automatic process, freeing up much-needed mental resources for other aspects of problem solving (Groen & Parkman, 1972; Siegler & Shrager, 1984). The ability to efficiently retrieve basic arithmetic facts has been shown to be integral to more complex, conceptual mathematical thinking and problem solving (Geary & Widaman, 1992). As Gersten and Chard (2001) state, "if too much energy goes into figuring out what 9 plus 8 equals, little is left over to understand the concepts underlying multi-digit subtraction, division, or complex multiplication." As discussed below research has demonstrated that declarative knowledge (e.g., memory for addition facts), procedural knowledge (or skills), and conceptual knowledge are mutually reinforcing, as opposed to being pedagogical alternatives. As discussed in greater detail in later sections, to obtain the maximal benefits of automaticity in support of complex problem solving, arithmetic facts and fundamental algorithms should be thoroughly mastered, and indeed, over-learned, rather than merely learned to a moderate degree of proficiency.

Young children are capable of far greater learning of mathematics than those in the United States typically attain.

Learning and development are incremental processes that occur gradually and continuously over many years (Siegler, 1996). Even during the preschool period, children have considerably greater reasoning and problem solving ability than was suspected until recently (Gelman, 2003; Gopnik, Meltzoff, & Kuhl, 1999). As stated in a recent report on the teaching and learning of science, "What children are capable of at a particular age is the result of a complex interplay among maturation, experience, and instruction. What is developmentally appropriate is not a simple function of age or grade, but rather is largely contingent on prior opportunities to learn" (Duschl, Schweingruber, & Shouse, 2007, p. 2). Claims based, in part, on Piaget's highly influential theory that children of particular ages cannot learn certain content because they are "too young," "not in the appropriate stage," or "not ready" have consistently been shown to be wrong (Gelman & Williams, 1998). Nor are claims justified that children cannot learn particular ideas because their brains are insufficiently developed, even if they possess the prerequisite knowledge for learning the ideas. As noted by Bruer (2002), research on brain development simply does not support such claims.

These findings have special relevance to mathematics learning. Research on students in East Asia and Europe show that children are capable of learning far more advanced math than those in the United States typically are taught (Geary, 2006). There is no reason to think that children in the United States are less capable of learning relatively advanced mathematical concepts and procedures than are their peers in other countries.

Practice retrieving information from memory can improve learning more than another opportunity to study.

Attending to information, encoding it into working memory, and eventually transferring it into long-term memory are only the initial steps in learning. The learner must also be able to retain the information in long-term memory storage until needed (e.g., on tests or on the job), sometimes over long periods of time, and be able to retrieve it from storage. One counterintuitive finding from these studies is that testing, which allows the learner to practice retrieving information from storage, has been found to improve performance more than the opportunity to study the material again. Such testing enhances both initial acquisition and long-term retention (Halff, 1977; Kinstch, 1968; Roediger & Karpicke, 2006a, 2006b; Runquist, 1983; Underwood, 1964). A key aspect of retrieval is the overlap between cues present at study and at test (the encoding specificity principle; Tulving & Thomson, 1973). For example, if variables within algebra problems are always stated in textbooks using *x* and *y*, but are then tested using other labels such as *j* and *k*, test performance will be reduced. Once information is stored, students must learn to recognize sometimes subtle cues (and to ignore irrelevant cues) in order to draw on the right knowledge in the right context.

Conceptual understanding promotes transfer of learning to new problems and better long-term retention.

Research has demonstrated that factors that enhance initial acquisition are not necessarily the same as those that maximize long-term retention (i.e., that minimize forgetting). For example, material that is too easy to understand can promote initial acquisition or learning, but it leads to lower retention than material that is harder to understand initially (e.g., Bjork, 1994). Challenging material causes the learner to exert more attentional effort and to actively process information, leading to superior retention. Similarly, transfer of learning is promoted by deeper conceptual understanding of learned material. Although this phenomenon was demonstrated in the early work of Gestalt psychologists (e.g., Wertheimer, 1959), it has since been verified repeatedly (for illustrative empirical studies on transfer and reviews of such studies, see Bassok & Holyoak, 1989; Reed, 1993; Wolfe, Reyna, & Brainerd, 2005). Transfer of learning refers to the ability to correctly apply one's learning beyond the exact examples studied to superficially similar problems (near transfer) or to superficially dissimilar problems (far transfer). Surprisingly, instruction using more abstract representations has been shown in some instances to benefit learning and transfer more than concrete examples (e.g., physical representations, such as manipulatives) (e.g., Sloutsky, Kaminski, & Heckler, 2005; Uttal, 2003). Thus, the cognitive processes that facilitate rote retention (e.g., of over-learned arithmetic facts), such as repeated practice, can differ from the processes that facilitate transfer and long-term retention, such as conceptual understanding. People's knowledge of how such factors affect cognition and thus how they can better monitor and control their learning—metacognition—also has been the subject of extensive research (e.g., Koriat & Goldsmith, 1996; Metcalfe, 2002; Nelson & Narens, 1990; Reder, 1987). Research has shown that there is much room for improvement in students' metacognitive judgments because they rely on misleading assumptions about their learning (e.g., using misleading cues such as retrieval fluency and familiarity, which are not perfectly correlated with strength of learning; see Benjamin, Bjork, & Schwartz, 1998).

2. Mental Representations

Although laws of memory apply to different kinds of content, just as the laws of physics apply to different kinds of objects, memories take different forms depending on their content. Declarative knowledge is explicit memory for specific events and information; procedural knowledge refers to implicit memory for cognitive (e.g., algorithms) and motor sequences and skills; and conceptual knowledge refers to general knowledge and understanding stored in long-term memory (see Hunt & Ellis, 2004, for further distinctions). Declarative, procedural, and conceptual knowledge seem to be represented in different ways in the brain (e.g., Schacter, Wagner, & Buckner, 2000). For example, a patient with brain damage can have amnesia for declarative knowledge, failing to remember his name and not recognizing his loved ones, but retain procedural skills such as piano playing or mathematical computation.

Using the mental number line: Counting supports learning addition.

One important mental representation in mathematics learning is the number line. The acquisition of counting, which forms the basis for arithmetic learning, is eventually mapped for successful learners onto an internal number line. One key use of this internal representation is for understanding the meaning of basic arithmetic operations. Counting, conceived as proceeding in steps up and down such an internal linear representation, provides a transition to learning arithmetic. Addition and subtraction can then be analogously conceived as proceeding in steps up and down that internal number line. The mental number line also plays a role in estimating the magnitudes of numbers in situations in which precise calculation is impossible (Siegler & Booth, 2005). For example, providing low-income children who attend Head Start centers an hour of practice playing numerical board games using consecutively numbered, linearly arrayed squares, dramatically improves their understanding of the mental number line and their estimation of numerical magnitudes (Siegler & Ramani, in press). In addition, explicitly instructing children from low-income backgrounds in number line skills using linearly organized board games (i.e., practicing with increments of only one step up or down) improves their procedural and conceptual arithmetic skills more than a year after this instruction, demonstrating both near and far transfer (e.g., Griffin, Case, & Siegler, 1994).

Mental models guide the acquisition of cognitive skills and the development of strategies, improving mathematics performance.

Mental models are ways of internally representing problems, often in the form of specific images. A mental number line is an example of a mental model (Case & Okamoto, 1996). The application of these mental models can be illustrated by thinking about fractions. A physical model, which can be internalized with practice and be used to think about fractions, is the familiar pie diagram; for example, $\frac{3}{4}$ might be represented by thinking of a pie cut into four equal pieces, three of which are highlighted. Analogous physical models can be constructed using folded paper, chips, and other physical objects. An alternative mental model for thinking about $\frac{3}{4}$ would be imagining two children who wanted to build a tower from their collection of Legos®. If one child supplied three of the Legos® and the other child a single Lego®, the first child would have supplied $\frac{3}{4}$ of the Legos®. Similarly, the ways in which children physically, and then mentally, represent the relation between divisors and quotients influence their skill at solving simple division problems (Squire & Bryant, 2003). For example, mentally picturing two sets of six objects helps children solve such problems as $\frac{12}{2}=6$. As Halford (1993) has pointed out, appropriate mental models—a mental picture of the concepts underlying the problem—provide a framework for problem solving that improves performance.

Verbatim memories of problem details are encoded separately from gist memories of the meaning of problem information; thinking in terms of gist often produces superior reasoning.

Experimentation on the relations between memory and reasoning has addressed how memory controls reasoning, why some forms of such reasoning are easier and more accurate than others, and what sorts of instruction benefit reasoning (e.g., Reyna & Brainerd, 1991). The most basic finding from this research is that there are two main types of memory, namely verbatim memory and gist memory (Brainerd & Reyna, 1993; Reyna & Brainerd, 1993). The importance of this distinction can be illustrated by a study of children's memory for numerical information within stories (Brainerd & Gordon, 1994). The verbatim level consisted of the actual numbers within the stories; the gist level consisted of various numerical relations, such as "more," "less," "most," "least," and "between." When told, for example, that Farmer Brown owned 3 dogs, 5 sheep, 7 chickens, 9 horses, and 11 cows, children accurately remembered that he had fewer dogs and more cows than any other animal. They were considerably less accurate in remembering how many of each type of animal he had (Brainerd & Gordon).

In some contexts, less precise gist memories are more important to performance than verbatim memories of the actual numbers and operations (Reyna & Brainerd, 1993). Many other features of these problems can be answered accurately and effortlessly by one or another type of gist knowledge. Estimation provides one such context. The late physicist Richard Feynman, for instance, argued that solving complex problems depends on seeing where solutions must lie–getting the gist of problems–more than on verbatim calculation (Leighton, 2006). Thus, being able to estimate that 74×97 must equal a little less than 7400 and thus cannot equal either 718 or 71,780, can help children recognize that they have made a mistake if they obtain either of those answers.

Psychological theory explains why ratio concepts, such as fractions, probabilities, and proportions, are especially difficult; this theory also provides straightforward ways to improve performance.

The importance of memory for gist extends to more complex mathematical relations as well, such as ratios, fractions, and probabilities (e.g., Hecht, Close, & Santisi, 2003; Reyna, 2004). For instance, in probability judgments, making accurate forecasts about the relative likelihood of occurrence of a set of events is usually quite difficult (Reyna & Brainerd, 1994), but it becomes much easier when gists are used (e.g., expressing the probabilities of the individual events in terms such as *more than half* or *less than half* (e.g., Brainerd & Reyna, 1995; Spinillo & Bryant, 1991). Based on these findings, interventions have been designed and tested with students ranging from young children to medical residents, and found to virtually eliminate common errors (e.g., Brainerd & Reyna, 1990, 1995; Lloyd & Reyna, 2001).

One reason why mathematics is so difficult to master is that it requires the accumulation of considerable verbatim knowledge, which often requires more effort to learn than the gist. Nonetheless, verbatim recall of facts, concepts, postulates, and other knowledge is an essential feature of a strong mathematics education, despite its often requiring a great deal of time, effort, and practice. Gist memory, or less precise, conceptual memory traces, has broad implications for learning because it is the form of memory that is typically relied on in reasoning. In short, a strong mathematics background requires a combination of gist and verbatim representations, with the importance of one or the other dependent on the goal at hand.

B. Social, Motivational, and Affective Influences on Learning

Research has shown that motivation enhances learning—and that some kinds of motivation are more effective than others. Motivation to persevere when intrinsic enjoyment is low should be distinguished from making learning enjoyable; the former may be especially important in sustaining the effortful learning needed to master difficult content. Perceived utility and willingness to engage in difficult learning is influenced by beliefs about the contributions of ability versus effort in learning, self efficacy (i.e., the belief that one has the specific skills needed to be successful, which differs from self esteem), and an array of other intrapersonal and social factors. In the following sections, the Task Group reviews major theories and findings related to these factors and how they influence learning mathematics and student achievement. Theoretical frameworks are reviewed that focus on learning goals, motivation to learn, attributions and beliefs about learning outcomes, mathematics anxiety, and sociocultural considerations. For more comprehensive coverage of theories and empirical data in this area, see Ames and Archer (1988), Barron and Harackiewicz (2001), Eccles and Wigfield (2002), Grant and Dweck (2003), Meece, Anderman, and Anderman (2006), Bandura (1993), Ellis, Varner, and Becker (1993), and Rieber and Carton (1987).

1. Goals and Beliefs About Learning

Children's goals and beliefs about learning are related to their mathematics performance. Children who adopt mastery-oriented goals show better long-term academic development in mathematics than do their peers whose main goals are to get good grades or outperform other children. They also are more likely to pursue difficult academic tasks. Students who believe that learning mathematics is strongly related to innate ability show less persistence on complex tasks than peers who believe that effort is more important. Experimental studies have demonstrated that children's beliefs about the relative importance of effort and ability can be changed, and that increased emphasis on the importance of effort is related to improved mathematics grades. The Task Group recommends extension of these types of studies.

Children's learning goals vary along several dimensions. One important dimension is whether the goals emphasize accomplishing a task or enhancing one's ego (Nicholls, 1984). Another important distinction is whether the goals emphasize mastery of the material or outperforming other students (Ames, 1990; Dweck & Leggett, 1988). Yet another important distinction is between performance approach goals (i.e., striving to surpass the performance of others) and performance avoidance goals (i.e., trying to avoid looking less knowledgeable or inferior) (Elliott & Harackiwiez, 1996; Midgley, Kaplan, Middleton, Maehr, & Urban, 1998).

Mastery and performance goals have received the most empirical attention. When pursuing mastery goals, students tend to choose tasks that are challenging and concern themselves more with their own progress than with outperforming peers. Mastery goal orientation should not be confused with Bloom's (1971; 1981) notion of *mastery learning.* The latter refers to an instructional approach whereby teachers lead students through a discrete set of stair step learning units with the progression predicated on pre-established criteria for proficiency at each step. When pursuing performance goals, students focus on outperforming others and thus prefer and seek tasks in which they are already competent (i.e., "easy," less challenging tasks). In the face of failure or incorrect performance, mastery-oriented students are likely to attribute the result to their own lack of effort or insufficient opportunities for mastery rather than to lack of ability; children who emphasize their lack of effort or opportunities are more likely to redouble their levels of effort when faced with later challenging problems (Ames, 1992; Ames & Archer, 1988). In contrast, when faced with demanding problems, performance-oriented students often conclude that they do not have the ability to do well in the domain, and thus tend to avoid challenging material when they begin to experience failure.

With respect to math outcomes, Wolters (2004) for example has shown that among middle school students a mastery orientation was positively related to engagement in learning and math grades, but this was not the case for a performance goal orientation. Elsewhere, Linnenbrink (2005) found that among fifth- and sixth-grade students working on a five-week math unit on statistics and graphing, those pre-tested as high in mastery orientation reported greater self-efficacy, personal interest in math, and more adaptive help seeking. These students performed significantly better on the math unit exam than those who were pretested as high in performance goal orientation.

In high school, children who tend to have mastery goals also tend to be high in self-efficacy. Such children also tend to obtain high grades in mathematics courses (Gutman, 2006). Moreover, parents' mastery goals are associated with better grades in mathematics courses by their children. Graham and Golan (1991) have shown that instructions that prompt a mastery orientation lead to higher academic outcomes than do performance-based instructions, when the task calls for deep processing of complex concepts.

Ames (1992) reviewed several types of academic contexts likely to foster mastery goal orientations in school. These include contexts that 1) provide meaningful reasons (e.g., personal relevance) for task engagement or developing understanding of content; 2) promote high interest and intermediate challenge; 3) emphasize gradual skill improvement; and 4) promote novelty, variety, and diversity.

Beliefs about learning and intelligence also influence mathematics performance. When faced with challenging problems, children who believe that intelligence is in large part created by their efforts to learn tend to do better than children who believe that intelligence is a fixed quality that cannot be changed (Dweck, 1999). Looking more specifically at mathematics achievement, Dweck and her colleagues recently showed that students who viewed their intelligence as a fixed trait fared more poorly across the transition to junior high than did their peers who believed that their intelligence was malleable and could be developed (Blackwell, Trzesniewski, & Dweck, 2007). Although both groups began junior high with equivalent mathematics achievement, their mathematics grades diverged by the end of the first semester of seventh grade and continued to move apart over the next two years. The superior performance of students who believed that intelligence is malleable was mediated by their greater emphasis on learning, their greater belief in the importance of effort, and their more mastery-oriented reactions to setbacks.

Blackwell et al. (2007) also conducted an intervention with a different group of seventh-graders with declining mathematics grades. Both the experimental and control groups received an eight-session workshop that taught them useful study skills. However, for the experimental group, several of the sessions also taught them a malleable theory of intelligence. (These sessions began with the article *You Can Grow Your Intelligence*, which likened the brain to a muscle; the article also described how neurons in the brain were transformed through learning. Students then learned how to apply this idea to their schoolwork.) Whereas the control group continued its downward grade trajectory, the experimental group showed a significant rebound in mathematics grades. Moreover, teachers (blind to condition) singled out three times as many students in the experimental group as having shown marked changed in motivation to learn mathematics.

2. Intrinsic and Extrinsic Motivation

Young children's intrinsic motivation to learn (i.e., desire to learn for its own sake) is positively correlated with academic outcomes in mathematics and other domains. However, intrinsic motivation declines across grades, especially in mathematics and the sciences, as material becomes increasingly complex and as instructional formats change. The complexity of the material being learned reflects demands of the modern workforce that may not be fully reconcilable with intrinsic motivation—the latter should not be used as the sole gauge of what is appropriate academic content. At the same time, correlational evidence suggests that the educational environment can influence students' intrinsic motivation to learn in later grades. The Task Group recommends studies that experimentally assess the implications of these correlational results, that is, studies aimed at more fully understanding the relation between intrinsic motivation and mathematics learning.

Intrinsic motivation to learn is the desire to learn for no reason other than the sheer enjoyment, challenge, pleasure, or interest of the activity (Berlyne, 1960; Hunt, 1965; Lepper et al., 2005; Walker, 1980). It is often contrasted to extrinsic motivation, in which the motivation to learn is to gain an external reward, such as the approval of parents and others, or the respect of peers. Thus, intrinsic motivation is related to mastery goals and extrinsic motivation to performance goals.

Several studies have shown that learning and academic achievement are positively correlated with intrinsic motivation (Lepper et al., 2005). For example, in a recent study by Lepper et al., it was found that across a sample of third- to eighth-grade students, an intrinsic motivation orientation was positively correlated with mathematics grade point average (GPA) and with performance on a mathematics achievement test, whereas an extrinsic motivation orientation was negatively correlated to these outcomes. However, there is evidence that intrinsic motivation declines as children progress through school and as material becomes more challenging (Gottfried et al., 2001). For example, Gottfried et al. found that from the ages of 9 to 16 years (although there was a slight increase for 17-year-olds), children's overall intrinsic motivation for academic learning declined, with particularly marked decreases in mathematics and the sciences. Findings like these have led some to propose that such a reduction in intrinsic motivation over time may compromise engagement in mathematics learning in the upper grades.

3. Attributions

Student beliefs about the causes of their success and failure have been repeatedly linked to their engaging and persisting in learning activities. *Self-efficacy*—a central concept in attributional theories—has emerged as a significant correlate of academic outcomes. However, the cause-effect relation between self-efficacy and mathematics learning remains to be fully determined, as does the relative importance of self-efficacy versus ability in moderating these outcomes. The Task Group recommends experimental and longitudinal studies that assess the relative contributions of these factors to mathematics learning.

Students can attribute their successes and failures to ability in (e.g., I'm just good/bad at) mathematics, effort (e.g., I worked/did-not-work hard enough), luck, or powerful people (e.g., the teacher loves/hates me). These attributions influence students' subsequent engagement in learning.

Self-efficacy can be defined as beliefs about one's ability to succeed at difficult tasks (Bandura, 1997). Mathematics self-efficacy moderates the effect of ability on performance. In other words, ability is important for mathematics learning but is not sufficient; self-efficacy or confidence in one's mathematics ability is also crucial for high levels of achievement. At times, self-efficacy is more influential than general mental ability in predicting high school mathematics performance (Stevens, Olivarez, & Hamman, 2006), although other studies suggest that ability may be more important than motivational influences in general (Gagné & St Père, 2002). Studies that simultaneously assess ability, prior content knowledge, motivation, and efficacy beliefs are needed to more firmly establish the relative contributions of these factors to mathematics learning and achievement.

4. Self-Regulation

Self-regulation is a mix of motivational and cognitive processes. It includes setting goals, planning, monitoring, evaluating, making necessary adjustments in one's own learning process, and choosing appropriate strategies. Self-regulation has emerged as a significant influence on some aspects of mathematics learning. Although the concept appears promising, research is needed to establish the relation for a wider range of mathematics knowledge and skills.

The concept of self-regulation includes aspects of both motivation and cognition. Among the processes that are associated with self-regulation are monitoring one's own actions, evaluating one's success, and reacting to discrepancies between one's outcomes and one's goals.

Fuchs et al. (2003), deploying an experimental design, provided evidence for the effectiveness of self-regulated learning strategies in enhancing mathematics problem solving. They focused on elementary school students' application of knowledge, skills, and strategies to novel mathematics problems. In the self-regulation condition, students were prompted to engage in self-regulated strategies by being required to check their answers, set goals of improvement, and chart their daily progress. These efforts to improve self-regulation improved the children's mathematics learning.

Another form of self-regulation involves choosing which strategy to use to solve problems. Children know and use a variety of strategies for solving mathematics problems. For example, to solve simple addition problems, elementary school children sometimes count from one, count from the larger addend, decompose the problem into two simpler problems, or retrieve the answer from memory. Individual differences in children's arithmetic strategy choices reflect differences in knowledge of answers to problems and also in degree of perfectionism. One group of children—labeled *good students* by Siegler (1988a)—has high knowledge and usually retrieves answers to problems. Another group of children—labeled *perfectionists*—has comparable knowledge but prefers to double-check their retrieved answers via counting strategies. A third group of children—labeled *not-so-good students*—has poor knowledge and often guesses at the answer. Perfectionists are toward the high end of self-regulation and not-so-good students are toward the low end. Children who fall into the *not-so-good student* group are more likely than the others to subsequently be labeled as mathematics disabled or not promoted to the next grade (Kerkman & Siegler, 1993).

5. Mathematics Anxiety

Anxiety about mathematics performance is related to low mathematics grades, failure to enroll in advanced mathematics courses, and poor scores on standardized tests of mathematics achievement. It also may be related to failure to graduate from high school. At present, however, little is known about its onset or the factors responsible for it. Potential risk factors include low mathematics aptitude, low working memory capacity, vulnerability to public embarrassment, and negative teacher and parent attitudes. The Task Group recommends research that assesses these potential risk factors; it also recommends development of promising interventions for reducing debilitating mathematics anxiety.

Mathematics anxiety refers to an emotional reaction, ranging from mild apprehension up through genuine fear or dread, in academic and everyday situations that deal with numbers, for instance taking a standardized achievement test, or figuring out a restaurant bill or change. Considerable research was done in the 1970s and 1980s on the relationships between mathematics anxiety, personality characteristics, and aspects of academic achievement, yielding a rather bleak picture (see Hembree, 1990). In brief, individuals with high mathematics anxiety perform poorly in school math, earn poor grades in math classes, take fewer elective mathematics courses in high school and college, and avoid college majors that rely on mathematics (e.g., mathematics, science, and engineering fields). There is a tendency, although weak, for women to exhibit higher levels of mathematics anxiety.

Mathematics anxiety research beginning in the 1990s has taken a more process-oriented approach to understanding the phenomenon, asking, "What are the cognitive consequences of mathematics anxiety?" The major discovery from this work is that during mathematics performance, those with mathematics anxiety focus many of their limited working memory resources on managing their anxiety reaction rather than on the execution of the mathematics procedures and processes necessary for successful performance (Ashcraft & Kirk, 2001). Difficult mathematics problems require considerable working memory resources for keeping track of intermediate solutions, retrieving facts and procedures, and so forth (LeFevre, DeStefano, Coleman, & Shanahan, 2005). These resources are limited to begin with, and thus are seriously compromised when the individual devotes substantial portions of them to the worry and negative thoughts associated with mathematics anxiety. This research aligns with other contemporary research on factors such as stress and stereotype threat (e.g., Beilock, Rydell, & McConnell, 2007), and their negative effects on high-stakes testing outcomes. As such, mathematics anxiety may be yet another factor leading to poorer-than-expected performance on proficiency and standardized tests of mathematics achievement (Ashcraft, Krause, & Hopko, 2007).

Interestingly, the research shows that highly math anxious individuals who undergo successful therapeutic interventions, especially cognitive-behavioral therapies, then show math achievement scores approaching the normal range. This suggests that their original math learning was not as deficient as originally believed, but instead that their math achievement scores had been depressed by their math anxiety during achievement testing itself. More precisely, in a meta-analysis, Hembree (1990) found that reductions in mathematics anxiety can result in significant (~ .5 standard deviations) improvements in mathematical test scores and in grade point average in mathematics courses. However, not all treatments are equally effective. Traditional individual or group counseling techniques appear to be relatively ineffective in reducing mathematics anxiety or improving mathematical performance. Similarly, changes in classroom mathematics curriculum, such as providing calculators or microcomputers to aid in problem solving, appear to be largely ineffective in reducing mathematics anxiety. A promising exception appears to be curricular changes that increase student's mathematical competence. Hutton and Levitt (1987) improved feelings of competence, or self-efficacy, by focusing on the relation between mathematical performance and good study habits, and by improving basic skills. These goals were achieved, in the context of an algebra class, through the use of a specially designed textbook. For each algebraic topic, the textbook presented a review of the basic arithmetic skills needed to solve the associated algebra problems. These basic skills were then practiced. Lectures and the text material were synchronized, such that the basic foundation of each lecture was presented as "skeletal notes" in the textbook. This feature was designed to improve students' note taking, and to focus them on essential features of the lecture. The intervention resulted in significant reductions in mathematics anxiety and was associated with algebraic skills that did not differ from those of children without mathematics anxiety; as noted, these two groups typically differ and thus no difference suggests a gain on the part of the students with mathematics anxiety.

Cognitive therapies that focus on the worry component of mathematics anxiety are also promising (Ellis et al., 1993). They are associated with moderate declines (~ .5 standard deviations) in mathematics anxiety, as well as modest (~ .3 standard deviations) improvements in mathematical performance (Hembree, 1990). These therapies focus on reducing the frequency of intrusive thoughts during mathematical activities, and on changing the individual's attributions about their performance. Poor performance that is attributed to a lack of ability will often result in avoidance of and lack of persistence on difficult mathematical tasks, as noted above. Changing attributions so that they focus on more controllable factors, such as preparation and hard work, often results in more persistent task-related behaviors and improvements in performance (Dweck, 1975). Ellis et al. argued that the treatment of mathematics anxiety should include building the student's basic competencies, knowledge, and skills. Increasing the competencies of students appears to reduce both the emotionality and worry components of mathematics anxiety, in addition to being an important goal in and of itself (e.g., Randhawa, Beamer, & Lundberg, 1993).

6. Vygotsky's Sociocultural Perspective

The sociocultural perspective of Vygotsky has been influential in education and characterizes learning as a social induction process through which learners become increasingly able to function independently through the guidance of more knowledgeable peers and adults. Aspects of this approach may add to the understanding of mathematics learning. However, a shortage of controlled experiments makes the usefulness of this approach for improving mathematics learning difficult to evaluate, and thus its utility in mathematics classrooms and mathematics curricula remains to be scientifically tested.

Vygotsky's sociocultural perspective posits that knowledge is first acquired in the interaction between the learner and other people, and that the knowledge is later internalized so that the learner can act on the knowledge in increasingly independent ways (see Rieber & Carton, 1987). Through this process of internalization, learners gain the knowledge and skills necessary for adequate functioning in their society (Wertsch, 1985).

From a sociocultural perspective, the most useful unit of analysis is not the child per se, but rather the child performing an activity in context. From this analytical frame, it is undesirable if not impossible to separate who the child is, from what the child does, from where the child does it. This framework calls for distinctive ways of construing learning processes, where notions such as zone of proximal development (the gap between what a learner can achieve independently and what the learner can achieve under the guidance of others), scaffolding (support from other people for problem solving activity), intersubjectivity (establishing a shared focus of attention), and apprenticeship (learning from more knowledgeable others) hold sway. Knowledge is not viewed as residing inside the child's head but rather as being distributed across the collectively held understandings of groups of people interacting with books, computers, worksheets, and other cultural tools. Knowledge acquisition is viewed as arising from participation in successful practices within a community of practice.

The evidence presented by socioculturalist scholars for their educational claims is not typically in the form of experiments or systematic empirical studies. Instead, detailed descriptions of people's everyday experiences in various contexts are provided and used to argue for particular educational arrangements. For example, Gonzales, Andrade, Civil, and Moll (2001) examined the informal mathematics knowledge shared among a group of community participants that was embedded in the everyday practical activity of sewing. It was demonstrated that through guided group discussions, the participants were able to link their informal mathematics knowledge to formal mathematics concepts and processes such as measurement, symmetry, geometric shapes, and angles, and to addition, subtraction, percentages, and proportions. Likewise, Kahn and Civil (2001) described how fourth- and fifth-grade children gained insight into domains like measurement (in this case the relation between area and perimeter) and graphing in the course of participating in a class-wide gardening project. The students' insights were said to be mediated by factors akin to guided participation and scaffolding. In addition, Gauvain (1993) has written extensively concerning how spatial thinking grows through participation in everyday practices. Although these descriptions are intriguing, lack of experimental studies makes it impossible to evaluate at present whether widespread adoption of such approaches would help or hinder mathematics learning and, if helpful, what specific areas of mathematics.

All told, concepts and processes such as zone of proximal development, scaffolding, and guided participation reflect core aspects of Vygotsky's sociocultural theory as related to instruction, and may hold important heuristic value. Yet to date, they have eluded measurement specificity and proven difficult to reliably quantify. Their ultimate utility in promoting effective evidence-based mathematics learning must await such specification and experimental validation.

7. Conclusions and Recommendations

Children's goals and beliefs about learning are related to their mathematics performance. Children who adopt mastery-oriented goals show better long-term academic development in mathematics than do their peers whose main goals are to get good grades or outperform other children. They also are more likely to pursue difficult academic tasks. Students who believe that learning mathematics is strongly related to innate ability show less persistence on complex tasks than peers who believe that effort is more important. Experimental studies have demonstrated that children's beliefs about the relative importance of effort and ability can be changed, and that increased emphasis on the importance of effort is related to improved mathematics grades. The Task Group recommends extension of these types of studies.

Young children's intrinsic motivation to learn (desire to learn for its own sake) is positively correlated with academic outcomes in mathematics and other domains. However, intrinsic motivation declines across grades, especially in mathematics and the sciences, as material becomes increasingly complex and as instructional formats change. The complexity of the material being learned reflects demands of a modern workforce that may not be fully reconcilable with intrinsic motivation. The latter should not be used as the sole gauge of what is appropriate academic content. At the same time, correlational evidence suggests that the educational environment can influence students' intrinsic motivation to learn in later grades.

The Task Group recommends studies that experimentally assess the implications of these correlational results, that is, studies aimed at more fully understanding the relation between intrinsic motivation and mathematics learning.

Student beliefs about the causes of their success and failure have been repeatedly linked to their engaging and persisting in learning activities. Self-efficacy has emerged as a significant correlate of academic outcomes. However, the cause and effect relation between self-efficacy and mathematics learning remains to be fully determined, as does the relative importance of self-efficacy versus ability in moderating these outcomes. The Task Group recommends experimental and longitudinal studies that assess the relative contributions of these factors to mathematics learning.

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Anxiety about mathematics performance is related to low mathematics grades, failure to enroll in advanced mathematics courses, and poor scores on standardized tests of mathematics achievement. It also may be related to failure to graduate from high school. At present, however, little is known about its onset or the factors responsible for it. Potential risk factors include low mathematics aptitude, low working memory capacity, vulnerability to public embarrassment, and negative teacher and parent attitudes. The Task Group recommends research that assesses these potential risk factors; it also recommends development of promising interventions for reducing debilitating mathematics anxiety.

The socio-cultural perspective of Vygotsky has been influential in education and places learning as a social induction process through which learners become increasingly able to function independently through the guidance of more knowledgeable peers and adults. Aspects of this approach may add to our understanding of mathematics learning. However, a shortage of controlled experiments makes the usefulness of this approach for improving mathematics learning difficult to evaluate, and thus its utility in mathematics classrooms and mathematics curricula remains to be scientifically tested.

Despite all that has been learned about the relation between these social/motivational goal orientations, attitudes, and beliefs and mathematics grades and achievement, too little is known about whether these influences reflect stable dispositions of students, or reflect teacher or peer influences in certain learning settings (Meece et al., 2006). The question of whether students in classroom settings have multiple goals or beliefs related to academic goals remains to be fully answered (Harackiewicz, Barron, Pintrich, Elliott, & Thrash, 2002; Brophy, 2005). In any case, the Blackwell et al. (2007) investigation, among others, indicates that beliefs about mathematics learning can be adaptively changed through targeted interventions. The Task Group recommends development and elaboration of these forms of intervention and assessment of ease with which they can be implemented by classroom teachers.

C. What Children Bring to School

Mathematical development begins in infancy, long before children go to school, and continues through the toddler and preschool years. The amount of mathematical knowledge that children bring with them when they begin school has large, long-term consequences for their further learning in this area. Thus, it is important to understand how mathematical knowledge typically develops before children start school, how children from different backgrounds and cultures vary in this knowledge, and how early mathematical learning can be improved.

1. Roots of Numerical Understanding

Mathematical development starts in infancy. Even infants between 1 and 4 months of age form nonverbal representations of the number of objects in very small sets. For example, when repeatedly shown two objects—two dots, two stars, two triangles—infants of this age gradually lose interest, and look for shorter and shorter times. However, when the number of objects is switched to one or three, they look longer, thus indicating that they noted the difference between sets with two objects and sets with one or three (e.g., Antell & Keating, 1983). This evidence suggests that sensitivity to number is innate to human beings.

Infants' surprising early numerical ability extends to a kind of nonverbal arithmetic. When 5-month-olds see a doll hidden behind a screen, and then see a second doll also placed behind the screen, they seem surprised and look longer when, through a trick, lifting the screen reveals one or three objects rather than two (Wynn, 1992). Presumably, they expected 1 + 1 to equal 2, and were surprised when it did not. A similar nonverbal form of subtraction is evident at the same age; when two objects are placed behind a screen and one object is removed, 5-month-olds look longer when lifting the screen reveals two objects rather than one. Whether these competencies are inherently numerical or not is debated (Cohen & Marks, 2002), but the basic finding has been replicated many times (e.g., Kobayashi, Hiraki, Mugitani, & Hasegawa, 2004).

In addition to these relatively precise nonverbal representations of very small numbers of objects, infants also display rudimentary estimation skills that allow them to discriminate between more and less numerous sets when the more numerous set has at least twice as many objects as the less numerous one. For example, they discriminate between sets of 16 and 8 objects, seeming to know that the set of 16 has more objects (Brannon, 2002; Xu & Spelke, 2000). These remarkable early nonverbal numerical abilities provide the foundation for learning about the verbal number system, including the number words, counting, numerical comparison, and more formal addition and subtraction.

2. Mathematical Understanding in the Preschool Period

a. Acquisition of Number Words and Counting

Many 2-year-olds in the United States know some number words, by age 3 or 4 years of age, many children can count (in the sense of accurately reciting the number words) from 1 to 10, and by the time they enter school, many children can count to 100 (U.S. Department of Education, NCES, 2001; Fuson, 1988; Miller, Smith, Zhu, & Zhang, 1995; Siegler & Robinson, 1982). Children also begin to learn to count objects at age 2 and a half or 3 years. At first, however, they acquire the superficial form of counting objects without understanding its purpose. Thus, when presented five objects and asked, "How many are there," many 2 and a half- and 3-year-olds will count to five but not answer the question. When again asked, "So how many are there," they will either again count to 5 without saying "There are 5" or say, "I don't know" (Le Corre, Van de Walle, Brannon, & Carey, 2006; Schaeffer, Eggleston, & Scott, 1974). This difficulty in connecting procedures with their goals and underlying principles is a persistent problem at all ages.

By age 4 or 5, when most children have had a reasonable amount of counting experience, they also come to understand the principles underlying the counting procedure: that each object must be labeled by one and only one number word, that counting requires the numbers to be recited in a constant order, and that the final word in the count indicates the number of objects in the set that has been counted (Gelman & Gallistel, 1978). Understanding these principles allows children to count in flexible ways, including, for example, starting the counting in the middle or at the right end of a row of objects if asked to do so, and to reject counts that skip an object or count an object twice (Frye, Braisby, Love, Maroudas, & Nicholls, 1989).

b. Ordering Numbers

Although it may seem surprising, being able to count from 1 to10 does not guarantee knowledge of the relative magnitudes of the numbers. Many 3- and 4-year-olds can count flawlessly to 10, but do not know that 8 is larger than 7 or that 7 is larger than 6 (Siegler & Robinson, 1982). By the time they enter kindergarten, however, most children know the relative magnitudes of numbers in this 1 to 10 range very well. Most children from middle-income backgrounds also have some knowledge of the order of numbers up to 100 when they enter school. When kindergartners are presented a number line with 0 at one end and 100 at the other end and asked to estimate the locations of numbers between 0 and 100 on the line, their estimates reflect the ordering of the numbers quite well, though not perfectly (Siegler & Booth, 2004).

c. Arithmetic

As with other numerical skills, children first show competence on addition problems with one to three objects. For example, if the experimenter asks a child to put three balls in an opaque tube, removes one of them, and then asks the child to remove the remaining balls, most 2 and a half- and 3-year-olds will reach into the tube exactly twice to pull out the remaining balls (Starkey, 1992). Children of this age usually fail, however, if the experimenter has the child put in four balls, removes one, and then makes the same request. The difficulty appears to involve limited ability to represent numbers precisely. Very young children show much greater ability to represent numbers approximately (Huttenlocher, Jordan, & Levine, 1994).

Most 4- and 5-year-olds can retrieve from memory the answers to at least a few basic addition and subtraction facts, such as 2 + 2 = 4, and also know a variety of other procedures for solving simple addition and subtraction problems. These include using fingers or objects to represent each addend and then counting them from one, representing the problem with fingers or objects and then recognizing how many of these are present, and counting from one without using objects or putting up fingers (Siegler & Shrager, 1984).

Even in the preschool period, children use these strategies in surprisingly adaptive ways. The harder the problem, the more likely 4- to 6-year-olds are to rely on counting or finger recognition strategies (Siegler & Shrager, 1984). This approach allows children to solve the easiest problems, such as 2 + 2, by using the fast approach of retrieving the answer from memory, and to solve problems that are too difficult to retrieve from memory via the slower but accurate alternative approaches of counting fingers or objects. The use of counting strategies on hard problems helps children generate the correct answer on those problems, which improves their likelihood of remembering it when the problem is presented later (Siegler, 1996).

Preschoolers also show some understanding of arithmetic concepts. For example, many 4- and 5-year-olds recognize that addition and subtraction are inverse operations. Thus, if presented problems of the form $A + B - B$, many preschoolers quickly answer "*A*" (Rasmussen, Ho, & Bisanz, 2003).

d. Measurement

During the preschool period, children acquire measurement strategies that are greatly oversimplified but that nonetheless reflect basic understanding of relations of equality, more than, and less than (Geary, 1994). When asked to divide up candies among friends, 2- and 3-year-olds typically give everyone some, without regard for whether each child receives the same number. In contrast, most 5-year-olds maintain exact numerical equivalence by using a "one for you, one for me, one for him, one for her" approach. They take this counting strategy too far, however, and use it even if one pile includes more large pieces of food than the other (Miller, 1984). Even 7- and 9-year-olds often use this strategy. Learning to restrict procedures to situations where they fit is another persistent challenge in mathematics learning.

e. Geometric Knowledge

During the preschool period, children also acquire rudimentary geometric knowledge. The large majority of 4- and 5-year-olds accurately identify circles and squares, and many also can identify triangles; by age 5, most also discriminate between squares and rectangles, and can describe some geometric attributes of those shapes (Clements, Swaminathan, Hannibal, & Sarama, 1999). Most children of these ages also have some skill in judging whether these basic figures are congruent; they usually adopt a strategy of comparing corresponding edges to do so (Clements, 2004).

Children's spatial knowledge also develops considerably in the preschool years. Most 5-year-olds can represent a location in terms of multiple landmarks, and from 5 to 7 years of age develop in their ability to maintain locations in challenging circumstances such as open areas (Newcombe & Huttenlocher, 2000). They use, implicitly, two coordinates in remembering direction, either polar or Cartesian, and can use simple external representation systems such as maps (Clements & Sarama, 2007b).

f. Number Sense

Through engaging in a variety of numerical activities, preschoolers, at least those from middle-income backgrounds, begin to develop number sense (Berch, 2005; Case & Sowder, 1990; Gersten & Chard, 1999; Jordan, Kaplan, Olah, & Locuniak, 2006). Number sense is the ability to approximate numerical magnitudes. The approximations can involve the numerical magnitude of specific dimensions of objects, events, or sets (e.g., "About how long is this line?" "About how many times have you been to New York?" "About how many people were at the play?"), or they can involve the results of numerical operations ("About how much is 24×94?"). Number line estimation tasks have proved particularly useful for investigating number sense. Such tasks involve presenting lines with a number at each end (e.g., 0 and 10) and no other numbers or marks in-between, and asking participants to locate a third number on the line (e. g., "Where does 7 go?").

Performance on this and other tasks used to measure number sense show that even before children enter school, children from middle-income backgrounds are developing a good sense of numerical magnitudes, whereas children from lower-income backgrounds have little sense of the numbers' magnitudes (Ramani & Siegler, 2008). This difference is important, because early number sense predicts subsequent ability to learn arithmetic in elementary school, above and beyond other important characteristics such as working memory (Locuniak & Jordan, in press). Measures of number sense also are strongly related to overall mathematics achievement (Booth & Siegler, 2006; Siegler & Booth, 2004). Although the number sense of children from low-income backgrounds typically lags behind that of peers from more affluent families, low-income children's number sense can be improved through playing linearly arranged numerical board games (Ramani & Siegler, 2008; Siegler & Ramani, in press).

3. Differences Among Individuals and Groups

Clear and systematic differences in children's mathematical competence emerge in the preschool period. The differences are present in counting, comparing magnitudes, adding, subtracting, and other aspects of numerical knowledge. These early-emerging differences among children appear to have important long-term consequences. A study that followed over many years large, nationally representative samples of U.S. children, as well as children from Canada and Great Britain, showed that mathematical knowledge during preschool and kindergarten is strongly predictive of mathematical knowledge in third grade, fifth grade, and eighth grade (Duncan et al., 2007). The relation is similarly strong for boys and girls and for children from low-income and middle-income backgrounds. It also is apparent in both math achievement test scores and teacher ratings of children's mathematical competence. Thus, children's mathematical knowledge differs substantially by the time they enter school and in ways that predict their mathematics achievement at least through middle school.

Differences in mathematical knowledge of U.S. children at the beginning of kindergarten reflect many aspects of the children's background. The Early Childhood Longitudinal Study (ECLS), which examined a large, representative sample of U.S. children, revealed several factors that predict children's mathematical knowledge when they enter kindergarten (U.S. Department of Education, NCES, 2001). One predictor is a mother's education; children of mothers with at least some college education usually have more knowledge of numbers and shapes than children whose mothers did not graduate high school. Another group of predictors involves risk factors such as single-parent families, families in which English is not the primary language spoken in the home, and families living in poverty. Children from families with fewer risk factors usually enter kindergarten with greater knowledge of numbers and shapes than children from families with more risk factors. A third predictor is race and ethnicity: white, non-Hispanic children and Asian children usually enter kindergarten with greater mathematical knowledge than black and Hispanic children.

The mathematical knowledge that children bring to school also varies with the country in which the child was raised. Children from East Asia generally have more mathematical knowledge when they enter school than do children in the United States. This superior knowledge seems to reflect the greater cultural emphasis on math learning within East Asian cultures. As the Japanese psychologist Giyoo Hatano commented, "Asian culture emphasizes and gives priority to mathematical learning; high achievement in mathematics is taken by mature members of the culture to be an important goal for its less mature members" (1990, pp. 110–111). Consistent with this observation, mothers in China rate doing well at math as being just as important for their children as doing well at reading, whereas mothers in the United States rate learning math as considerably less important (Miller, Kelly, & Zhou, 2005). Also reflecting the greater East Asian emphasis on math, in one study that compared Chinese and U.S. children from similar backgrounds who were just beginning kindergarten, the Chinese children generated three times as many correct answers to addition problems (Geary, Bow-Thomas, Fan, & Siegler, 1993). The difference was due to the Chinese children having memorized more correct answers to problems and to their using more advanced strategies when they could not retrieve the answer from memory. Preschoolers in China also count much higher, aided by the greater regularity of number words in their language (Miller et al.). Knowledge of shapes and other geometric information, memory for numbers, and other mathematical skills are also more advanced for Chinese than for U.S. preschoolers (Starkey et al., 1999). Although almost all studies show this pattern, a few have not; for example, Song and Ginsburg (1987) found that U.S. preschoolers outperformed Korean preschoolers in informal math knowledge.

4. Improving Early Mathematical Knowledge

A variety of instructional programs have been developed to improve the mathematical knowledge of U.S. preschoolers, especially preschoolers from low-income backgrounds. Several of these programs have met with considerable success. Project Rightstart and its successor Number Worlds (Griffin, 2004) focus on helping young children form an appropriate mental representation of numbers, akin to a mental number line; on using this mental representation to think about sets of real-world objects and arithmetic operations on those sets; and on familiarizing children with the language of numbers and mathematics. The

Berkeley Math Readiness Project (Klein & Starkey, 2004; Starkey, Klein, & Wakeley, 2004) provides preschool children with experience in counting and numerical estimation; arithmetic, spatial, geometric, and logical reasoning; measurement; and other aspects of mathematics. The Building Blocks program (Clements, 2002; Sarama, 2004; Sarama & Clements, 2004) uses computer software tools to help preschoolers acquire geometric and numerical ideas and skills. All of these produce substantial positive effects on children's mathematical knowledge. For example, in one study, Griffin's Number Worlds curriculum produced median effect sizes (Cox Index for standardized mean differences between experimental and control group) of 1.79 for 6 measures on the posttest and 1.40 for 13 measures on a later follow-up. The Berkeley Math Readiness curriculum produced an overall effect size of .96 (Hedges' g) among low-income children, and the Building Blocks program produced an overall effect size of .77 (Hedges' g) on 9 measures of numerical understanding and 1.44 on 8 measures of geometrical understanding. These are not the only programs that have been shown to increase preschoolers' mathematical competence, but they are good examples of the types of promising efforts that are being made in this direction (for a more comprehensive review of these and other programs aimed at enhancing preschoolers' mathematical competence, see Sarama & Clements). Research is needed to establish the longer-term effects of these programs.

5. Conclusions and Recommendations

Most children develop considerable knowledge of numbers and other aspects of mathematics before they begin kindergarten. Even in kindergarten, children from single-parent families with low-parental education levels and low incomes have less mathematical knowledge than do children from more advantaged backgrounds. The mathematical knowledge that children from both low- and middle-income families bring to school influences their learning for many years thereafter, probably throughout their education. A variety of promising instructional programs have been developed to improve the mathematical knowledge of preschoolers' and kindergartners, especially those from at-risk backgrounds. Research that scales up these interventions and evaluates their utility in preschool and early kindergarten settings is urgently needed, with a particular focus on at-risk children.

D. Mathematical Development in Content Areas

This section provides a review of the cognition literature as related to learning in the core mathematical content areas identified in the *Report of the Task Group on Conceptual Knowledge and Skills*. At the most general level, these include whole number arithmetic, fractions, estimation, geometry, and algebra. The quantity and quality of research on this learning differs considerably across the mathematical content areas. The Task Group notes areas in which substantive conclusions about learning or obstacles to learning can be drawn, and key mathematical areas in which a better understanding of learning is needed but for which the research base does not allow strong conclusions to be drawn. At the end of the review for each content area, the Task Group presents Conclusions and Recommendations.

Due to limited time, space, and resources, the coverage in this review is far from exhaustive. Nonetheless, the literature was thoroughly reviewed across all theoretical perspectives on mathematics learning. The studies included in the review were the ones that met the highest criteria of methodological rigor, as documented in Section IV, Methodology, in this report.

1. Whole Number Arithmetic

Children's learning of whole number arithmetic is a critical step in their mathematics education and a complex undertaking that extends for many years and engages multiple memory and cognitive systems. Core areas of competency include knowledge of basic arithmetic facts, skill at using standard procedures or algorithms for solving complex problems, estimating answers, and knowledge of key concepts (National Council of Teachers of Mathematics, 2006). For some of these core areas, such as simple addition, there is a substantive research base from which reliable descriptions of skill development can be provided, and inferences regarding at least some of the factors that facilitate or impede this development can be drawn. At the same time, there are other core areas, such as division algorithms, for which there is comparatively little empirical research, and thus the Task Group cannot make strong statements regarding the progression of skill development or the factors that influence this development.

Debate is common regarding whether mathematics education and related research studies should focus on conceptual knowledge or procedural skills (Baroody, Feil, & Johnson, 2007; Star, 2005, 2007). Empirical studies that have simultaneously assessed both of these aspects of mathematical competency reveal interdependence in children's development of declarative knowledge (e.g., addition facts), procedural knowledge (e.g., arithmetical algorithms), and conceptual knowledge (e.g., understanding the base-10 system). Aspects of skill development for each of these different types of competencies may require different prior knowledge, different instructional techniques, and different patterns of practice for mastery (Cooper & Sweller, 1987; Kalyuga, Chandler, Tuovinen, & Sweller, 2001; Sweller, Mawer, & Ward, 1983), yet their development is often interrelated (Rittle-Johnson et al., 2001). Children's use of one algorithm or another, or the detection of a computational error can be influenced by their understanding of related concepts, and the execution of algorithms can provide a context for their conceptual learning (Geary, Bow-Thomas, & Yao, 1992; Fuson & Kwon, 1992b). Children's skill at estimating is firmly linked to their computational skills (Dowker, 2003), and their ability to solve different types of complex word problems is dependent on different mixes of declarative, procedural, and conceptual competencies (Fuchs et al., 2006; Hecht et al., 2003).

For ease of presentation, the Task Group covers skill progression separately for these different competencies; nonetheless, it includes a few explicit examples of their interrelationships. The associated cognitive studies involve a detailed and time-intensive assessment of children's problem solving and learning and thus do not typically include large, nationally representative samples. The smaller-scale cognitive studies have, nevertheless, produced findings that have been replicated by many research groups and oftentimes in many nations. The Task Group's focus is on these replicated outcomes.

a. Acquisition of Arithmetic Facts

Addition and subtraction

One of the most thoroughly studied areas in children's mathematical learning involves descriptive assessments of developmental and schooling-based changes in the ways children solve simple addition and subtraction problems (Ashcraft, 1982; Carpenter & Moser, 1984; Geary, 2006; Geary, Bow-Thomas et al., 1996), as well as theoretical (e.g., computer simulations) and quantitative studies of the cognitive mechanisms underlying these changes (Shrager, & Siegler, 1998; Siegler, 1987; Siegler, 1988a). These studies and studies in other domains have clearly indicated that children's problem solving does not involve a step-by-step progression from use of one procedure to the next, but rather involves a mix of procedures and memory-based processes (e.g., direct retrieval of a fact) at most ages (Siegler, 1996). Learning involves a change in the mix of strategies used during problem solving, as well as improvement in the speed and accuracy with which individual procedures and memory-based processes are executed (Delaney, Reder, Staszewski, & Ritter, 1998; Geary, Bow-Thomas et al.). The focus here is on the mix of procedures and processes children use when they solve simple addition and subtraction problems and on their progression toward the learning of basic facts.

The Task Group notes that the learning and subsequent retrieval of basic facts does not involve the representation of isolated problem-answer combinations in long-term memory. Rather, this knowledge is embedded in a network of number- and arithmetic-related information. The use of the term *fact retrieval* simply refers to the goal of remembering the correct answer; it does not imply that associated problems, numbers, and answers are unrelated to other forms of knowledge, such as knowledge of general magnitude of the answer.

Paths of acquisition

Concepts. Young children's ability to solve formal addition and subtraction problems, such as $5 + 3 =$; or $7 - 2 =$, requires an integration of their emerging knowledge of the properties of associativity and commutativity (described below) with their counting knowledge and counting procedures (Ohlsson & Rees, 1991; Rittle-Johnson, & Siegler, 1998). Although there is some evidence for such an integration, the relation between these conceptual and procedural aspects of children's arithmetical learning has not been as thoroughly studied as the independent development of these competencies. For instance, there are many studies of children's emerging counting procedures and concepts (e.g., Briars, & Siegler, 1984; Fuson, 1988; Gelman, & Meck, 1983; LeFevre et al., 2006) and many studies of children's procedural development in addition and subtraction (described below), but only a few studies that have explicitly attempted to examine the link between these competencies (e.g., Geary et al., 1992).

Procedures. By the time children in the United States enter kindergarten, the most common procedures used to solve simple addition problems involve finger counting; some problems will be solved by counting out loud or mentally, and some children will know a few basic facts (Siegler, & Shrager, 1984). Counting procedures vary in sophistication—in terms of supporting conceptual knowledge and working memory demands—and kindergarten children typically rely on the least sophisticated of these procedures, referred to as *counting-all*, whereby children count both addends starting from 1. With the more sophisticated procedure called *counting-on*, children state the value of one addend (suggesting they understand the cardinality principle) and then count a number of times equal to the value of the other addend, counting 5, 6, 7, 8 to solve 5 + 3 = (Fuson, 1982; Groen & Parkman, 1972). Preliminary studies suggest that children's shift from counting-all to counting-on is related, in part, to improvements in their understanding of counting concepts (Fuson; Geary et al., 1992; Geary et al., 2004).

The frequent use of counting procedures results in the development of memory representations of basic facts (Siegler & Shrager, 1984); the act of counting 5, 6, 7, 8 to solve 5 + 3 facilitates the formation of an association in declarative memory between the addends and the answer generated by the counting. Once formed, these representations support the use of memory-based problem-solving processes. The most common of these are direct retrieval of arithmetic facts and decomposition. The latter involves reconstructing the answer based on the retrieval of a partial sum; for example, the problem 6 + 7 might be solved by retrieving the answer to 6 + 6 and then adding 1 to this partial sum (Siegler, 1987). A similar pattern is evident with children's skill progression in subtraction (Carpenter & Moser, 1983, 1984; Siegler, 1989). As with addition, children initially use a mix of strategies but largely count, often using their fingers or physical objects (i.e., manipulatives) to help them represent the problem and keep track of the counting. Children also rely on their knowledge of addition facts to solve subtraction problems, which is called addition reference (9 – 7 = 2, because 7 + 2 = 9) or use other related information (see Thornton, 1990). The most sophisticated processes involve decomposing the problems into a series of simpler problems and directly retrieving the answer (Fuson & Kwon, 1992a).

Declarative information. The primary declarative information contributing to the fast and efficient solving of simple addition and subtraction problems is knowledge of basic facts. The representation of these facts in long-term memory enables the use of direct retrieval and decomposition to solve these problems. Cognitive studies indicate that, unlike their peers in East Asian countries (), many college students in the United States have not memorized all of the basic addition and subtraction facts and thus often resort to use of backup strategies (Campbell & Xue, 2001; Geary, 1996; Geary & Wiley, 1991; Geary, Frensch, et al., 1993).

Multiplication and division

Paths of acquisition

Concepts. The core associative, commutative, distributive, and identity concepts as related to multiplication are described in a separate section below.

Procedures. Trends in children's ability to solve simple multiplication problems mirror those described for addition and subtraction, although formal skill acquisition begins in the second or third grade, at least in the United States. The initial mix of strategies is grounded in children's knowledge of addition and counting, including use of repeated addition and counting by *n* (e.g., Campbell & Graham, 1985; Mabbott & Bisanz, 2003; Siegler, 1988b; Thornton, 1978, 1990). Repeated addition involves representing the multiplicand, the number of times indicated by the multiplier, and then successively adding these values; when presented with 2 × 3, the child adds 2 + 2 + 2. The counting by *n* strategy is based on the child's ability to count by 2s, 3s, 5s, and thus is dependent on memorization of these counting sequences. Somewhat more sophisticated strategies involve the use of rules (identity element in this example; see below), such as $n \times 0 = 0$, and decomposition (e.g., $12 \times 2 = 10 \times 2 + 2 \times 2$). As with addition and subtraction, the use of these procedures appears to result in the formation of problem and answer associations in long-term memory (Miller & Paredes, 1990).

In comparison with the other operations, considerably less research has been conducted on skill progression in division. The research that has been conducted indicates that children rely heavily on their knowledge of addition and multiplication (Ilg & Ames, 1951; Robinson, Arbuthnott, et al., 2006). Robinson, Arbuthnott, et al. found that fourth-graders solved more than half of simple division problems by means of an addition-based procedure; to solve $\frac{20}{4}$, they repeatedly added the divisor until the dividend was reached, $4 + 4 + 4 + 4 + 4 = 20$, and then counted the divisors. Fourth-graders will sometimes solve the problem through reference to the corresponding multiplication problem ($5 \times 4 = 20$ for this example) or retrieve a division fact (e.g., $\frac{6}{3} = 2$). By seventh grade, the majority of the problems are solved by multiplication reference, although retrieval and the addition-based procedure are still used to solve some problems. Unlike the three other operations, use of direct retrieval did not increase across grade level; about 15% of division problems were solved by direct retrieval in grades 4 through 7.

Declarative information. As with addition and subtraction, the primary declarative information contributing to the fast and efficient solving of simple multiplication and division problems is knowledge of basic facts, that is, the representation of these facts in long-term memory. Studies of college students in the U.S. and Canada [computational skills are similar for students from these countries (Tatsuoka, Corter, & Tatsuoka, 2004)] suggest that many of these adults have not mastered all basic multiplication facts (LeFevre et al., 1996), and may continue to rely on multiplication reference to solve larger division problems (e.g., $\frac{72}{9}$) (Campbell, 1999; LeFevre & Morris, 1999; Robinson, Arbuthnott, & Gibbons, 2002). In contrast, college students who received their primary education in China can quickly and accurately retrieve the answers to all multiplications problems—though they rely on the commutative relation between problems to facilitate retrieval of some problems (e.g., 9×6 is retrieved based on 6×9)—and most simple division problems (Campbell & Xue, 2001; LeFevre & Liu, 1997). The implication is that many, perhaps most, U.S. children have not achieved fluency with simple multiplication and division.

Obstacles to mastery

In keeping with the broader methods and literature described in the section in this report entitled General Principles: From Cognitive Processes to Learning Outcomes (e.g., Ericsson, Krampe, & Tesch-Römer, 1993; Newell & Rosenbloom, 1981), the learning of simple and complex arithmetic has been studied using a variety of speed-of-processing, behavioral, and brain imaging methods (Charness & Campbell, 1988; Frensch & Geary, 1993; Klapp, Boches, Trabert, & Logan, 1991; Rickard, Healy, & Bourne, 1994; Royer, Tronsky, Chan, Jackson, & Marchant, 1999). These studies consistently find that practice results in faster solutions to basic problems and fewer errors, as well as related reductions in the working memory resources needed for problem solving and changes in the brain regions supporting this problem solving (described in Brain Sciences and Mathematical Learning, Section F in this report). The cognitive mechanisms underlying these changes include increased use of memory-based processes, more rapid execution of problem-solving procedures, and faster retrieval of relevant information from long-term memory (e.g., Delaney et al., 1998; Rickard, 1997). Skilled use of procedures also appears to require an understanding of associated concepts (Geary et al., 1992; Ohlsson & Rees, 1991).

The experimental studies have revealed that for most individuals, the ease of learning and retrieving arithmetic facts varies by the type of operation. The learning of addition and multiplication facts occurs with less practice than the learning of subtraction and division facts (Campbell & Xue, 2001; Rickard, 2004, 2005; Rickard et al., 1994). One reason for this operation effect is that the commutative principle for addition and multiplication (i.e., $a + b = b + a$; $a \times b = b \times a$) facilitates the learning of the associated facts; the learning of one combination of addition (e.g., $3 + 4$) or multiplication pairs (e.g., 7×2) contributes to the learning of the commuted pair (i.e., $4 + 3$, 2×7); but this relation does not hold for subtraction and division. Graduate students educated in China directly retrieve answers to smaller-valued subtraction and division (e.g., $\frac{28}{4}$) problems, but often solve larger-valued problems (e.g., $\frac{56}{8}$) through reference to the corresponding addition or multiplication problem, respectively (Campbell & Xue). A similar pattern is found for North American (Canada and United States) college students, but often extends to smaller-valued subtraction and division problems (LeFevre & Morris, 1999; Mauro, LeFevre, & Morris, 2003). This type of "mediated" retrieval is faster and more efficient than the use of procedures but still requires more time and an additional cognitive step—thus increased opportunity to commit an error—than direct retrieval of the answer. Finally, Rickard (2005) found that skill at factoring is related to knowledge of multiplication facts and that the practice of factoring (e.g., when presented with 21, the participant produces 7, 3) speeds subsequent retrieval of multiplication facts.

In studying the cognitive bases of children's arithmetic learning, researchers have not only examined how problem-solving approaches change with practice but also how these approaches vary across grade level and follow introduction of the operation (e.g., multiplication) in the school curriculum (Geary, 1996; Geary, Bow-Thomas, et al., 1996; Lemaire & Siegler, 1995; Miller & Paredes, 1990; Royer et al., 1999; Siegler, 1988b, 1989; Siegler & Jenkins, 1989; Steel & Funnell, 2001). The results of such studies are consistent with the experimental research: Fast and efficient problem solving is achieved with shifts from frequent use of counting or other procedures to direct retrieval of basic facts or use of decomposition. As with the experimental studies, children appear to learn addition and multiplication facts more easily than they learn subtraction and division facts, although comparatively little is known about children's learning of division.

Studies of children in the United States, comparisons of these children with children from some other nations, and even cross-generational changes within the United States indicate that many contemporary U.S. children do not reach the point of fast and efficient solving of basic arithmetic problems (Fuson & Kwon, 1992a; Koshmider & Ashcraft, 1991; Geary, Salthouse, et al., 1996; Geary et al., 1997; Schaie, 1996; Stevenson et al., 1985; Stevenson, Lee, Chen, Lummis, et al., 1990; for discussion see Loveless & Coughlan, 2004).

This point is particularly evident with comparisons of U.S. children to children educated in East Asia. Stevenson, Lee, Chen, Lummis, et al. (1990) assessed the speed and accuracy with which 480 first- or fifth-grade children from the United States and 264 same-grade children from China solved simple addition problems (e.g., 5 + 9), and found first-graders from China accurately solved three times as many problems (in 1 minute) than their U.S. peers (d = 2.97). The difference was smaller (1.6:1, d = 1.69) but still substantial for fifth-graders. The pattern was replicated for the basic subtraction skills of children from China and the U.S. for 6th- (d = 2.05) and 12th- (d = 2.05) graders matched or equated on cognitive ability (Geary et al., 1997). Although cognitive studies of how the children solved the problems have not been conducted for all arithmetical operations, the available evidence for addition suggests the differences in efficiency are related to less frequent use of retrieval by U.S. children, use of less sophisticated counting procedures, and slower retrieval and procedural execution when the problems are solved the same way (Geary, Bow-Thomas, et al., 1996).

The reasons for these differences are likely to be multifaceted, including language-related differences in the structure of number words, parental involvement in mathematics learning, and curricula (Miller et al., 1995; Miura, Okamoto, Kim, Steere, & Fayol, 1993; Steel & Funnell, 2001; Stevenson, Lee, Chen, Stiegler, et al., 1990). For instance, the structure of Asian-language number words where the teen values are stated as ten one, ten two, may facilitate, with teachers' guidance, the use of decomposition strategies to solve simple addition and subtraction problems (Fuson & Kwon, 1992a). Cross-national differences in mathematics curricula have not been directly tied to the cognitive studies of children's arithmetic learning. Nonetheless, results from several smaller-scale studies suggest such a link: In a review of the frequency of presentation of simple addition problems in first- to third-grade mathematics textbooks in the United States, Hamann and Ashcraft (1986) found that easier problems (e.g., 3 + 4) were presented much more frequently than harder problems (e.g., 8 + 7). In contrast, Geary (1996) found the opposite pattern in workbooks used to learn addition in China; a similar pattern of easier mathematics problems being presented in U.S. textbooks relative to same-grade textbooks from other nations has been reported by other researchers (Fuson, Stigler, & Bartsch, 1988). For third-grade children from the United States and China, the speed with which individual addition facts were retrieved from long-term memory was correlated (r's = .34 to .49; d's = 0.74, 1.12) with the cumulative (first- to third-grade) frequency with which the problems were presented in their respective countries. Whether children are learning addition in China or the United States, fast and efficient problem solving is related to frequency of prior exposure to the problem.

b. Learning Arithmetical Algorithms

Addition, subtraction, multiplication, and division

In this section, the four arithmetic operations are considered together because so little is known about children's learning of multiplication algorithms and division, and because what is known suggests similar obstacles to mastery across operations. The learning of algorithms requires a combination of an explicit conceptual understanding of related concepts (e.g., base-10); an understanding of when the algorithm should and should not be used; and, eventually, the ability to use the algorithm quickly and efficiently.

Paths of acquisition

Concepts. A central concept related to the use of arithmetical algorithms is the base-10 system and the corresponding understanding of place value and "trading" across columns (Blöte et al., 2001; Fuson & Kwon, 1992b). Coming to understand the base-10 system and place value is highly dependent on instruction (Hiebert & Wearne, 1996). Studies conducted in the United States have repeatedly demonstrated that many elementary-school children do not fully understand the base-10 structure of multidigit written numerals (e.g., understanding the place value meaning of the numeral) or number words (Fuson, 1990). As a result, many of these children are unable to effectively use this system when attempting to solve complex arithmetic problems. It appears that many children require instructional techniques that explicitly focus on the specifics of the repeating decade structure of the base-10 system and that focus on clarifying often confusing features of the associated notational system (Fuson & Briars, 1990; Varelas & Becker, 1997). An example of the latter is that sometimes "2" represents two units; other times it represents two tens; and, still other times it represents two hundreds (Varelas & Becker). Unlike East Asian languages where the base-10 structure is transparently represented in the associated number words (e.g., 21 is stated as two ten one), the English language number word system may actually lead to confusions about this relation (Miura et al., 1993). The development of base-10 knowledge is also facilitated by understanding that basic units ("ones") can be aggregated to form higher-order ones ("tens"), and prior understanding of cardinality, min counting, (i.e., stating the value of the larger addend and counting a number of times equal to the value of the smaller addend) and skill at decomposing numbers (Saxton & Cakir, 2006).

Procedures. The solving of arithmetic problems that are more complex than the simple problems described above, such as 23 + 6 or 12 × 73, involves the application of prior arithmetical skills and knowledge, the incorporation of new knowledge, and the learning of new procedures or algorithms.

When learning complex addition problems, children initially rely on the knowledge and skills acquired for solving simple addition problems, as reviewed in Siegler (1983); problems can be solved by means of counting, decomposition, or regrouping, as well as the formally taught columnar procedure (Ginsburg, 1977; Reys, Reys, Nohda, & Emori, 1995; Siegler & Jenkins, 1989). The decomposition or regrouping strategy involves adding the tens values and the units values separately; the problem 23 + 45 would involve the steps 20 + 40, 3 + 5, and then 60 + 8 (Fuson & Kwon, 1992b). The most difficult process in terms of time needed to solve the problem and frequency of errors involves regrouping or "trading", as in the problem 46 + 58. As described in the Obstacles to Mastery section in this section, several factors contribute to the difficulty of regrouping.

A similar pattern is found when children are first learning to solve complex subtraction problems; they rely on their knowledge of simple subtraction and addition when using counting or decomposition to solve the problem (Siegler, 1989). As with complex addition, the process of regrouping, as with 33 - 17, is the most common source of difficulty (Fuson & Kwon, 1992b). There are comparatively few cognitive studies of children's learning of the algorithms for solving complex multiplication and division problems, as noted. Studies of adults reveal they solve most complex multiplication problems by using the standard algorithm and partial products (Figure 1), and that the carry operation is the most time-consuming process (Geary, Widaman, & Little, 1986; Tronsky, 2005).

Figure 1: Multiplication Algorithms

Partial Products	Standard Algorithm
32	32
x 53	x 53
6	96
90	1600
100	1696
1500	
1696	

Although they did not provide detailed results on the problem-solving steps children use to solve division problems (e.g., $\frac{345}{23}$), Pratt and Savoy-Levine's (1998) study of contingent tutoring (i.e., providing different levels of support ranging from hints to explicit demonstration) is insightful. In one component of this study, fourth- and fifth-grade children from Canada—recall that the computational skills of U.S. and Canadian children are comparable (Tatsuoka et al., 2004)—solved four division problems and were scored on the accuracy of executing four problem solving steps: estimating the quotient; multiplying the divisor and the quotient; subtracting the product from the dividend; and obtaining the remainder. Before tutoring, these children correctly executed less than four of 32 problem-solving steps across four problems (there were eight steps/problem); one type of tutoring substantially increased accuracy but other types did not. The overall pre-tutoring accuracy rate of 12% is substantially lower than the 25% to 72% correct found for fourth-graders from Japan for problems of similar complexity (Reys et al., 1995).

Declarative information. Adults who are skilled at using arithmetical algorithms can describe the steps they used in the execution of the algorithms (Tronsky, 2005). Mastery of algorithms, however, may involve commitment of the associated steps to procedural memory, rather than to explicit declarative memory. With mastery, it is expected—based on studies of procedural learning in other domains and the studies that have been conducted in arithmetic (Delazer et al., 2003; Pauli et al., 1994; Tronsky)—that the algorithms can be executed automatically and without need for explicit recall and representation of each problem-solving step in working memory.

Obstacles to mastery

As the complexity of the arithmetical problem increases, the number of potential obstacles to mastery increases. The learning of arithmetical algorithms and their fluent execution once learned are influenced by process constraints, conceptual knowledge, errors of induction, and current context. Process constraints include the individuals' working memory capacity (DeStefano & LeFevre, 2004; Hitch, 1978), and the fluency with which component skills embedded within the algorithm can be executed (e.g., ease of retrieving basic facts) (Fuchs et al., 2006; Royer et al., 1999; Starch, 1911). Conceptual knowledge, especially an understanding of the base-10 system and place value, influences how the individual organizes the component processes that compose the algorithm, and facilitates the flexible use of alternative algorithms and the transfer of algorithms to the solving of novel problems (Blöte et al., 2001; Fuson & Kwon, 1992b; Hiebert & Wearne, 1996). During algorithmic learning, children and adults often make errors of induction based on prior learning of related algorithms or related concepts (Ben-Zeev, 1995; VanLehn, 1990). Contextual factors vary from external factors that reduce process limitations (e.g., scratch paper) or that exacerbate (e.g., high-stakes testing) these limitations, as well as factors (e.g., teacher, worked examples) that may help the individual recall relevant concepts (Beilock et al., 2004; Cary & Carlson, 1999).

The solving of complex arithmetic problems, especially during the early phases of learning, requires the retention of intermediate results in working memory while the individual processes the next problem-solving step. These demands require attentional control and working memory resources, and are a potential source of problem solving failure (e.g., Ashcraft & Kirk, 2001; DeStefano & LeFevre, 2004; Hitch, 1978; Logie, Gilhooly, & Wynn, 1994). Experimental manipulations of problem complexity and results from the use of dual-task procedures—asking the individual to engage in an activity that occupies one component of working memory (e.g., repeating nouns) during arithmetical problem solving—suggest the central executive component of working memory is a core source of processing limitations. The phonological loop and visuospatial sketch pad also can pose limitations for some aspects of problem solving (DeStefano & LeFevre); the execution of the carry or borrow procedure is particularly time consuming, and places added demands on the central executive and phonological loop. Working memory resources improve as children mature (Cowan et al., 2002), and can be functionally improved at any age with practice of the algorithm (Beilock et al., 2004; Tronsky, 2005) and with use of external memory aids (e.g., scratch paper) (Cary & Carlson, 1999).

Practice reduces the working memory demands of the problem because it results in the formation of procedural memories, such that the algorithm can be executed without the need to explicitly recreate and represent the sequence of steps in working memory. External aids reduce these demands because intermediate steps can be noted externally (e.g., on scratch paper or with a worked example) rather than in working memory. Practice to the point of automaticity reduces the disruptive effects of anxiety on problem-solving performance. In high-stakes situations, as when performance will be evaluated by others, anxious individuals tend to have thoughts regarding their competency intrude into working memory (Ashcraft & Kirk, 2001; Beilock et al., 2004); these intrusions functionally lower working memory capacity and thus increase the likelihood of committing an error. Beilock et al. experimentally demonstrated that this "choking under pressure" occurs much more often when problem solving requires use of infrequently practiced algorithms; in their studies, errors were rare for frequently practiced algorithms in both low-pressure and high-pressure situations.

Although practicing algorithms has the benefit of eventual automatic execution and reduced working memory demands, practice without conceptual knowledge can result in reduced flexibility in use of alternative algorithms (Blöte et al., 2001; Hiebert & Wearne, 1996). In an experimental study of algorithmic learning in second-graders in the Netherlands, Blöte et al. found that the combination of direct instruction of algorithms in the context of learning associated concepts resulted in a better ability to flexibly use one algorithm or another, depending on the structure of the problem, than did direct instruction of algorithms without a conceptual context. The Task Group discusses the importance of conceptual knowledge in more depth in the next section, Core Arithmetical Concepts; it is noted here that conceptual understanding in one area of arithmetic can sometimes facilitate transfer of algorithms to related problems but at other times can interfere with algorithmic learning (Ben-Zeev, 1995; VanLehn, 1990). For instance, learning the commutative property for addition can lead to the overgeneralization of this property to subtraction; leading students to infer that since 92 – 17 = 75, 17 – 92 = 75.

At other times, errors in executing algorithms are related to a poor understanding of the base-10 system and place value (Fuson & Briars, 1990; Fuson & Kwon, 1992b). Because they do not understand the base-10 concept and place value, many children do not understand that the 1 traded from the units- to the tens-column, for instance when solving 24 + 38, actually represents 10 and not 1; in this case, they write 52 as the answer, instead of 62. Children may not execute the carry procedure at all (leading to answers such as 24 + 38 = 512), or they may ignore place holding 0 values and carry across columns (e.g., 407 + 309 = 806). A similar type of algorithmic error has been found with complex subtraction (VanLehn, 1990; Young & O'Shea, 1981), but much less is known about algorithmic development in complex multiplication and division.

In the earlier mentioned assessment of the speed and accuracy of the arithmetical problem solving of first- and fifth-grade children from the United States and China, Stevenson, Lee, Chen, Lummis, et al. (1990) found that fifth-graders from China solved more than twice as many multidigit (e.g., 34 + 86) addition problems in 1 minute as did their U.S. peers (d = 1.91). A similar pattern was found comparing multidigit subtraction skills of children from China and the United States for 6th- (d = 1.89) and 12th- (d = 1.82) graders that matched or equated on general cognitive ability (Geary et al., 1997). The latter study found a smaller gap for multidigit addition than that found by Stevenson et al., but the differences were still substantial in both 6th- (d = 1.22) and 12th- (d = 1.30) grades. The same pattern was found for multidigit multiplication problems (e.g., 23 x 6), whereby fifth-graders from China solved more than twice as many problems in 1 minute as did their U.S. peers (d = 1.57; Stevenson, Lee, Chen, Lummis, et al.). The source of these fluency differences is not entirely understood but is related at least in part to a better understanding of the base-10 system and place value in East Asian than in U.S. students. It also is likely related to differences in the grade placement, the quantity and quality of algorithmic practice, and the extent to which this practice is integrated with concept learning (Fuson & Kwon, 1992b; Fuson et al., 1988).

c. Core Arithmetical Concepts

The core arithmetical concepts that children should come to understand and apply during problem solving are the associative and commutative properties of addition and multiplication (described below), the distributive property of multiplication [e.g., $a \times (b + c) = (a \times b) + (a \times c)$], identity elements for addition ($a + 0 = a$) and multiplication ($b \times 1 = b$), and the inverse relation between addition and subtraction, and between multiplication and division. The availability of research on children's understanding and skill at using these concepts is quite variable across these topics. There is, for instance, considerable work on children's understanding of commutativity as related to addition, but comparatively little work on children's understanding of identity elements and the inverse relation between multiplication and division.

Associativity and commutativity

The commutativity property concerns the addition or multiplication of two numbers, and states that the order in which the numbers are added or multiplied does not affect the sum or product ($a + b = b + a$; $a \times b = b \times a$). The associativity property concerns the addition or multiplication of three numbers, and again states that the order in which the numbers are added or multiplied does not affect the sum or product [$(a + b) + c = a + (b + c)$; $(a \times b) \times c = a \times (b \times c)$]. Empirical research on children's understanding of these concepts has focused on the commutative property of addition (Baroody, Ginsburg, & Waxman, 1983; for review see Resnick, 1992), although some research has been conducted on associativity (Canobi et al., 1998, 2002). Different approaches have been used in this research:

A. An informal understanding is sometimes inferred when preschool children's physical manipulation of sets of objects or responses to such manipulations is consistent with these concepts (for review, see Resnick, 1992). A child might watch as sets of different objects (e.g., red candy, blue candy) are given to different dolls in different orders. Implicit knowledge of commutativity is inferred if the child indicates the dolls received the same amount.

B. A formal understanding is inferred when the child can explicitly state that answers to problems are equal (e.g., 14 + 78 = 78 + 14) and can justify his or her answer using the appropriate concept, that is, that number order does not affect the answer.

C. An intermediate level of knowledge is inferred when a child's solving of formal problems is consistent with an implicit understanding of the concept or the child provides a partial explicit justification (Baroody et al., 1983). If the problems 3 + 14 and 14 + 3 are presented one after the other, and the child counts to solve the first problem (e.g., 14, 15, 16, 17) and quickly states the same answer without counting to solve the second problem, an implicit understanding of commutativity is inferred. A partial justification might involve the child stating that the problems are the same, but does not include statements regarding number order.

Paths of acquisition

Concepts. By 4 to 5 years of age, many children understand that sets of physical objects can be decomposed and recombined into smaller and larger sets, and that the order of these manipulations is not important; that is, they implicitly understand commutativity in this context (Klein & Bisanz, 2000; Sophian, Harley, & Martin, 1995; Sophian & McCorgray, 1994; Canobi et al., 2002). This implicit knowledge is limited to two sets of objects, indicating that most children of this age do not implicitly understand associativity. Moreover, many of these children may not link commutativity, as expressed in manipulation of physical sets, to addition of specific quantities. About half of kindergartners implicitly recognize commutative relations in simple addition problems (e.g., 3 + 2 = 3 + 2), as do the majority of first-graders (Baroody et al., 1983; Baroody & Gannon, 1984). Many second- and third-graders will begin to provide partial explicit explanations of commutative relations, but it is not well understood when and under what instructional conditions children come to explicitly understand commutativity as a formal arithmetical principle. Some kindergarten children recognize associative relations when presented with sets of physical objects, and many first- and second-graders implicitly understand associative relations when they are presented as addition problems (Canobi et al., 1998, 2002). These studies have also demonstrated that an implicit understanding of associativity does not emerge until after children implicitly understand commutativity.

In comparison to addition, much less is known about children's implicit and explicit knowledge of commutativity and associativity as related to multiplication. In a study with third-graders who were just being introduced to multiplication, Baroody (1999) found that practice at solving multiplication problems (e.g., 3×4) made the solving of unfamiliar commuted problems (e.g., 4×3) faster and less error prone than other unfamiliar problems. This type of finding is consistent with the adult studies on retrieval of multiplication facts but is not sufficient to demonstrate an explicit conceptual understanding of the commutative property as related to multiplication.

Declarative information. The core concept of commutativity and associativity is that the order in which two (commutativity) or three (associativity) numbers are added or multiplied does not affect the result. Although elementary school children's justifications for a problem-solving approach often reflect a partial understanding of this equivalence (Baroody et al., 1983), many children do not explicitly state this core concept as a justification. It is not known when and under what instructional conditions children can express these concepts algebraically (e.g., $a \times b = b \times a$). It is also important for children to come to understand that commutativity and associativity do not apply to subtraction and division; children's problem-solving errors in subtraction suggest they often draw the incorrect inference that the principles apply to these operations as well (VanLehn, 1990).

Obstacles to mastery

The relation between children's implicit and explicit knowledge of commutativity and associativity is not fully understood. Resnick (1992) proposed that children's implicit understanding of commutativity and associativity provides the foundation for their explicit understanding of these concepts, but evidence for such a relation is mixed (Baroody et al., 1983). Many children implicitly or explicitly infer that commutativity applies to subtraction and thus often make errors; since $7 - 3 = 4$, it is inferred that $3 - 7 = 4$ (Young & O'Shea, 1981); the use of these "buggy rules" (i.e., use of a procedure that is correct for one type of problem to solve another type of problem for which the procedure is not appropriate) varies, however, as children may use them to solve one problem and then use a correct procedure to solve another (Hatano, Amaiwa, & Inagaki, 1996). These forms of error have not been as extensively studied for associativity with subtraction or to the misapplication of these principles to division, but similar confusions are likely.

Distributive property, identity properties, and inversion

There is not a sufficient amount of research on children's understanding of the distributive property of multiplication to draw conclusions at this time; e.g., $a(b + c) = ab + ac$. In one of the few studies of children's understanding of the distributive property (conducted in the United Kingdom), Squire, Davies, and Bryant (2004) found that less than 5% of fifth-graders could solve various forms of distributive property problems at above chance levels, as compared to 44% to 52% solving similar forms of commutative problems at above chance levels. The majority (> 74%) of sixth-graders correctly solved various forms of commutative principle problems, but only 22% to 44% of these children correctly solved various forms of distributive property problems.

Error patterns were systematic and suggested that the children confused addition and multiplication when solving distributive problems. Some of the distributive items were presented as word problems with an underlying form of if $a \text{ x } b = c$, then $(a + 1) \text{ x } b = c + b$. The first statement in a corresponding word problem item might be presented as 67 candies in each of 25 bags = 1675 candies. The next statement might then be presented as 68 candies in each of 25 bags = *M* candies. Children were provided with six potential answers and were given a limited amount of time to choose one of these. If the children understood the distributive property then they would choose the answer that is 1675 + 25, or 1700. If they approached the problem as an addition of 1 to both sides of the equation, then they would choose 1675 + 1 or 1676. Nearly all of the fifth-graders and the majority of sixth-graders committed this type of error.

As with the distributive property, there is not enough research to draw firm conclusions about children's understanding of identity elements. Studies of adults' mental arithmetic indicate that identity problems in addition ($a + 0 = a$) and multiplication ($b \times 1 = b$) are solved more quickly and accurately than are other problems, suggesting these may be solved by means of a rule (Miller, Perlmutter, & Keating, 1984). These findings, however, do not address the issue of whether these adults explicitly understand the mathematical concept of an identity element nor do they address the more focal issue of how children come to understand this concept. Studies of children's conceptual understanding of and ability to apply the distributive property and identity elements are clearly a priority for future research.

Inverse relations are an integral part of many aspects of mathematics. Children's first encounter with such a relation is with addition and subtraction; e.g., $a + b = c$, $c - b = a$. Studies of knowledge of the inverse relation between addition and subtraction have revealed an implicit understanding for many children by the time they enter kindergarten (Baroody, & Lai, 2007; Klein, & Bisanz, 2000; Vilette, 2002), and a growing implicit use of this relation with schooling, as reflected in problem-solving performance (Bryant, Christie, & Rendu, 1999; Gilmore & Bryant, 2006; Siegler & Stern, 1998). Developmental and experimental studies indicate that the majority of children implicitly use addition and subtraction inversion in their problem solving (as measured by a shorter time needed to solve inversion problems as compared to similar problems that cannot be solved with inversion) before they can explicitly state this relation. This pattern is common in many areas (Siegler, & Araya, 2005). An ability to explicitly state some aspect of this relation is found in many children by the end of the elementary school years (Robinson, Ninowski, & Gray, 2006).

However, many weaknesses in children's and even adults' (Robinson, & Ninowski, 2003) understanding of inverse relations are evident; many adults and most children do not have a firm grasp of the inverse relation between multiplication and division, nor do they appear to understand the concept of inversion at an abstract mathematical level. For instance, Robinson, Ninowski et al. (2006) found that knowledge of the inverse relation between addition and subtraction in sixth- and eighth-graders did not transfer to multiplication and division; they seemed to understand these relations separately for addition and subtraction and multiplication and division but did not link them together through the more general concept of inverse relations in mathematics.

d. Conclusions and Recommendations

American students do not meet the goal of fast and efficient solving of basic arithmetic combinations or execution of standard algorithms, and their competence in these areas is well below that of students in many other countries. American students have a poor grasp of most core arithmetical concepts; most American students do not understand the distributive property of multiplication, and they do not know identity elements or the inverse relation between division and multiplication, among other deficits. Mastery of these core concepts is a necessary component of learning arithmetic and is needed to understand novel problems and to use previously learned procedures to solve novel problems. Debates regarding the relative importance of conceptual knowledge, procedural skills, and the commitment of arithmetical facts to long-term memory are misguided. The development of conceptual knowledge and procedural skills is intertwined, each supporting the other. Fast access to number combinations, prime numbers, and so forth supports problem solving because it frees working memory resources that can then be focused on other aspects of problem solving.

Classroom

The development of measures that support the teacher's ability to make formative assessments of children's procedural and conceptual competencies in all key areas of whole number arithmetic should be a research priority.

Training

Teachers. For teachers to take full advantage of the above noted types of formative assessments, they must have a better understanding of children's learning and the sources of children's conceptual and procedural errors in the content areas they are teaching. As an example, many errors on conceptual tasks are systematic and can provide information on how students are misunderstanding the concept. These errors can be used in formative assessments and to focus instruction. However, as noted, for teachers to make full use of these common errors in children's arithmetic learning, they must understand how children learn arithmetic and how children conceptualize and misconceptualize core concepts.

The development of courses in mathematical cognition for inclusion in teacher training programs will be necessary to address this goal.

Researchers. Programs that support cross-disciplinary pre-doctoral and postdoctoral training in cognition, education, and mathematics are needed to ensure a sufficient number of researchers that study children's mathematical learning, and have the background needed to bridge the gap between laboratory studies and classroom practice.

Curricula

The fast and efficient solving of arithmetic combinations and execution of procedures requires considerable practice that is distributed over time. The consistent failure of American children to achieve mastery of these topics is a strong indication that most current curricula in the United States do not provide these experiences. Although definitive conclusions cannot be drawn at this time due to lack of relevant, large-scale experimental studies, the research that has been conducted suggests that effective practice should include a conceptually rich and varied mix of problems, with several features:

1) Present more difficult problems (e.g., 9 + 7) more frequently than less difficult problems (e.g., 3 + 1); this is because long-term retention of difficult problems requires more practice.
2) Highlight the relations among problems.

   For example, the inverse relation between addition and subtraction:

   4 + 7 =

   11 - 4 =
3) Order practice problems in ways that reinforce core concepts.

   For example, identity elements:

   3 x 0 =

   0 x 8 =

   6 x 0 =
4) Include key problems that support formative assessments.

   Such problems can reveal students' misconceptions and problem-solving errors:

   7 - 4 =

   4 - 7 =

   Errors on the second problem (i.e., 4 - 7 = 3) are common because children infer that the commutative relation they learned for addition also applies to subtraction.

   Errors on these types of problems may be diagnostic of this incorrect inference, which can then be addressed as part of classroom instruction.

U.S. students' poor knowledge of core arithmetical concepts—the distributive property, identity elements, the inverse relation between division and multiplication, among others—is unacceptable and indicates a substantive gap in the mathematics curricula that must be addressed.

Research

Although much is known about some areas of children's arithmetical cognition and learning, further research is needed in the areas of children's learning of complex algorithms (e.g., division algorithm); the relation between conceptual knowledge and procedural learning; and on the learning of core concepts, including the base-10 number system, the distributive property of multiplication, and identity elements, among others.

Studies are needed that focus on the translation of cognitive measures of children's learning into formative assessments that are easily understood by teachers and used in the classroom.

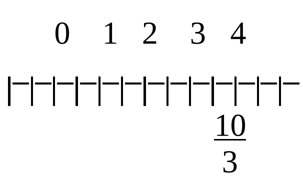
Funding priorities that target areas of deficit in children's arithmetical cognition and learning are recommended, along with priorities that encourage projects that bridge the gap between basic research and classroom practice.

2. Fractions, Decimals, and Proportions

Fractions, decimals, and proportions are introduced into the mathematics curriculum as early as elementary school or its equivalent in different countries, and yet solving problems with these quantities remains difficult for many adults. Nevertheless, understanding and manipulating fractions is crucial for further progress in mathematics and for tasks of everyday life, such as computing interest on a loan or deciding among risky medical treatments (e.g., Kutner, Greenburg, & Baer, 2006; Reyna & Brainerd, 2007; Wu, 2006). Central to the charge of this Panel, knowledge of fractions and related concepts has been described as "fundamental to the learning of algebra" (p. 1; Wu, 2007). In a nationally representative sample, teachers of Algebra I rated students as having the poorest preparation in "rational numbers and operations involving fractions and decimals" among 15 areas of mathematics, surpassed only by "solving word problems" (Hoffer, Venkataraman, Hedberg, & Shagle, 2007, Table 3).

a. Definitions and Interpretations

Mathematically, the definition of a fraction begins with the concept of a number line (Wu, 2007). A fraction is defined as a point on the number line, based on the concept of a part-whole relation, with the unit segment [0,1] (the segment from 0 to 1) serving as the whole. The fraction $\frac{1}{3}$, for example, is obtained by dividing [0,1] into three equal parts. Every segment on the number line, not just the unit segment, can be similarly divided into three equal parts. More generally, m/n consists of *m* adjoining short segments of $\frac{1}{n}$ each (e.g., thirds). The example of $m = 10$ and $\frac{1}{n} = \frac{1}{3}$ is shown below:



From this mathematical definition of a fraction, other definitions can be derived, such as the division interpretation (i.e., $\frac{m}{n} = m \div n$).

Psychologically, however, there are at least five interpretations or "subconstructs" of fractions (Kieren, 1988), defined as follows by Sophian (2007, pp. 114–115):

- measure, as representing magnitudes that can be intermediate between whole numbers of units (e.g., magnitudes between 0 and 1);
- quotient, as numerical values obtained by dividing one whole number by another;
- ratio number, as representing the relative magnitude of two non-overlapping quantities (as in a recipe that calls for 3 eggs for every 2 cups of flour);
- multiplicative operator, as representing an extending/contracting or stretching/shrinking function applied to some object, set, or number (so that, e.g., taking $\frac{2}{3}$ of a quantity stretches that quantity by a factor of 2 but then shrinks the "stretched" quantity by a factor of 3 so that the end result is smaller than the original quantity); and
- part-whole, as representing one or more parts of a whole, where the parts are formed by partitioning the whole into a number of equal units (see Behr, Harel, Post, & Lesh, 1992).

There are a number of properties of fractions that are related to one or more of these interpretations, such as inversion (that fractions become smaller as denominators become larger, assuming that the numerator is held constant; (Sophian, Garyantes, & Chang, 1997), that the effect of denominator magnitude is multiplicative (e.g., Thompson & Saldanha, 2003), that segments are infinitely divisible or dense (that there are infinite fractions between two endpoints on a number line; e.g., Smith, Solomon, & Carey, 2005), and others. Unlike mathematical definitions, which can be explained, derived, or, with the help of theorems, proved to be related to one another in precise ways, the relations among different psychological interpretations or properties are unclear. Mathematically, although a precise definition of fractions using the number line makes it possible to derive other properties of fractions, empirically, a student might successfully perform tasks that fit one psychological interpretation of fractions but fail others that are mathematically equivalent (or derivable). How interpretations relate is a question that can be answered empirically; a taxonomy of interpretations based on a process model of underlying causal mechanisms could be produced through hypothesis-driven experimentation (see Platt, 1964). However, current scientific theory is not sufficiently developed to fully answer this important question of how the understanding of different properties and interpretations of fractions are related to one another.

Furthermore, because of the lack of clarity concerning how psychological interpretations of fractions relate to one another, scholars frequently differ in the meanings they attach to such terms as *conceptual knowledge* of fractions, emphasizing varied interpretations and properties. Fortunately, researchers generally provide operational definitions of conceptual knowledge (as well as of computational facility) by precisely specifying the tasks that subjects are asked to perform. For example, subjects might be asked to judge the relative magnitude of two fractions with identical denominators but different numerators, or vice versa. Other tasks include judging equivalence of fractions, translation of pictures (e.g., of pizzas with different portions shaded) into numerically expressed fractions, ordering fractions according to magnitude, judging which of two pairs of fractions are closer in magnitude, and computation (e.g., adding, subtracting, multiplying or dividing fractions).

Certain tasks are more diagnostic than others with respect to assessing specific aspects of conceptual knowledge of fractions. For example, if subjects judge $\frac{2}{3}$ and $\frac{4}{5}$ to be equivalent they likely do not understand that the relation between numerator and denominator is multiplicative rather than additive (Sophian, 2007). Similarly, if subjects select a container with 3 winning chips out of 7 chips rather than a container with 1 winning chip out of 2 chips (the so-called numerosity effect or ratio bias), they are failing to take the magnitude of the denominator into full account (e.g., Acredolo, O'Connor, Banks, & Horobin, 1989; Hoemann & Ross, 1982; Reyna & Brainerd, 1994, 2008). Although no task is process pure in the sense that it cleanly measures one and only one psychological process, specific empirical tests have been devised to identify processes that underlie judgments involving fractions, decimals, and proportions (see Kerkman & Wright, 1988; Siegler, 1981, 1991; Surber & Haines, 1987). Therefore, in the remainder of this review, conceptual knowledge is identified with respect to performance on specific tasks that are designed to diagnose comprehension of aspects of knowledge about fractions, decimals, or proportions. Psychometric studies have distinguished computational ability from conceptual knowledge and, thus, research concerning the former is also reviewed by the Task Group.

b. Extent of the Problem

Computations involving fractions and decimals have proved challenging for every group that has been tested in the U.S. Difficulties emerge when such concepts are introduced in elementary school, and they persist through middle school, high school, and into adulthood, extending beyond those with learning disabilities in mathematics (e.g., Hecht et al., 2007; Mazzocco & Devlin, in press; U.S. Department of Education, NCES, 2003; Sophian, 2007; Stafylidou & Vosniadou, 2004). The percentage of middle school students who have difficulties with fractions and decimals, which has been estimated at 40%, far exceeds the cumulative incidence of MLD, as the Task Group reviews in the section on Learning Disabilities (Barbaresi et al., 2005; Hope & Owens, 1987; U.S. Department of Education, NCES, 1990; Smith, 1995). To illustrate, the 1990 National Assessment of Educational Progress documented that only 53% of 7th-graders and only 71% of 11th-graders could correctly subtract two mixed fractions with unlike denominators, despite the fact that such content is typically taught in elementary school (U.S. Department of Education, NCES, 1990). Recent assessments paint a similar picture. On the 1996 and 2005 NAEP tests, only 65% and 73% of eighth-graders, respectively, were able to correctly shade $\frac{1}{3}$ of a rectangle; on the 2004 NAEP test, only 55% of eighth-graders could correctly solve a word problem involving dividing one fraction by another.

Adults also perform poorly on problems involving fractions, decimals, and proportions. The most recent report of the National Assessment of Adult Literacy (NAAL) assessed literacy and numeracy in 2003 for 19,000 U.S. adults, who completed realistic tasks (Kutner et al., 2006). More adults scored in the Below Basic level on the quantitative scale (22%) of the NAAL than on any other scale, such as those measuring document or prose literacy (Kutner et al.). Most studies of adult "numeracy" assess the ability to perform simple computations or quantitative judgments concerning decimals, probabilities, percentages, and frequencies (e.g., Fagerlin, Zikmund-Fisher, & Ubel, 2005; Lipkus, Samsa, & Rimer, 2001; Schwartz, Woloshin, Black, & Welch, 1997; Woloshin, Schwartz, Byram, Fischhoff, & Welch, 2000; for a review, see Reyna & Brainerd, 2007). For example, one question from the Newest Vital Sign (NVS) test assesses the ability to calculate percentages of ingredients based on information from a nutrition label for ice cream (Weiss et al., 2005). Another well-known numeracy scale includes 11 questions, all of which pertain to fractions, decimals, and percentages (Lipkus et al.). Only 46% of adults in one sample and 24% in another were able to answer a question from this scale correctly that involved converting frequencies to percentages (i.e., In the Acme Publishing Sweepstakes, the chance of winning a car is 1 in 1,000. What percent of tickets win a car?). The percentage of adults who answered three such questions correctly ranged from 15% or 16% (Lipkus et al.; Schwartz et al.) to 38% (Black, Nease, & Tosteson, 1995; Woloshin et al.), including samples that were mostly college-educated. Scores on these tests have been found to relate to important real-world outcomes, such as patients' knowledge, health behaviors, health outcomes, and medical costs (American Medical Association Ad Hoc Committee on Health Literacy, 1999; Baker, 2006; Berkman et al., 2004; Estrada, Martin-Hryniewicz, Peek, Collins, & Byrd, 2004; Institute of Medicine, 2004).

In sum, it is clear that a broad range of students have difficulties with fractions, and these problems continue after graduation for many adults (e.g., Hecht et al., 2007; Mazzocco & Devlin, in press). The failure to attain basic facility with fractions constitutes an obstacle to progress to more advanced topics in mathematics, including algebra (although direct evidence for this link is lacking, but see e.g., Hecht, 1998; Hecht et al., 2003; Heller, Post, Behr, & Lesh, 1990; Loveless, 2003) and, presumably, to career paths that require mathematical proficiency (e.g., National Science Board Commission on 21st Century Education in Science, Technology, Engineering, and Mathematics (STEM, 2006)), as well as potentially interfering with life-and-death aspects of daily functioning, such as compliance with medication.

c. Paths of Acquisition

Informal, implicit knowledge

In order to assess competence accurately, it is important to separate children's understanding of formal fractional notation (i.e., what the line between two numbers in a fraction such as $\frac{1}{3}$ means) from their intuitive ability to understand fractional relations and perform calculations using fractional quantities (e.g., Mix et al., 1999). Illustrating the difficulty in understanding notation, children frequently add numerators and denominators together without regard for the notational convention that each numerator-denominator combination refers to a single quantity (e.g., $\frac{3}{4} + \frac{1}{2} = \frac{4}{6}$) (Carpenter et al., 1978; see also Silver, 1986; Resnick & Ford, 1981). When such notational constraints are removed, young children reveal a nascent ability to understand ratios (Geary, 2006; Mix et al.; Sophian, 2000). Preschool children's experiences with and understanding of part-whole relations among sets of physical objects, such as receiving $\frac{1}{2}$ of a cookie or having to share 1 of their 2 toys, may contribute to an early understanding of simple ratios (Correa, Nunes, & Bryant, 1998; Geary; Mix et al.).

For example, avoiding the use of conventional notation, Goswami (1989) gave 4-, 6-, and 7-year-olds a series of analogy problems using shaded portions of geometric shapes such as $\frac{1}{2}$ of a circle: $\frac{1}{2}$ of a rectangle: $\frac{1}{4}$ of a circle: ?, and the children selected an answer from among five alternatives. A simpler version of the task was also presented in which proportions did not change across shapes (e.g., $\frac{1}{2}$ of a diamond: $\frac{1}{2}$ of a circle: $\frac{1}{2}$ of a square: ?) and children selected from among four alternatives. Four-year-olds performed significantly above the chance level of 25% correct in the simpler task (56% correct), and 6- and 7-year-olds performed nearly perfectly (86% and 91% correct, respectively). However, performance for 4-year-olds was only 31% correct in the harder version of the task, though significantly above chance-level performance, and 6-year-olds remained far from perfect at 74% correct. Thus, the ability to recognize equivalent fractions undergoes significant development in early childhood, but basic competence emerges before children enter formal schooling.

Similarly, the ability to manipulate fractions—to engage in a kind of informal computation with fractions that does not involve conventional notation—is also present early. In a study of simple part-whole relations, Mix et al. (1999) administered a nonverbal task that assessed children's ability to mentally represent and manipulate $\frac{1}{4}$ segments of a whole circle. The results indicated that children as young as 4-years-old could calculate with fractional amounts of less than or equal to one, as shown by their ability to recognize fractional manipulations. For instance, if $\frac{3}{4}$ of a circle was placed under a mat and $\frac{1}{4}$ of the circle was removed, the children recognized that $\frac{1}{2}$ of a circle remained under the mat. However, it was not until 6 years of age that children began to understand manipulations that were analogous to mixed numbers; for instance, placing $1\frac{3}{4}$ circles under the mat and removing $\frac{1}{2}$ a circle. These results suggest that about the time children begin to show an understanding of part-whole relations in other contexts (Sophian et al., 1995; Sophian & McCorgray, 1994; Resnick, 1992), they demonstrate a rudimentary understanding of fractional relations. Although it is possible that children in the Mix et al. study represented the $\frac{1}{4}$ sections of the circles as single units and not as parts of a whole, this seems unlikely because solution of whole number and fraction problems differs in important ways.

Correa et al. (1998) argue that sharing forms the basis for preschool age children's ability to partition a quantity into roughly equal parts through a process of distributive counting (see also Hunting & Davis, 1991; Miller, 1984). Preschoolers also know that the term *half* refers to one of two parts (Hunting & Davis) and can use the notions of greater than half versus less than half to recognize which of two proportions are closer to a target proportion (Spinillo & Bryant, 1991; see also Singer-Freeman & Goswami, 2001; Spinillo & Bryant, 1999). However, knowing how to share equal amounts or that some fractions are greater or lesser than half is not the same as understanding the inverse relation among quantities (e.g., between numerator and denominator in a fraction when the value of the fraction is held constant or between terms in division).

For example, Sophian et al. (1997) asked 5- and 7-year-old children to determine which of two sharing scenarios would yield a larger portion for a recipient. Consistent with the idea that sharing forms an early basis for understanding fractions, children understood that if different total quantities were shared among the same number of recipients, the recipients who shared the larger total quantity would get larger portions than those who shared the smaller total quantity. However, the children also expected that sharing *equal* total quantities among different numbers of recipients would result in larger portions when shared with a larger number of recipients, compared to fewer recipients. The latter expectation violates the inverse rule that the larger the number of shares, the smaller the size of each share, or vice versa, when total amounts are held constant. About half of 6-year-olds were able to understand the inverse relation between divisor and quotient in Correa et al.'s (1998) sharing task, which is well below the age at which division is formally taught in schools.

Acredolo et al. (1989), Schlottman (2001), and others using sensitive techniques that do not require explicit numerical computation have also shown an early appreciation for the inverse relation between numerators and denominators in probability and other ratio concepts (e.g., Hoemann & Ross, 1982; Reyna & Brainerd, 1994; Reyna & Ellis, 1994). These techniques, such as functional measurement, make it possible to discern whether the perceived relation between numerator and denominator is multiplicative rather than additive; by first grade, most students correctly perceived the relation to be inverse and multiplicative (see also Jacobs & Potenza, 1991). Notably, Sophian et al. (1997) found that in a study subsequent to the one reported above, children were able to appreciate and generalize the inverse relation after just a few trials demonstrating how changes in the denominator affected the size of each share, suggesting that some level of competence was already present to build on.

In sum, studies of nonverbal or implicit knowledge of fractions show an intuitive awareness of fractions based on part-whole relations, notions of sharing, and a limited conception of inverse, multiplicative relations between numerators and denominators (or divisors and quotients) in the preschool years. Like place value in decimals (e.g., .1 vs. .0001), the symbolic notation for fractions is not yet correctly interpreted and must be explicitly taught. Despite evidence of early basic competence, these studies show considerable change in performance between ages 4 and 7 (and beyond, in some studies) and significant differences between performance with whole numbers and fractions, with competence with fractions lagging substantially behind competence with whole numbers even on relatively simple tasks (e.g., Mix et al., 1999).

Formal, mathematical knowledge

Studies of elementary and middle school-aged children have focused on the acquisition of conceptual knowledge, computational skills (e.g., multiplying fractions), and the ability to use both of these abilities in conjunction with reading comprehension to solve word problems involving fractional quantities (Byrnes & Wasik, 1991; Hecht et al., 2007; Rittle-Johnson et al., 2001). Conceptual knowledge tasks have included identifying which of several fractions is largest, judging relative magnitude (e.g., 21/18 > 1), translating pictorial representations into equivalent formal fractional representations, and vice versa. Computational tasks have involved adding, subtracting, multiplying and (rarely) dividing fractions using pictures (e.g., providing pictorial representations of answers to pictorial problems), numbers, and verbal descriptions, as in word problems (e.g., Hecht et al.). Scores on items assessing conceptual knowledge have consistently been shown to explain unique variance (beyond general intellectual and reading abilities) in performance on computational fraction problems, word problems that include fractions, and estimation tasks with fractional quantities (e.g., Byrnes & Wasik; Hecht, 1998; Hecht et al., 2003; Sophian, 2007; Hecht et al., 2007).

Consistent with these findings and illustrating the close connection between conceptual and procedural (computational) abilities, Hecht (1998) reported that fully 82% of 1,474 errors on fraction computation problems could be classified as involving a faulty procedure, as opposed to wild guesses, no attempt, or calculation errors. Children's accuracy at recognizing formal procedural rules for fractions (e.g., when multiplying, that both numerators and denominators are multiplied) and automatic retrieval of basic arithmetic facts also predicted computational skills (i.e., accuracy in adding, multiplying, and dividing proper and mixed fractions), above and beyond the influence of intelligence, reading skills, and conceptual knowledge (see Hecht et al., 2007 for a review).

In a follow up study, Hecht et al. (2003) investigated effects of conceptual knowledge of fractions, basic arithmetic skills, working memory capacity, and on-task time in mathematics class. Outcome measures included the computation of fraction sums and products; the estimation of fraction sums; and the solution of one-step word problems involving fraction addition, multiplication, or division. On-task time referred to paying attention to instruction or engaging in other forms of on-task behaviors in the mathematics classroom, which other studies have shown correlates with the acquisition of academic skills (Bennett, Gottesman, Rock, & Cerullo, 1993 in two of six samples; McKinney & Speece, 1986; Wentzel, 1991); for example, Chen, Rubin, and Li (1997) reported a correlation of .52 for sixth- and eighth-graders between on-task time and academic achievement (.52 is the concurrent simple correlation at Time 2; a cross-lagged correlation of .47 was also reported). For fraction problems, Hecht et al. found that on-task time influenced performance through its effect on conceptual knowledge. Presumably, children who engaged in more on-task behavior in class were better able to acquire and practice conceptual understanding of fractions that then contributed to their ability to solve fraction computation, estimation, and word problems. For fraction computation, on-task time influenced performance through its effect on simple arithmetic knowledge as well. That is, on-task time was associated with better knowledge of simple arithmetic, and this arithmetic knowledge contributed to better performance on fraction computation problems.

Note that on-task time refers to focused attention and practice, rather than motivation. Motivation also has positive effects on fraction learning. Schunk (1996) showed that fourth-graders who had a learning goal (trying to learn how to solve fraction problems) rather than a performance goal (trying to solve fraction problems) had higher self-efficacy, skill, and other achievement outcomes, such as number of fraction problems solved. Children in the first of Schunk's experiments were also assigned either to a self-evaluation condition (they judged their fraction capabilities at the end of each of six learning sessions) or they did not engage in self-evaluation (instead answering an attitudes question at the end of the six sessions). In the second experiment, all children engaged in self-evaluation. In both experiments, the learning goal with or without self-evaluation led to higher motivation and achievement outcomes than the performance goal. Performance goals with self-evaluation were more effective than without self-evaluation.

Taken together, these studies suggest that motivation and on-task time contribute to superior conceptual knowledge of fractions, which broadly benefits computation, estimation, and skill at solving word problems (see Hecht et al. 2003 for detailed models). Basic skills (i.e., arithmetic knowledge, reflected in rapid retrieval of basic arithmetic facts) were also important, but they more narrowly benefited the solution of fraction computation problems. Basic skills were related to fraction computation even when other factors were controlled for, such as conceptual knowledge, reading ability, on-task time, and working memory. Therefore, early levels of basic arithmetic skills may predict those children who will later have difficulty with fractions, and building such skills (e.g., Goldman, Mertz, & Pellegrino, 1989; Jordan, Hanich, & Kaplan, 2003) may enhance performance on fraction computation problems.

Several studies have examined the relation between conceptual and procedural knowledge (computational ability), and their results echo findings in other domains of mathematics learning (e.g., Hecht et al., 2003; Rittle-Johnson & Alibali, 1999; Rittle-Johnson & Siegler, 1998). Rittle-Johnson et al. (2001) demonstrated that children's skill at solving decimal fractions was related to both their conceptual and procedural knowledge of fractions and that learning conceptual and procedural knowledge occurred iteratively. That is, conceptual knowledge predicted gains in procedural skills, and vice versa (Rittle-Johnson et al.; Sophian, 1997); Byrnes and Wasik (1991), in contrast, did not find that procedural knowledge affected conceptual knowledge, but failure to detect an effect is not evidence against it. Specifically, in two experiments with fifth- and sixth-graders, Rittle-Johnson et al. found that conceptual knowledge of decimal fractions at pretest (with initial procedural knowledge controlled for) predicted changes–as a result of instruction–in procedural competence from pretest to posttest. These changes in procedural competence (again controlling for initial scores on the procedural knowledge pretest) in turn predicted changes in conceptual knowledge from pretest to posttest. The iterative model of gradual, bidirectional influence of conceptual and procedural knowledge on development has been supported in multiple domains of learning. This model explains why children might be able to pass one test of conceptual knowledge and yet fail another test; because children have intuitions about part-whole relations, for example, does not mean that they fully understand conventional fractions.

The mechanism linking conceptual and procedural knowledge appears to be children's ability to represent the decimal fraction on a mental number line, supported by correlational evidence from Experiment 1 and causal evidence from Experiment 2 of Rittle-Johnson et al.'s research (2001). This linear representation undergoes development from childhood to adulthood and has been linked to both whole number magnitude estimation (Siegler & Opfer, 2003) and fractional magnitude estimation (e.g., Opfer, Thompson, & DeVries, 2007). Adults have been shown to successfully use a mental number line to represent relative magnitude and to solve inference problems, and recent neuropsychological evidence points to an internalized representation that preserves spatial features of a physical line, such as left-to-right orientation (e.g., Bouwmeester, Vermunt, & Sijtsma, 2007; Trabasso, Riley, & Wilson, 1975; Zorzi, Priftis, & Umiltá, 2002). Prompting children to think of decimal fractions as composite representations (e.g., a certain number of tenths, a certain number of hundredths and so on) rather than common unit representations (e.g., .45 as 45 hundredths), and then mapping those representations spatially onto the number line, led to large gains in procedural knowledge (Rittle-Johnson et al., 2001). In addition, children who began with low conceptual knowledge benefited more from representational supports than children who began with higher conceptual knowledge.

Some scholars have argued that frequencies are "privileged" mental representations from an evolutionary perspective and have used this concept to explain common errors in fraction and decimal use (e.g., Brase, 2002). The claim is not dissimilar from Gelman's (1991) ideas about negative transfer from counting whole numbers, that "a frequentist representation that tends to parse the world into discrete, countable units" (p. 406, Brase) explains difficulties in dealing with part-whole relations as opposed to part-part relations (e.g., Sophian & Wood, 1997; Sophian & Kailihiwa, 1998). However, recent research disentangling effects of frequentistic representations from clarification of class-inclusion (or part-whole) relations has shown that using frequencies per se does not reduce errors (for reviews, see Barbey & Sloman, in press; Reyna & Brainerd, 2008). Making part-whole relations transparent (e.g., by using Venn diagrams or distinctively labeling classes that are nested or overlapping), however, has been found to reduce errors for children and for adults in problems involving fractions, decimals, percentages, and frequencies (e.g., Girotto & Gonzalez, 2007; Reyna & Brainerd, 1994; Reyna & Mills, in press). For more advanced reasoners who have acquired conceptual knowledge, representations that make part-whole relations salient or transparent, in contrast to making part-part relations salient, virtually eliminate errors for simple magnitude judgments (Brainerd & Reyna, 1990, 1995; Lloyd & Reyna, 2001).

Proportional reasoning involves many of the same elements as fractions, decimals, and other ratio concepts but it requires, in addition, the coordination of two ratio quantities. By this definition, judging the equivalence or relative magnitude of two fractions with unequal numerators and denominators is an example of proportional reasoning. Thus, many of the studies reviewed thus far concern *proportional reasoning* although they are not labeled as such. Early, informal competence can be detected if children are able to use perceptual cues, in particular surface area, to judge relative numerosity (Rousselle, Palmers, & Noel, 2004). Using carefully controlled stimuli, Rousselle et al. showed that 3-year-olds responded above chance, even for large numerosities, by using an analog mechanism that codes continuous perceptual dimensions. In another study involving visual displays rather than numbers or notation, 3- to 4-year-olds were able to match proportions of pizzas (divided into

8 slices) and boxes of chocolates (consisting of 4 pieces), even when the numbers did not match (e.g., matching $\frac{4}{8}$ to $\frac{2}{4}$) (Singer-Freeman & Goswami, 2001). Sophian (2000) showed that 4- to 5-year-olds were able to identify corresponding spatial ratios based on relational information rather than the exact form of the stimuli (e.g., matching large and small rectangles of similar proportions, and rejecting rectangles that matched on only one dimension). Sophian and Wood (1997) used a more difficult task involving “conflict” problems in which 5-to-7-year-olds matched sample pictures either to a test stimulus that preserved the part-whole relation or one that preserved the part-part relation. By age 7, children were able to use part-whole relations to compare proportions. Jeong, Levine, and Huttenlocher (2007) found that 6-, 8-, and 10-year-olds failed a proportional reasoning task when discrete quantities were used, but even the youngest children showed some success when proportions involved continuous quantities. Children’s greater success with continuous quantities was related to the use of erroneous counting strategies in two discrete conditions: they counted the parts (and compared them) rather than comparing parts to wholes. In the continuous conditions, children were presumably able to rely on their earlier developing ability to perceptually compare relative surface areas.

Hence, although there is some debate about exact ages and some variation across tasks, the studies indicate that children prior to formal schooling can recognize proportional analogs when they can perceptually compare relative amounts of surface area. In contrast, when problems involve numbers and simple ratios, children generally perform poorly until 7 or 8 years of age, although gaps in understanding remain after these ages (Dixon & Moore, 1996; Fischbein, 1990; Kieren, 1988; Moore, Dixon, & Haines, 1991; Nunes, Schliemann, & Carraher, 1993; Singer, Kohn, & Resnick, 1997). Ahl, Moore, and Dixon (1992) compared the relation between informal, intuitive and formal numerical proportional reasoning in fifth-graders, eighth-graders, and college-aged subjects. In a temperature mixing task, varying amounts of water (1, 2, or 3 cups) at varying temperatures (20˚, 40˚, 60˚, and 80˚) were added to a container that was either cool (40˚) or warm (60˚), and children were asked what the resulting temperature would be. In the intuitive version of the task, quantities and temperatures were described verbally (e.g., cold, cool, warm, and hot), but in the numerical version, numbers were used (and students were told to use mathematics). Half of the students received the intuitive task first, and the other half received the numerical task first. Performing the intuitive task first improved performance in the numerical task; performance in the intuitive task did not change when it followed the numerical task. Therefore, students were able to use their intuitive understanding—elicited without numbers—to inform their numerical performance.

In sum, studies of elementary and middle school-aged children’s abilities to solve fraction problems indicate that conceptual knowledge broadly determines performance in such tasks as estimation, word problems, and even computation. Procedural knowledge, too, influenced performance on such tasks, and worked hand-in-hand with conceptual knowledge to determine the benefit derived from instruction. A key mechanism linking conceptual and procedural knowledge is the ability to represent fractions on a mental number line, which also supports reasoning performance in adults. On-task time, motivation, working memory, and well-learned basic arithmetic skills (in addition to general intelligence and reading ability) were also determinants of performance. Studies of preschool and older children’s ability to solve proportional reasoning problems mirror findings for fraction problems inasmuch as intuitive or pictorial versions of tasks are mastered early, in the preschool period. The tendency to rely on perceptual cues (comparing relative surface areas) continues in the elementary years, and children perform better using such intuitive strategies compared to numerical strategies. Among older elementary and middle school students, receiving an intuitive version of a proportional reasoning problem aids performance on a numerical version, but not vice versa.

d. Obstacles to Mastery

Many observers have remarked on the contrast between the relative ease of learning to count, which is engaged in spontaneously and seems to build easily on prior intuitions, and the relative difficulty of learning fractions (e.g., Moss, 2005; Sophian, 2007). Indeed, some have attributed the difficulties children have with fractions to the lack of fit with properties of counting (Gelman, 1991); for example, 3 > 2 and therefore children infer that 1/3 > 1/2. Because of the property of infinite divisibility, fractions, unlike counting numbers, do not form a sequence in which each number has a fixed successor. Therefore, it has been argued that the one-to-one and stable-order principles that are important to counting are misleading when children attempt to generalize from whole numbers to fractions.

Gelman (1991) examined kindergarten and first-grade children's interpretations of pictorial and numeral representations of fractions to determine whether children try to generalize from counting to fractions. Consistent with points made earlier about lack of familiarity with notation, young children read fraction symbols such as "$\frac{1}{2}$" as combinations of whole numbers (e.g., "one and two" rather than "one-half"). As others have reported, children also incorrectly judged fractions with larger denominators to be larger than those with smaller denominators (e.g., that "$\frac{1}{4}$" was more than "$\frac{1}{2}$"). Finally, most children were unable to correctly place pictorial representations of proper and mixed fractions (e.g., $\frac{1}{3}$ of a circle, $1\frac{1}{2}$ circles) on a number line on which the values 0, 1, 2, and 3 were marked. Consistent with Gelman's analysis that each of these effects had to do with negative transfer from knowledge of whole numbers, Vamvakoussi and Vosniadou (2004) found that just over half ($\frac{9}{16}$) of a sample of ninth-graders expressed the view that fractions form a series (such as $\frac{1}{8}, \frac{2}{8}, \frac{3}{8}$ and so on) rather than being infinitely divisible. However, strong conclusions cannot be drawn from Vamvakoussi and Vosniadou's relatively small sample, and a subsequent study by Smith et al. (2005) found that elementary school children were able to express the idea of infinite divisibility when prompted (e.g., endorsing the idea that one could divide numbers in half forever without ever getting to zero).

Although the concept of infinite divisibility is of interest because it distinguishes fractions from whole numbers, this does not mean that children do not inappropriately apply knowledge about whole numbers to fractions regardless of what they believe about divisibility. It appears that when children do not understand the conventions of reading fractions, they overgeneralize from their knowledge of whole numbers. For instance, they often judge the relative magnitude of two fractions as corresponding to the relative magnitudes of the numbers within them, which is sometimes correct (e.g., $\frac{2}{5} \langle \frac{3}{5}$ and $\frac{2}{3} \langle \frac{4}{5}$), but sometimes is not correct (e.g., $\frac{2}{3} \rangle \frac{2}{5}$, $\frac{3}{4} \rangle \frac{5}{8}$, and $\frac{3}{4} = \frac{6}{8}$ although 3 < 5, 4 < 8, and 3 < 6) (Sophian, 2007). Similarly, they add compound fractions by reading left to right, such as adding 2 + $\frac{3}{8}$ to get $\frac{5}{8}$ (Mack, 1995; cf. Sophian, 2007). The reliance on knowledge of counting and whole numbers leads to predictable errors in judging relative magnitudes or equivalence of fractions. However, it is not clear that this negative transfer occurs because of conflicts with innate counting mechanisms. Rather, it may stem from lack of knowledge of conventional notation, an argument that is strengthened by the demonstration of accurate intuitions when such notation is not used.

Another potential obstacle to mastery of fractions is the use of pictorial representations in early demonstrations, without sufficient emphasis on the nature of wholes in part-whole relations and the importance of equal-sized parts (Sophian, 2007). For example, if fractions are represented as slices of pizza as they often are, it becomes difficult to conceptualize improper ($\frac{6}{5}$) fractions. The number line representation presented earlier would seem to be more robust, easily representing quantities less than and greater than one. However, little research has been conducted comparing the relative effectiveness of different representational formats and whether, for example, pizza slices or other part-whole pictorial representations introduce difficulties when children move to fractions beyond the unit segment. Despite assertions made about the relative merits of different formats (e.g. using discrete vs. continuous quantities to represent fractions), few experiments with sufficient sample sizes and appropriate dependent measures (i.e., learning outcomes) have been conducted. A straightforward, randomized assignment experiment, for instance, pitting initial instruction using the number line against pizza slices or other pictorial formats could be used to answer this question. Both near (using problems resembling examples from training) and far (using superficially different problems) transfer could then be assessed.

In addition to pictures, words seem to influence the mental representations that children form concerning fractions. Several studies have confirmed that being a speaker of English, Croatian, or other languages that do not demarcate parts and wholes in fraction names is an obstacle to mastery of fractions. In East Asian languages, the part-whole relation is reflected in the corresponding names for fractions; for example, “one-fourth” is “of four parts, one” in Korean (Geary, 2006). Children whose languages demarcate parts and wholes in fractions names are able to demonstrate conceptual knowledge (e.g. they are able to correctly associate numerical fractions with pictorial representations) prior to formal instruction in fractions. For example, Miura, Okamoto, Vlahovic-Stetic, Kim, and Han (1999) found that 6- and 7-year-old Korean children grasped the part-whole relations represented by simple fractions (e.g., $\frac{1}{2}, \frac{1}{4}$) before formal instruction in first and second grade and before Croatian- and English-speaking children, whose languages do not have transparent word names for fractions. Such evidence is correlational, however, and subject to alternative interpretations based on differences in culture and experience. However, Paik and Mix (2003) demonstrated that when nontransparent, whole-number representations were used, U.S. and Korean children made similar errors in a fraction-identification task (although Korean children still scored better overall). When presented with fraction names that explicitly referred to parts and wholes on analogy with Korean names, U.S. children's performance improved and their scores exceeded those of the same-grade Korean children. In order for such labeling to be effective without additional training, it must build on fundamentally sound intuitions. These studies introduce a manipulation—fraction names with explicitly marked parts and wholes—that resembles the class-inclusion effects noted earlier (e.g., using Venn diagrams and tagging sets) because they, too, highlight part-whole relations. Both kinds of interventions are effective without additional training, suggesting that confusion about parts and wholes in working memory, rather than a total lack of conceptual knowledge, is responsible for unaided errors (e.g., Brainerd & Reyna, 1990; 1995).

As noted earlier, working-memory limitations are an obstacle to mastery of fractions (although we use the language of limited capacity, interference rather than capacity may offer a more satisfactory explanation of developmental and individual differences) (Dempster, 1992). English and Halford (1995) analyzed the working-memory demands of different tasks, and argued that different fraction interpretations entail different information-processing demands. A ratio interpretation, for example, a 2:3 ratio between red and blue chips involves just binary relations, because only two subsets need to be related to each other. In contrast, conceiving of the same array as corresponding to the fraction $\frac{2}{5}$ entails "ternary" relations, because three sets are related, the total set of all chips and each of its subsets, red chips and blue ones. Assessing equivalence relations between two fractions, as in the expression $\frac{1}{2} = \frac{3}{6}$, entails "quaternary" relations, because relations among all four quantities must be considered; judging relative magnitudes of fractions with unequal denominators and numerators, such as $\frac{5}{14} \rangle \frac{11}{34}$, makes similar demands. Quotient interpretations of fractions (e.g., sharing 3 pizzas among 4 people) are more demanding of memory resources than part-whole interpretations because they involve a more complex series of mappings (see Sophian, 2007). Formally similar tasks can have different information-processing demands. For example, area models of fractions (such as a partitioned rectangle) are assumed to be lower in demands for memory resources than set models (such as an array of red and blue chips) because the whole is more salient in the area model and the nonselected parts (e.g., the nonshaded segments) are less salient. English and Halford's claims once again reinforce the importance of making part-whole (or class-inclusion) relations salient or transparent.

Individual differences in working memory have been associated with performance on fraction tasks (e.g., Hecht et al., 2003; see Hecht et al., 2007, for a review). Effects of working memory were independent of effects of conceptual knowledge, which means that both factors are important and neither can be reduced to the other. Specifically, in the Hecht et al. (2003) study, individual differences in *working memory* were assessed with a counting-span task and individual differences in *conceptual knowledge* were assessed with tasks involving providing numerical representations of pictorially presented fractions and vice versa, providing a pictorial representation of the sum of two pictorially-represented fractions, or identifying the larger of two numerically-represented fractions. The effects of working memory on fraction computation were mediated by differences in fifth-graders' mastery of basic arithmetic facts (assessed by measures of accuracy and speed of retrieving basic addition and multiplication facts). That is, working memory uniquely contributed to variability in basic arithmetic knowledge (e.g., direct retrieval requires less working memory resources than counting to solve basic problems), and basic arithmetic, in turn, influenced fraction computation. Working memory directly influenced the solution of word problems, without any mediation through effects of basic arithmetic knowledge or conceptual knowledge (and when factors such as reading ability were also controlled for). Working memory predicted accuracy at, for example, setting up or translating word problems, as the Task Group elaborates in the section on Algebra. Although working memory is described as an individual difference, that does not mean that it cannot be changed or that strategies cannot be learned that make the most of whatever capacity an individual has, so that performance surpasses that of individuals with greater basic capacity. For example, strategies such as chunking (recoding a multidimensional concept into fewer dimensions) or segmentation (breaking a task into a series of steps, each of which is not too resource demanding) can reduce the working-memory demands of a task (Sophian, 2007). Moreover, conceptual knowledge carried the greatest weight in predicting performance on all three outcome measures (computation, estimation, and word problems), whereas working memory only affected word problems and only indirectly affected computation through knowledge of basic arithmetic facts (see Table 4, p. 290; Hecht et al., 2003).

In sum, despite evidence of early appreciation of part-whole relations prior to formal schooling, children lack sufficient conceptual knowledge of conventional fractions, which is a stumbling block to performance on such fraction tasks as estimation, computation, and word problems. Conceptual knowledge is assessed using a variety of tasks, such as judging equivalence or rank ordering quantities according to magnitude, but it should be pointed out that these tasks do not tap identical competence; tasks such as rank ordering decimals and fractions may be harder than judging equivalence (Mazzocco & Devlin, in press). When students do not understand conventional fraction notation, they will often generalize inappropriately from whole number counting to fractions. However, they seem to have a rudimentary understanding of infinite divisibility, so the generalization from counting has exceptions, and they can build on intuitions about part-whole relations. Different representational formats, such as pictures and fraction names that separate parts and wholes, allow those intuitions to be tapped to support better performance, prior to explicit instruction. Intuitive versions of proportional reasoning problems are solved earlier, are easier, and improve performance on subsequent numerical problems. Even complicated operations, such as division, seem to be supported by earlier kinds of knowledge, for example, about sharing. Representations that make part-whole relations salient or transparent, in contrast to making part-part relations salient, improve performance across tasks and age groups. Effects of different representational formats (e.g., discrete objects vs. portions of shapes) on more advanced problem solving, such as adding improper and mixed fractions, has yet to be definitively determined. Among other differences, students with low working memory capacity are less able to bring arithmetic facts to mind quickly and automatically, without drawing on mental resources (e.g., counting to solve basic problems) that could be used for other aspects of complex problem solving, compared to typically achieving students (Hecht et al., 2007). Despite the ubiquity of differences in working-memory capacity for low- and high-achieving groups in studies of mathematics learning, however, recent reviews of the literature on fractions assign greater weight to lack of conceptual knowledge in accounting for performance (e.g., Hecht et al.; Sophian, 2007). Conceptual knowledge has been shown to promote procedural knowledge (or computational ability), and vice versa, and development progresses iteratively, with gains in conceptual and procedural knowledge reinforcing, and bootstrapping, one another.

e. Conclusions and Recommendations

A basic interpretation of a fraction is a part-whole relation of two or more values, although there are other interpretations of fractions. Fractions can be represented as proper fractions (e.g., $\frac{1}{8}$), mixed numbers (e.g., $2\frac{1}{8}$), or in decimal form (e.g., 0.8), but are often represented using pictures during early instruction. Difficulty with fractions is pervasive and is an obstacle to further progress in mathematics, and, thus, is likely to constrain achievement in science and pursuit of scientific careers (e.g., Sadler & Tai, 2007). The inability to understand and compute fractions, decimals, and proportions has important real-life implications, and has been linked to poor health outcomes, among other harmful effects.

Classroom

The learning of arithmetic facts provides a foundation for learning fractions. Committing such facts to memory reduces working memory demands of problem solving and thus allows attention to be focused on other problem features. Therefore, children should begin fraction instruction with the ability to quickly and easily retrieve basic arithmetic facts. Instruction focusing on conceptual knowledge of fractions is likely to have the broadest and largest impact on problem-solving performance (provided that it is aimed at accurate solution of specific problem types that tap conceptual knowledge). Procedural knowledge is also essential, however, and although it must be learned separately, is likely to enhance conceptual knowledge and vice versa. Successful interventions reported in the scientific literature could be transferred easily to classrooms. These interventions include using fraction names that demarcate parts and wholes, using pictorial representations that are mapped onto the number line, and linking composite representations of fractions to representations of the number line. Conceptual and procedural knowledge about fractions less than one do not necessarily transfer to fractions greater than one (i.e., improper and mixed fractions), and must be separately instructed. Appropriate intuitions about sharing, part-whole relations, and proportional relations can be built on in classrooms to support acquisition of conceptual and procedural knowledge of fractions.

Training

Teachers. Training of teachers should include sufficient coverage of the scientific method so that teachers are able to critically evaluate the evidence for proposed pedagogical approaches and to be informed consumers of the scientific literature (who can keep up with advances in scientific knowledge after graduation from training programs). Teachers should be aware of common conceptions and misconceptions involving fractions, based on the scientific literature, and of effective interventions involving fractions. Thus, training should include comprehensive courses on cognitive development focusing on mathematics learning that draw on the primary literature in this area (i.e., refereed journal articles).

Future researchers. Many of the best researchers in the basic science of mathematics learning are currently not engaged in directly relevant educational research. New funding should be provided to train future researchers, to begin new interdisciplinary degree programs with rigorous quantitative training, and to establish support mechanisms for career shifts for rigorous researchers that are similar to K awards from the National Institutes of Health.

Curriculum

The curriculum should allow for sufficient time on task to ensure acquisition of conceptual and procedural knowledge of fractions and of proportional reasoning, with the goal for students being one of learning rather than performance. However, there should be ample opportunity in the curriculum for accurate self-evaluation. The curriculum should include representational supports that have been shown to be effective, such as number line representations, and encompass instruction in tasks that tap the full gamut of conceptual and procedural knowledge, such as ordering fractions on a number line, judging equivalence and relative magnitudes of fractions with unequal numerators and denominators, estimation, computation, and word problems. The curriculum should make explicit connections between intuitive understanding and formal problem solving.

Research

Basic. Studies suggest that preschool and early elementary-school children have a rudimentary understanding of very simple fractional relations, but the mechanism underlying this knowledge is not yet known. The relation between this informal, often implicit knowledge, and the learning of formal mathematical fractional concepts and procedures is not well understood, and is an area in need of further study. Similarly, the mechanisms that contribute to the emergence of formal competencies in school are not fully understood, but involve a combination of instruction, working memory, and the bidirectional influences of procedural knowledge on the acquisition of conceptual knowledge and conceptual knowledge on the skilled use of procedures. Therefore, research is needed that tests specific hypotheses designed to uncover these mechanisms, including linking earlier intuitive understanding with later formal problem solving. In addition, research on understanding and learning of fractions should be integrated with what is known and with emerging knowledge in other areas of basic research, such as neuroscience, cognition, motivation, and social psychology. Research on mental representations and retrieval in memory, as well as on intuitive versus analytical reasoning, are especially relevant and currently not integrated with research on fractions. Ironically, the absence of a coherent and empirically supported theory of fraction tasks (i.e., how tasks are related to one another in terms of underlying processes) is a major stumbling block to developing practical interventions to improve performance in this crucial domain of mathematics. Such a theory would, for example, provide scientific guidance concerning how instruction in different fraction tasks should be ordered.

Classroom. Classroom-relevant research need not be conducted physically in classrooms, and constraints on funding that require that relevant research be performed in classrooms should be removed. Conversely, many interventions demonstrated to be effective in experiments should be scaled up and evaluated in classrooms. In order to produce a steady supply of high-quality research that is relevant to classroom instruction, a pipeline of research must be funded that extends from the basic science of learning to field studies in classrooms. Incentives should also be provided to encourage partnerships between basic and applied researchers, and to support research that includes both laboratory and field-based research, in a way that will provide converging operations.

3. Estimation

Estimation is an important part of mathematical cognition, one that is pervasively present in the lives of both children and adults. Consider just a few everyday examples. How many people were at the game? How fast can that Lamborghini go? About how much is 65 × 29? Estimation may be used more often in everyday life than any other quantification process.

In addition to its pervasive use, estimation is quite strongly related to other aspects of mathematical ability, such as arithmetic skill and conceptual understanding of computational procedures, and to overall math achievement test scores (Booth & Siegler, 2006; Dowker, 2003; Hiebert & Wearne, 1986; LeFevre, Greenham, & Waheed, 1993; Geary, Hoard, Byrd-Craven, Nugent, & Numtee, 2007). It usually requires going beyond rote application of procedures and applying mathematical knowledge in flexible ways. This type of adaptive problem solving is a fundamental goal of contemporary mathematics education.

Yet another basis of the importance of estimation is practical—most school age children are surprisingly bad at it and even many adults are far from good. Standardized scores on the part of the NAEP that tests estimation proficiency are below those for the mathematics test as a whole (Mitchell, Hawkins, Stancavage, & Dossey, 1999). This limited proficiency, together with the pervasiveness of estimation in everyday life, its relation to general mathematical ability, and its embodying the type of flexible problem solving that is viewed as crucial within modern mathematics education, have led the National Council of Teachers of Mathematics (NCTM) to assign a high priority to the goal of improving estimation skills within each revision of its Math Standards since 1980 (e.g., NCTM, 1980, 2000), as well as in its recent *Focal Points* (NCTM, 2006).

Despite the importance of estimation both in and out of school, far less is known about it than about other basic quantitative abilities, such as counting and arithmetic. One reason for the discrepancy is that estimation includes a varied set of processes rather than a single one. Some estimation tasks, for example estimating the distance between two cities or the cost of a bag of groceries, require knowledge of measurement units such as miles or dollars. Other estimation tasks, for example estimating the number of coins in a jar or the answers to arithmetic problems, do not. Similarly, some uses of estimation, for example estimating the cost of a pizza or the speed of a Lamborghini, require prior knowledge of the entities whose properties are being estimated (i.e., pizzas, Lamborghinis). Other uses, such as estimating the length of a line on a page or the number of fans at a game, do not.

In this discussion, the Task Group focuses on *numerical estimation*, the process of translating between alternative quantitative representations, at least one of which is inexact and at least one of which is numerical. This category includes many prototypic forms of estimation. For example, computational estimation involves translating from one numerical representation (e.g., 75 × 29) to another (about 2,200). Number line estimation either requires translating a number into a spatial position on a number line (e.g., given: 0_____________100, place a mark on the line where 71 would fall) or translating a spatial position on a number line into a number. Numerosity estimation requires translating a nonnumerical quantitative representation (e.g., a visual representation of the approximate volume and density of candies in a jar) into a number (e.g., about 300 marbles.) Because this task group's focus is on the learning of mathematics, excluded from consideration are tasks that require knowledge external to mathematics, in particular knowledge of measurement units (e.g., pounds, hours, miles) or real-world entities (e.g., population of Russia, number of people with AIDS). We also exclude from consideration trivial applications of estimation, such as rounding to the nearest 10, which unfortunately are the predominant focus of instruction in estimation in many U. S. classrooms.

a. Understanding the Goals of Estimation

Many children have highly distorted impressions of the goals of estimation, especially the goals of computational estimation. As noted by LeFevre et al. (1993), accurate computational estimation requires understanding of the simplification principle (the understanding that mental arithmetic is easier with simple operands) and the proximity principle (the understanding that the main aim of estimation is to obtain estimates close in magnitude to the correct answer). LeFevre et al. found that fourth- and sixth-graders understood the principle of simplification, but they showed little if any understanding of the importance of generating an estimate close in magnitude to the correct answer. When asked to define estimation, most said that it was "guessing" or indicated that they did not know. When asked to estimate the products of multidigit multiplication problems, only 20% of fourth-graders produced reasonable estimates (estimates that varied systematically with the product).

Sowder and Wheeler (1989) found that even middle and high school students typically do not understand that the goal of estimation is to generate estimates that are close to the correct value, rather than following some prescribed procedure. They based this conclusion on the reluctance of even ninth-graders to accept that both of two alternative estimates could be acceptable and on their infrequent use of compensation to correct for distortions introduced by rounding. When asked to generate estimates, some students went as far as to calculate the correct answer and then to round to a nearby number. The problem seemed to be that the children viewed estimation as a rigid algorithmic procedure that required following preset rounding rules rather than as a flexible attempt to approximate the magnitude of an answer using whatever means made sense in the particular situation. This blind execution of an algorithm is reflected in Sowder and Wheeler's observation that, "Some students in both Grades 5 and 7 objected to rounding 267 to 250 rather than 300, arguing, 'You're always taught to go up if it's past five,' or 'Seven is above five, so you have to go up, not down (p. 144).'"

On the other hand, Sowder and Wheeler (1989) also noted that by fifth grade, the large majority of children, when presented hypothetical estimation procedures in which rounding was or was not followed by compensation for the distortions introduced by rounding, recognized that rounding with compensation was superior. This finding suggests that some conceptual understanding of the importance of the proximity principle is present by fifth grade. Instruction in estimation clearly needs to convey to students earlier and more consistently that the purpose of estimation is to generate values close in magnitude to the correct value.

b. Development of Estimation Skills

Computational estimation

Development of computational estimation skills (the ability to answer an arithmetic problem with the goal of approximating the correct magnitude rather than calculating the exact answer) begins surprisingly late and proceeds surprisingly slowly. In one study, more than 75% of third- and fifth-graders did not agree that two alternative estimates of the sum of two addends could both be acceptable (Sowder & Wheeler, 1989). Similarly, Dowker (1997) found that early elementary school children often could perform exact computations in a numerical range but could not estimate answers in the same range. Thus, 7- and 8-year-olds who were able to compute the correct answer on problems with sums less than 100 failed to generate reasonable estimates (estimates within 30% of the correct answer) on 33% of these problems.

Computational estimation does improve considerably, albeit gradually, with age and experience. Adults and sixth-graders are more accurate than fourth-graders in estimating the sum of 2 three-digit addends (Lemaire & Lecacheur, 2002), sixth- and eighth-graders are more accurate than fourth-graders in estimating the sums of long strings of addends (Smith, 1999), and fourth-graders are more accurate than second-graders in estimating the sums of two-digit addends (Booth & Siegler, 2006). Similarly, adults are more accurate than eighth-graders, who in turn are more accurate than sixth-graders, in estimating the products of multidigit multiplication problems (LeFevre et al., 1993). Improvements in the speed of estimation of the answers to both addition and multiplication problems follow a similar course to improvements in accuracy over the same age range (Lemaire & Lecacheur).

From early in the development of computational estimation, individual children use a variety of strategies (Reys, 1984). Evidence for such strategic variability comes both from observations of ongoing behavior and from immediately retrospective self-reports (LeFevre et al., 1993; Sowder & Wheeler, 1989). The following is a list of some of the most common estimation strategies for addition and multiplication (Dowker, Flood, Griffiths, Harriss, & Hook, 1996; LeFevre et al.; Reys et al., 1982; Reys et al., 1991; Sowder & Wheeler):

1) *Rounding*: Converting one or both operands to the closest number ending in one or more zeroes (e.g., on 297 × 296, both multiplicands might be converted to 300).
2) *Truncating*: Changing to zero one or more digits at the right end of one or more operands (e.g., on 297 × 296, both multiplicands might be converted to 290).
3) *Prior compensation*: Rounding the second operand in the opposite direction of the first before performing any computation (e.g., on 297 × 296, 296 might be rounded to 290 rather than 300 to compensate for the effect of rounding 297 to 300).

4) *Postcompensation*: Correcting later for distortion introduced by earlier rounding or truncation (e.g., on 297 × 296, multiplying 300 × 300 and then subtracting 2% of 90,000).
5) *Decomposition*: Dividing numbers into simpler forms (e.g., on 282 × 153, multiplying 280 by 100 and then by 1.5).
6) *Translation*: Simplifying an equation (e.g., by changing the operation, on 44 + 53 + 51 + 47, multiplying 50 × 4).
7) *Guessing.*

As might be expected, some of these strategies are used more often than others. Rounding is the most common approach (Lemaire et al., 2000; LeFevre et al., 1993; Reys et al., 1982; Reys et al., 1991). Compensation tends to be among the least frequent approaches, although it is among the most useful. For example, in Lemaire et al.'s study of estimation of multidigit sums, fifth-graders used rounding on 64% of trials and compensation on 2%.

This use of multiple strategies is not a result of individuals using only one approach but differing in what that approach is. Instead, both children and adults often know and use a variety of computational estimation strategies. This is especially true among mathematically sophisticated individuals. For example, Dowker et al. (1996) examined the multiplication and division estimates of four groups of adults: mathematicians, accountants, and students majoring in psychology or English at Oxford University. The strategies that they used were remarkably diverse: For example, the 176 participants used 27 different strategies for solving the single problem 4645 ÷ 18. Individuals in each of the four groups averaged more than five strategies apiece. Strategic variability was evident even within a single person solving the same problem on two occasions. When problems were presented to participants a second time, mathematicians used a different strategy on 46% of items and psychology students on 37%.

Both children and adults adapt their strategy choices to problem characteristics. One form that this adaptation takes is to use rounding more often on problems where it introduces less distortion. For example, on multidigit addition problems, the closer an addend is to the nearest 10, and therefore the less distortion introduced by rounding, the more often fourth- and sixth-graders round (Lemaire & Lecacheur, 2002). Similarly, on multidigit multiplication problems, sixth-graders, eighth-graders, and adults more often round both of the multiplicands when each includes two or three digits, but often only round the larger multiplicand when the smaller one is a single digit (LeFevre et al., 1993). This choice pattern minimizes distortion, because rounding two or three digit multiplicands to the nearest 10 changes the product by a smaller percentage than rounding single-digit multiplicands to the nearest 10.

The range and appropriateness of computational estimation strategies increase with age and mathematical experience. Adults use a considerably greater variety of multiplication strategies than do sixth- or eighth-graders (LeFevre et al., 1993). Similarly, mathematicians and accountants, who have unusually extensive numerical experience, use a greater variety of appropriate estimation strategies than do even the highly selected psychology and English students at Oxford University (Dowker et al., 1996). The latter two groups used a greater variety of inappropriate estimation strategies than did the former two, which indicates that ability to generate *appropriate* variants is what distinguishes the mathematicians and accountants, rather than greater variation per se.

A second type of change in strategy use involves the sophistication of the strategies that are used. Use of compensation, a strategy that requires a good conceptual understanding of estimation, shows especially substantial growth. In estimating the answers to multidigit addition problems, far more ninth-graders use post-compensation than do third- or fifth-graders (Lemaire et al., 2000; Sowder & Wheeler, 1989). The quality of strategy choices in multidigit multiplication also increases with age and mathematical experience. LeFevre et al. (1993) provided the example of strategy choices on 11×112. Among adults, 75% rounded the problem to 10×112, a computationally tractable approach that yields an answer within 9% of the correct answer. Although this approach would seem well within the capabilities of sixth-graders (LeFevre et al.), no sixth-grader used it. Instead, they rounded either to 10×100 or to 10×110.

All of the studies reviewed in this section involve computational estimation with whole numbers. Far less is known about computational estimation with fractions. Two findings that have emerged are that even high school students are very poor at computational estimation with fractions and that the main problem seems to be inadequate conceptual understanding of the magnitudes of fractions (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1981; Hecht et al., 2007). When more 13- and 17-year-olds estimate that $\frac{12}{13} + \frac{7}{8}$ is roughly equal to 19 than estimate that it is roughly equal to 2, there clearly is a serious problem in their understanding of the relation between fractional notation and the magnitudes that are being estimated (Carpenter et al.).

Number line estimation

The number line task has proved highly informative, not only for improving understanding of estimation but also for providing useful information about children's understanding of the decimal number system more generally. On this task, children are presented a line with 0 at one end, another number such as 100 or 1,000 at the other end, and no other numbers or hatch marks in between. The child is presented a new line and number to be estimated on each trial, until the child has estimated the magnitudes of numbers throughout the range (e.g., 0–1,000). Then each estimate is translated into a numerical value, and the relation between the number presented and the estimate is examined for the full set of numbers. Ideally, the estimated value should increase linearly with the actual value in a 1:1 fashion, in accord with the equation $y = x$. Thus, on a number line with 0 at one end and 1,000 at the other, the estimate for 20 should be 2% of the way between 0 and 1,000, the estimate for 230 should be 23% of the way, the estimate for 760 should be 76% of the way, etc.

Although this task seems easy, elementary school children's estimates consistently depart from correct values in predictable ways. Moreover, similar departures from correct estimates are seen at the same ages on other types of estimation tasks; the deviations are indicative of broader difficulties with mathematics. This number-line task and findings have inspired an educational intervention that succeeded in improving a broad range of numerical skills in low-income preschoolers (Ramani & Siegler, 2008).

Number line estimation improves steadily during the elementary school years, with accuracy at any given age being greater on smaller numerical scales than on larger scales. On 0–10 number lines, Petitto (1990) found that the percent of absolute error decreased from 14% late in first grade to 4% late in third grade. On 0–100 number lines, the same children's percent of absolute error decreased from 19% late in first grade to 8% late in third grade. On 0–1,000 number lines, Siegler and Opfer (2003) found that percent absolute error improved from 21% in second grade to 14% in fourth grade, 7% in sixth grade, and 1% in adulthood. The number line estimates of children from low-income backgrounds and children with learning disabilities in mathematics are far less accurate than those of typically achieving children from middle-income families, though they also improve with age (Geary, Hoard, Nugent, & Byrd-Craven, in press; Siegler & Ramani, in press). The superiority of the estimates on the smaller scale (0–100) indicates that at least through fourth grade, children use their knowledge of particular numbers, rather than general understanding of the decimal system, to estimate.

Children use two primary mental representations of numerical magnitude on number line estimation tasks. One common approach (the correct one with whole numbers) is to use a linear representation, that is, a representation in which numerical magnitude increases linearly with the size of the number. Another common approach is to employ a logarithmic representation, in which representations of numerical magnitudes increase logarithmically with numerical size. When children use such a logarithmic representation on a number line estimation task, the spatial positions they choose increase very quickly in the low range of numbers and then increase only slowly in the upper part of the range. For example, Siegler and Opfer (2003) found that on 0–1,000 number lines, differences between second-graders' estimates for 5 and 86 were much larger than the differences between their estimates for 86 and 810. Such logarithmic representations of quantities and other magnitudes are common across many species and tasks (Dehaene, 1997), and for good reason: To a hungry animal, the difference in importance between 5 and 86 pieces of food often is far larger than the difference between 86 and 810 pieces. This is not the case within the formal number system, however.

With age and experience, children progress from using the less accurate logarithmic representation to the more accurate linear one on the number line task. For example, kindergartners' median estimates on 0–100 number lines are better fit by the logarithmic function than by the linear function, first-graders' estimates are fit equally well by the two functions, and second-graders' estimates are better fit by the linear function (Geary et al., in press; Laski & Siegler, in press; Siegler & Booth, 2004). The same progression is seen in children with learning disabilities in mathematics, but it occurs more slowly and at older ages (Geary et al.). On 0–1,000 number lines, second-graders' estimates are fit better by the logarithmic function, whereas fourth-graders' are fit better by the linear function (Booth & Siegler, 2006; Opfer & Siegler, 2007). The same child often uses different representations depending on the scale of numbers they are asked to estimate. For roughly half of the second-graders in Siegler and Opfer (2003), the best fitting function for number line estimates was linear on the 0-100 line but logarithmic on the 0–1,000 line.

Number lines can be used to examine estimates of the magnitudes of fractions as well as whole numbers. Results from such studies, like studies of computational estimation with fractions, show poor understanding of fractional magnitudes at all ages. Fifth- and sixth-graders' estimates of the magnitudes of decimal fractions do not even maintain the correct rank order (Rittle-Johnson et al., 2001). Even many adults in the United States have poor understanding of the magnitudes of common fractions. Opfer et al. (2007) found that most adults estimate the magnitudes of common fractions with numerators of "1" as a linear function of the distance between their denominators (even though the magnitudes of such fractions actually follow a logarithmic function). For example, adults estimate the magnitudes of $\frac{1}{1}$ and $\frac{1}{60}$ to be much closer than those of $\frac{1}{60}$ and $\frac{1}{1440}$), even though the magnitudes of the fractions in the initial pair are more than 60 times as discrepant.

c. Individual Differences in Estimation

Both children and adults show substantial individual differences in skill at computational estimation (Dowker, 2003) that are associated with broader individual differences in mathematical understanding. Proficiency at computational estimation correlates positively, and often substantially, with mathematics SAT scores (Paull, 1972), mathematics achievement test scores (Booth & Siegler, 2006; Siegler & Booth, 2004), performance on other estimation tasks (Booth & Siegler), and arithmetic fluency scores (Dowker, 1997, 2003; LeFevre et al., 1993).

Accuracy and linearity of number line estimation also is highly associated with general mathematical ability. Significant and substantial correlations—typically between r = .50 and r = .60—have been found between mathematics achievement test scores and linearity of number line estimates among kindergartners, first-graders, and second-graders on 0–100 number lines (Geary et al., in press; Siegler & Booth, 2004) and among second-, third-, and fourth-graders on 0–1,000 number lines (Booth & Siegler, 2006). Individual differences in linearity of number line estimates also are closely associated with individual differences in linearity on other estimation tasks (Booth & Siegler). These results suggest that performance on a variety of estimation tasks reflects a common underlying representation of numerical magnitude and that the closer this representation is to the formal linear mathematical system the better the overall mathematics achievement.

d. Improving Children's Estimation

Findings with number line estimation have raised the question: What leads children to shift from logarithmic to linear representations of numerical magnitude? One common activity that seems likely to contribute is playing board games with linearly arranged, consecutively numbered, equal-size spaces (e.g., Chutes and Ladders©). Such board games provide multiple cues to both the order of numbers and the numbers' magnitudes. In the games, the greater the number in a square, the greater a) the distance that the child has moved the token, b) the number of discrete moves the child has made, c) the number of number names the child has spoken, d) the number of number names the child has heard, and e) the amount of time since the game began. The linear relations between numerical magnitudes and these visuospatial, kinesthetic, auditory, and temporal cues provide a broadly based, multi-modal foundation for a linear representation of numerical magnitudes.

To determine whether playing number board games produces improvements in numerical understanding, Siegler and Ramani (in press) and Ramani and Siegler (2008) randomly assigned preschoolers at Head Start centers, all of whom came from low-income families, to play one of two board games. The games differed only in the board that children encountered. One board included linearly arranged, equal-spaced squares that progressed from 1–10 from left to right. The other board was identical except for the squares varying in color rather than number. Each child played the number board game or the color board game with an experimenter for four 15-minute sessions within 2 weeks.

Playing the numerical board game for this 1 hour period increased the Head Start children's proficiency not only at number line estimation but also at three other key numerical skills: counting, identifying printed numerals, and comparing the relative sizes of numbers. The gains in all four skills remained when children were tested nine weeks after the game playing experience. Gains were comparable for African-American and white children; they also were comparable for children who, relative to their low-income peers, entered the game with more or less numerical knowledge. Classmates who played the color board version of the game did not improve on any of the skills. The effect sizes of differences between the groups were substantial: *d*'s between .69 and 1.08 on the four measures on the immediate posttest and between .55 and .80 on the 9-week follow-up.

Ramani and Siegler (2008) also tested whether board game experience in the everyday environment is related to numerical knowledge and whether it might contribute to the knowledge differences between children from low- and middle-income backgrounds. They asked children from the initial experiment, as well as age peers from middle-income backgrounds, about their experience playing board games, card games, and video games at their own and other people's homes. The children from middle-income homes reported having more experience playing board games and card games in both contexts (though less experience playing video games). Of particular interest, the number of board games that the Head Start children reported playing at their own and other people's homes correlated positively with their skill at all four numerical tasks examined in the study. In contrast, the preschoolers' experience playing card games and video games was only minimally related to their numerical knowledge. Thus, playing numerical board games appears to be a promising (and inexpensive) way to improve low-income preschoolers' numerical knowledge and to reduce discrepancies in the numerical knowledge that children from low- and middle-income homes bring to school.

A different procedure has been found effective for improving elementary school children's number line estimation. By second grade, a large majority of children generate linear representations of magnitudes in the 0–100 range but logarithmic ones in the 0–1,000 range. Opfer and Siegler (2007) reasoned that a dramatic error of a number line estimate in the 0–1,000 range might lead children to search for an alternative approach, that their representations of numbers in the 0–100 range provided such an alternative, and that the children would draw the analogy to the 0–100 range and quickly improve their estimates in the 0–1,000 range. This proved to be the case. Providing the second-graders with feedback on their estimate of the single number 150—the number where the logarithmic and linear functions are most discrepant—led 80% of the children to shift from a logarithmic to a linear approach after that single feedback problem. Almost all of these children continued to use the linear approach on all subsequent trials. Thus, feedback on well-chosen problems is another means of improving children's estimation.

Improving elementary school children's numerical representations also can improve their skill at learning arithmetic. Presenting first-graders with accurate number line representations of the magnitudes of addends and sums enabled the children to recall the correct answer more often than children who were told the correct answer but were not presented the number line representations (Booth & Siegler, in press). Providing the number line representations also led to errors being closer in magnitude to the correct answer. Thus, numerical magnitude representations influence learning of arithmetic as well as a variety of other numerical skills and knowledge.

e. Conclusions and Recommendations

Numerical estimation is an important part of mathematical cognition. It is used frequently by both children and adults, in both academic and nonacademic contexts; is closely related to arithmetic skill, conceptual understanding of computational operations, and mathematics achievement test performance; and receives a considerable amount of attention in elementary school mathematics textbooks and classroom instruction. Moreover, estimation performance often reveals both subtle and gross deficiencies in numerical understanding.

Despite its importance and the substantial attention that it receives, most children's proficiency at estimation is poor. This in part reflects the emphasis in many classrooms on rounding procedures, to the exclusion of conceptually richer aspects of estimation, such as compensating for the distortions introduced by rounding. Many students do not even know that the goal of estimation is to generate values that are close to the correct value or that there is often more than one reasonable estimation procedure.

From kindergarten or first grade onward, most children's estimates of the magnitudes of whole numbers accurately reflect the rank order of the numbers. However, children from low-income backgrounds often do not even know the rank order of the numbers 0–10 when they enter school. Proficiency develops first in the 0–10 and 0–100 ranges, and then in the 0–1,000 and larger ranges. However, many elementary school students fail to discriminate adequately among the magnitudes of numbers in the hundreds or thousands.

Studies of estimation of the magnitudes of fractions show little if any understanding, even among middle school and high school students. Estimates often do not even maintain the rank order of the fractions' magnitudes. There is a strong need to develop effective procedures for remedying most students' lack of understanding of fractional magnitudes.

Classroom

Teachers should broaden instruction in computational estimation beyond rounding. They should insure that students understand that the purpose of estimation is to approximate the correct value and that rounding is only one of several means for accomplishing this goal.

Teachers should provide examples of alternative procedures for compensating for the distortions introduced by rounding, should emphasize that there are many reasonable procedures for estimating rather than just a single correct one, and should discuss reasons why some procedures are reasonable and others are not.

Teachers in Head Start and other facilities serving preschoolers from low-income backgrounds should be made aware of the usefulness of numerical board games for improving the children's knowledge of numbers and of the importance of such early knowledge for long-term educational success.

Teachers should not assume that children understand the magnitudes represented by fractions even if the children can perform arithmetic operations with them, because the arithmetic competence may only represent execution of memorized procedures. Examining children's ability to perform novel estimation tasks, such as estimating the positions of fractions on number lines, can provide a useful tool for assessing children's knowledge of fractions. Providing feedback on such number line estimates can improve children's knowledge of the fractions' magnitudes.

Training

Teachers in preservice and in-service programs should be informed of the tendency of elementary school students not to fully understand the magnitude of large whole numbers, should be taught how to assess individual students' understanding, and should be taught research-based techniques for improving the children's understanding.

Teachers should be made aware of the inadequate understanding of the magnitudes of fractions of elementary school, middle school, and high school students. The teachers also should be familiarized with the usefulness of feedback on number line estimates of the magnitudes of fractions for overcoming these difficulties.

Curriculum

Textbooks need to explicitly explain that the purpose of estimation is to produce accurate approximations. Illustrating multiple useful estimation procedures for a single problem, and explaining how each procedure achieves the goal of accurate estimation, is a useful means for achieving this goal. Contrasting these procedures with others that produce less accurate estimates, and explaining why the one set of procedures produces more accurate estimates than the other, is also likely to be helpful.

Research

Research is needed regarding simple instruments that teachers can use in the classroom for assessing children's estimation skills, and regarding instruction that can efficiently improve children's estimation.

Research is needed on how the inadequate representations of whole number numerical magnitudes that have been identified by studies of estimation influence learning of other mathematical skills, such as arithmetic.

Research is needed on how children can be taught to accurately estimate the magnitudes of fractions and on how learning to estimate those magnitudes influences acquisition of other numerical skills involving fractions, such as arithmetic and algebra.

Research is needed on how estimation is used by students (e.g., to solve complex problems, to improve test performance) and by adults in everyday life and professional tasks (e.g., to rule out implausible answers and thus reduce human error).

4. Geometry

Geometry is the branch of mathematics concerned with properties of space, and of figures and shapes in space. Euclidean geometry is the domain typically covered in mathematics curricula in the United States, although a separate year-long course is not usually taught until high school. Units on geometry as well as measurement are frequently included in middle school mathematics classes, whereas only the latter tends to be emphasized in the elementary grades.

a. Geometry Performance of U.S. Students on International Mathematics Assessments

Although geometric concepts and skills are typically taught in both elementary and middle school classrooms in the United States, international assessments indicate that the achievement levels of U.S. students are comparatively poor in this mathematical domain. To begin with, the 2003 Trends in International Mathematics and Science Study (TIMSS) showed no significant improvement in geometry for U.S. eighth-graders between 1999 and 2003, despite significant gains in algebra during this same time period (Gonzales et al., 2004). Moreover, of the five mathematical content areas assessed by TIMSS (number, algebra, geometry, measurement, and data), U.S. eighth-graders' performance in geometry items was weakest (Mullis et al., 2004).

Similarly, a report from the American Institutes for Research (Ginsburg et al., 2005) reexamined the 2003 mathematics performance of U.S. students on the TIMSS fourth- and eighth-grade assessments, as well as the Program for International Student Assessment (PISA)—relative to a common set of 11 other countries which had also participated in these studies (including Australia, Hong Kong,[1] Japan, and New Zealand, among others). The content areas that were evaluated included 1) number/quantity, 2) algebra/change and relationships, 3) measurement, 4) geometry/space and shape, and 5) data/uncertainty. The United States ranked 8th, 9th, and 9th out of the 12 countries on the TIMSS-4, TIMSS-8, and PISA, respectively. And again the performance levels of U.S. students were found to be significantly weakest in the area of measurement in Grade 4 and in geometry in Grade 8, as compared against the average U.S. score across all content areas. Furthermore, the United States was found to devote only about half as much time to the study of geometry as the other countries.

[1] Hong Kong is a Special Administrative Region (SAR) of the People's Republic of China.

b. Importance of Geometry for Learning Algebra

Given that the primary charge to the National Mathematics Advisory Panel concerns preparation for and learning of algebra, one may ask what geometry has to do with the acquisition of algebraic concepts and skills? Moreover, as the teaching of high school level geometry usually follows the first course in algebra, what if any geometric concepts should be learned in the middle school years, if not earlier, to ensure that students are best prepared to acquire a thorough understanding of key algebraic concepts and expressions? As noted in the Conceptual Knowledge and Skills Task Group report, the single aspect of geometry that is most directly relevant for early learning of algebra is that of similar triangles. In particular, the proof that the slope of a straight line is independent of the two points selected depends logically on considerations of the properties of similar triangles. This is because the corresponding angles of similar triangles are congruent and their corresponding sides are proportional. Therefore, the Conceptual Knowledge and Skills Task Group contends that it is crucially important for students to be given the opportunity to acquire these and other essential facts about similar triangles prior to the formal study of algebra. Furthermore, they point out that whereas students do not need to learn to construct the proofs of these theorems until they take a course in Euclidean geometry, they should nonetheless be able to make use of them.

Consistent with this perspective, the NCTM's (2006) *Focal Points* underscores (as do some state frameworks) the importance of these ideas in its section on algebra and connections to geometry for eighth-graders: "Given a line in a coordinate plane, students should understand that all 'slope triangles' triangles created by a vertical 'rise' line segment (showing the change in y), a horizontal 'run' line segment (showing the change in x), and a segment of the line itself—are similar. They also [should] understand the relationship of these similar triangles to the constant slope of a line" (p. 20).

What are the essential aspects of similar triangles? Acknowledging the need for learning how the relations between various properties of triangles underlie the fact that the slope of a straight line is independent of the two points selected, the question arises as to what kinds of concepts students need to acquire to understand the "basic aspects" of similar triangles? Certainly, to comprehend that the corresponding sides of similar triangles are proportional requires at minimum an understanding of length, equal angles, right triangles, and correspondence, as well as the crucial concepts of ratio and proportion. At this point, the Task Group notes that the difficulties associated with acquiring a sound conceptual understanding of ratio and proportion in and of themselves (as outlined in the section on Fractions in the this report) clearly constitute a significant obstacle to mastering how the slope of a straight line is derived from the properties of similar triangles.

Moreover, some additional difficulties may arise from the way in which the concept of similarity is often defined for students in school mathematics. For example, Wu (2005) has argued that rather than defining the similarity of figures as "same shape but not necessarily the same size," the most mathematically accurate and potentially effective way to define it is two figures are similar if one figure is congruent to a dilated version of the other. Naturally, understanding this definition would necessitate learning the meanings of congruence and dilation. Although a common way of defining congruence in school mathematics is "same size and same shape," Wu contends that a more mathematically correct and grade-level appropriate (i.e., for middle school students) definition is a composition of translations, reflections and rotations. It follows that to make sense of this definition students would first have to learn the meanings of these various transformations of the plane (more commonly referred to as slides, flips and turns, respectively). Furthermore, students would have to learn the meaning of dilation—a transformation of the plane that expands (or contracts) all points away from (or toward) a central point by a common scale factor. This mathematically accurate definition is clearly rather complicated in comparison with the more commonly used definition: a transformation that changes a figure's size, while its shape, orientation, and location remain the same. To sum up, in order to understand the mathematics underlying the proof that the slope of a straight line is independent of the choice of the points selected, students must successfully develop a conceptual understanding of the following: points, lines, length, angle, right triangle, correspondence, ratio, proportion, translation, reflection, rotation, dilation, congruence, and similarity.

c. Limitations of the Relevant Research-Based Literature

For the purposes of the present section, it is important to understand the developmental course that children take in learning the concepts that are required for understanding the properties of similar triangles. Whereas empirical studies of the key components of congruence, similarity, transformations, and so forth have indeed been conducted, it is difficult to draw firm, scientific conclusions from the relevant research literature. The reasons for this include, among others, 1) numerous studies of convenience samples with small numbers of participants, 2) the frequent use of a single age group or grade level, 3) the almost complete lack of longitudinal data, 4) an overemphasis on interview data and anecdotal reports, 5) a lack of rigor in study designs with comparatively limited use of experimental manipulations, and perhaps of greatest concern 6) a paucity of programmatic and cumulative efforts that could yield a clearer picture of the development of geometric thinking and reasoning. Thus for the most part, the Task Group is in agreement with Clements and Sarama's (2007a) conclusion following a recent extensive and intensive review of the relevant literature, with a focus on early childhood mathematics:

> Although far less developed than our knowledge of number, research provides guidelines for developing young children's learning of geometric and spatial abilities. However, researchers do not know the potential of children's learning if a conscientious, sequenced development of spatial thinking and geometry were provided throughout their earliest years. Insufficient evidence exists on the effects (efficacy and efficiency) of including topics such as congruence, similarity, transformations, and angles in curricula and teaching at specific age levels. Such research, and longitudinal research on many such topics, is needed (p. 517).

Nevertheless, the Task Group reviewed influential theories and literatures on children's geometric learning and provided directions for future research in this area.

d. Paths of Acquisition

Piaget's theory of spatial development. One of the earliest and most influential theories of the development of spatial and geometric concepts was put forth by Piaget and Inhelder (1967), who proposed that young children initially conceptualize space and spatial relations topologically as characterized by the following properties: proximity, order, separation, and enclosure. With development, children subsequently begin to represent space in a projective fashion, that is, in relation to different points of view, and then sometime between middle and late childhood the Euclidean conceptual system emerges permitting preservation of metric relationships such as proportion and distance.

Although numerous studies have been carried out to test the validity of this theory of "topological primacy," the consensus of investigators who have reviewed the empirical literature is that evidence supporting this developmental model is comparatively weak. One central criticism of this theory has been that Piaget and Inhelder's uses of terms such as *topological*, *separation*, and *proximity* are mathematically erroneous (Clements & Sarama, 2007a; Kapadia, 1974). For example, as Clements and Sarama point out, Piaget and Inhelder (1967) maintain that children do not synthesize the concepts of proximity, separation, order, and enclosure to construct the notion of continuity until the emergence of the formal operations stage (at approximately 11 or 12 years of age). However, in direct contrast to comprising a synthesis of these four properties, continuity is itself a central concept in topology (McCleary, 2006). And thus as Clements and Sarama note, the claim that this concept does not develop until early adolescence undermines the argument for the primacy of topological concepts (Darke, 1982; Kapadia). Concomitantly, these authors indicate that classifying figures as topological or Euclidean is problematic given that all figures possess attributes of both.

Clements and Sarama (2007a) conclude that although the empirical evidence does in fact suggest that the spatial abilities of children develop considerably throughout the school years, young children are more competent than hypothesized by Piaget and Inhelder as they can reason about spatial perspectives as well as distances. Indeed, Liben (2002) cites research suggesting that implicit Euclidean concepts are present perhaps as early as birth or soon thereafter, and that visual experience may not even be necessary for this system to develop. Furthermore, she points out Piaget himself subsequently replaced his topological primacy model with a different theory (i.e., intra, inter, and transfigural relations; Piaget & Garcia, 1989).

More recently, Dehaene, Izard, Pica, and Spelke (2006) tested adult and child participants from an isolated Amazonian community to determine whether they possess intuitive geometric conceptions, notwithstanding their lack of formal schooling, experience with maps, and a language containing an abundance of geometric terms. These investigators demonstrated that both children and adults spontaneously made use of foundational geometric concepts, including points, lines, parallelism, and right angles when trying to identify intruders in simple pictures, and used distance and angular relationships in geometrical maps to locate hidden objects. Finally, although a comparison group of American adults performed at a higher level overall than the Amazonian adults, the two groups showed similar profiles of difficulty. Dehaene et al. concluded that the existence of core Euclidean geometrical knowledge in all humans is inconsistent with Piaget's hypothesis of a developmental progression from topology to projective to Euclidian geometry.

A final body of evidence emanating from research with adults has yielded findings that are also inconsistent with the developmental trajectory proposed by Piaget and Inhelder. According to Newcombe and Huttenlocher (2006), although the Piagetian approach has stimulated much research on spatial thought for many decades, investigators who have focused on spatial cognition in adults have questioned the accuracy of characterizing mature spatial thought as explicitly Euclidean. They go on to describe how a good deal of evidence supports the assertion of cognitive psychologists that the spatial representations of adults are "inevitably erroneous, biased, and fragmentary" (p. 738).

Taken together, the mathematical inaccuracies of Piaget's topological primacy thesis along with the mounting, negative empirical evidence to date leads the Task Group to conclude that this theory lacks the kind of compelling support needed to make it useful for continuing to inform the design and testing of instructional approaches in geometry.

The van Hiele model of the development of geometric reasoning. The van Hiele model (1986) has been the dominant theory of geometric reasoning in mathematics education for the past several decades. According to this model the learner moves sequentially through five levels of understanding:

> ***Level 0: Visualization/Recognition*—**Students can name common geometric figures but usually recognize them only by their shapes as a whole, not by their parts or properties.
>
> ***Level 1: Description/Analysis*—**Students can judge a shape to be a certain type of figure based on its properties and can analyze component parts of the figures but cannot explain the interrelationships between figures and properties; they still do not understand definitions.
>
> ***Level 2: Informal Deduction or Ordering*—**Students can form definitions, establish interrelationships of properties within and among figures, and follow informal proofs but cannot construct one.
>
> ***Level 3: Formal Deduction*—**Students understand the significance of deduction as a way of establishing geometric theory within an axiomatic system, and comprehend the interrelationships and roles of undefined terms, axioms, definitions, theorems, and formal proof.
>
> ***Level 4: Rigor*—**Students can reason formally about different axiom systems.

The majority of high school geometry courses are taught at Level 3.

Battista (2007) has recently carried out a review and analysis of the strengths and weaknesses of this theory, and new developments pertaining to it, including 1) extending the level descriptors from two-dimensional to three-dimensional shapes, 2) reexamining the nature of levels, 3) elaborating the levels and proposing alternatives though related conceptions, 4) considering the idea that different types of reasoning develop simultaneously but at different rates, 5) judging whether the developmental periods should be viewed as stages or levels, and 6) evaluating extant methods of assessment. Battista concludes that the van Hiele theory provides a generally valid description of the development of students' geometric reasoning, especially pertaining to the learning of shapes (see Clements & Battista, 1992, for a detailed review of the supporting evidence).

Cognitive processes underlying the van Hiele levels. From the standpoint of the Task Group's analysis, empirical examinations of the cognitive processes underlying van Hiele levels of geometric reasoning is a much more challenging endeavor. Nonetheless, such an approach is crucial for making further progress in this area, as well as designing appropriate assessment tasks for use in both research and practice. The Task Group thus concurs with Battista's (2007) comment that "…it is one thing to devise broad categories of behavioral descriptors; it is another to determine the cognitive processes underlying these categories of behaviors. This has been and will continue to be, a major challenge facing researchers" (p. 854).

Although theories such as the hierarchical interactionism model developed by Clements and colleagues (Clements & Battista, 1992; Clements & Sarama, 2007a) represent a major step in this direction, the Task Group agrees with Battista's (2007) perspective on the state of the science:

> Although a number of theories and studies have been reviewed in an attempt to describe the cognitive processes by which students progress through the early van Hiele levels, this area of research is still in its infancy. This is due in great part because researchers are investigating cognitive processes that cannot be observed. To achieve progress in this domain, it is important for mathematics education researchers to heed the work of researchers in other fields such as cognitive science and neuroscience. Such research can provide valuable insights into these difficult-to-observe processes (pp. 858–859).

Numerous advances have been made in recent years regarding the development of spatial cognition, including topics such as spatial visualization, spatial relations, spatial orientation, spatial perception, spatial memory, spatial reasoning, and spatial and visual imagery. Excellent reviews of this rich research literature can be found in Liben (2002), Newcombe and Huttenlocher (2000; 2006), and Tversky (2004). Nevertheless, comparatively few cognitive or developmental psychologists have explicitly studied the development and learning of Euclidean geometric concepts and skills. Having said this, it should be noted that Koedinger, Anderson, and colleagues have been applying theory-based cognitive processing approaches to instructional interventions in geometry for many years (see Ritter, Anderson, Koedinger, & Corbett, 2007 for an overview of this and related work, as well as recent work by Kao & Anderson, 2006, 2007; and Kao, Roll, & Koedinger, 2007). Finally, research on the cognitive neuroscience of geometric reasoning is just beginning to get off the ground (Kao & Anderson, 2006).

e. Obstacles to Mastery

Earlier, the Task Group mentioned how difficulties in learning about ratio and proportion can present obstacles to understanding the meaning of similarity of triangles. Additionally, research on some intriguing geometric misconceptions are described below which are also relevant to understanding characteristics of shapes, albeit with respect to the concept of area. As such, future efforts to design instructional approaches for overcoming these kinds of errors may assist students in acquiring concepts crucial to understanding the definition of similarity.

Same-perimeter/same-area misconception. Dembo et al. (1997) examined a common misconception involving the relationship between area and perimeter—namely, that shapes with the same perimeter must have the same area. In fact, shapes with the same perimeter frequently have different areas, and the area increases as the shape becomes more regular. Thus, as these authors note, for rectangles having a constant perimeter, area increases as the figure approaches a square and decreases as it approaches a line. Dembo et al. tested the effects of schooling on this misconception by comparing the performance of two groups of Israeli students: one that attended ultra-orthodox schools and had received virtually no instruction in math and science, and the other that attended mainstream schools and received extensive instruction in these areas.

Surprisingly, the ultra-orthodox 12- to 14-year-old group correctly solved the geometric misconception problems more frequently than did their mainstream peers. The authors suggested two possible explanations of this unexpected finding. According to one perspective, the relatively strong performance of the ultra-orthodox students may have resulted from a curriculum which cultivated proficiency in applying general cognitive strategies and in carefully implementing rules of analytical reasoning to solve problems. Alternatively, early conventional instruction in geometry and related topics may actually have had an adverse effect on mainstream students' geometric reasoning. That is, initial formal instruction may have inadvertently promoted this misconception as a consequence of students being presented with the concepts of perimeter and area pertaining to the same shapes and kinds of problems—and during one and the same course. As these investigators point out, since the same factors are used to compute perimeter and area for many types of shapes (e.g., length of the sides for squares, rectangles, and right triangles), students may deduce that area and perimeter are determined by the same variables, leading them to erroneously infer that when the perimeter remains the same under some transformation, the area must as well.

Illusion of linearity. De Bock and colleagues have recently reviewed numerous studies demonstrating what has come to be known as the "illusion of linearity." Essentially, this phenomenon consists of a misconception that the linear (or proportional) model can pertain to situations where it is in fact not applicable. More specifically, many students incorrectly believe that that if the perimeter of a geometric figure is enlarged *k* times, its area (and/or volume) is enlarged *k* times as well (De Bock, Verschaffel, & Janssens, 1998, 2002; De Bock, Van Dooren, Janssens, & Verschaffel, 2002; Freudenthal, 1983; Modestou, Gagatsis, & Pitta-Pantazi, 2004). Apparently, this misconception emerges not only in geometry, but also in elementary arithmetic (Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2005), probability (Van Dooren, De Bock, Depaepe, Janssens, & Verschaffel, 2003), algebra and calculus (Esteley, Villarreal, & Alagia, 2004).

Remarkably, a series of studies has shown that even with considerable scaffolding (e.g., supplying drawings, presenting the problem in another format, or providing meta-cognitive hints), the vast majority of 12- to 16-year-old students fail to solve these problems due to a strong tendency to inappropriately apply linearity (De Bock et al., 1998; De Bock, Van Dooren, et al., 2002; Modestou et al., 2004). Van Dooren, De Bock, Janssens, & Verschaffel (2005) note that additional research suggests that this tendency is attributable to a set of closely related factors, including "the intuitiveness of the linear model, shortcomings in students' geometrical knowledge, inadaptive attitudes and beliefs towards mathematical (word) problem solving, and a poor use of heuristics" (p. 266).

De Bock, Verschaffel, Janssens, Van Dooren, and Claes (2003) explored the potential utility of two additional factors with 13- to 16-year-olds: 1) the authenticity of the testing context (i.e., prefacing the test with well-chosen, meaningful video fragments and linking all test items directly to these), and 2) the integrative use of drawings (i.e., having students draw a reduced copy of the figure described in the problem before trying to solve it). Neither of these manipulations improved performance. Additionally, both factors actually produced a negative effect. After exploring several possible explanations for this unanticipated finding, the authors conclude that, "Most likely, only a long term classroom intervention, not only acting upon students' deep conceptual understanding of proportional reasoning in a modeling context, but also taking into account the social, cultural and emotional context for learning, can produce a positive effect in defeating the illusion of linearity" (p. 460).

f. Conclusions and Recommendations

Classroom

Teachers should recognize that from early childhood through the elementary school years, the spatial visualization skills needed for learning geometry have already begun to develop. In contrast to Piagetian theory, young children appear to possess at least an implicit understanding of basic facets of some Euclidean concepts, although proper instruction is needed to ensure that children adequately build upon and make explicit this core knowledge for subsequent learning of formal geometry. Additionally, whereas children can reason to some extent about the properties of and relationships among different shapes, their developing abilities to acquire more detailed information about the metrics of these properties and the changes that occur under various transformations in the plane is by no means simple and straightforward.

Training

Teachers. Teachers need to learn more about the latest research concerning the development of children's spatial abilities in general and their geometric conceptions and misconceptions in particular. Acquiring knowledge of the spatial skills children bring to school with them, the limitations of these early developing competencies, and their use and misuse of shape words and names can help teachers capitalize on children's strengths and aid them in overcoming their weaknesses.

Future researchers. The next cohort of researchers who will be investigating geometry learning need to have a firm grounding in cognitive development and spatial information processing, in addition to mathematics education. Although some math education researchers have explicitly linked their work to advances in these areas, future progress in studying students' geometry learning will require a blend of content knowledge, proficiency with multiple research methods, and theoretical sophistication across several different disciplines. In addition, research teams composed of people who each possess a relevant area of expertise may be even more likely to help advance the study of geometric reasoning and evidence-based approaches to instruction in this domain.

Curriculum

Early exposure to common shapes, their names, and so forth appears to be beneficial for developing young children's basic geometric knowledge and skills. However, comparatively little is known about what the long-term effects would be of including a foundational treatment of more complex geometric concepts in preschool through second-grade curricula. Moreover, despite the widespread use of mathematical manipulatives such as geoboards, dynamic software, and so forth during the elementary school years, rigorous evidence is lacking as to precisely when and how these should be implemented to help children acquire a foundational understanding of concepts such as congruence, similarity, transformations, and angles. Finally, while a judicious reliance on manipulatives may enhance the initial acquisition of some concepts under specified conditions, students must eventually transition from concrete (hands-on) or visual representations to internalized abstract representations. The crucial steps in making such transitions are not clearly understood at present.

Research

***Basic*.** Longitudinal studies are needed to assess more directly how developmental changes in spatial cognition can inform the design of instructional units in geometry. Studies are needed to demonstrate whether and to what extent knowledge about similar triangles enhances the understanding that the slope of a straight line is the same regardless of the two points chosen, thus leading to a more thorough understanding of linearity.

***Classroom*.** More research is needed that specifically links cognitive, theory-driven research to classroom contexts. At the same time, cognitive theorizing pertaining to geometry learning needs to take into account more facets of classroom settings if it is to eventually have a large impact on the design of instructional approaches.

5. Algebra

This Task Group acknowledges the existence of arguments for early algebra learning, that is, implicit knowledge in elementary-school children's solving of arithmetic and other problems (Carraher & Schliemann, 2007). At this point, it is not known if the early algebra achievement of elementary school children reflects an actual implicit understanding of aspects of algebra, or if their performance is the result of the mathematical relation between algebra and arithmetic and not an indication of accumulating implicit knowledge. In either case, the Task Group focuses on explicit algebra content typically encountered in middle school to high school algebra courses. The bulk of the cognitive literature related to learning of this content focuses on simple linear equations and word problems; the Task Group summarizes the major findings from these studies below. The research literature for many of the remaining conceptual and procedural competencies identified within the Major Topics of School Algebra listed in the *Report of the Task Group on Conceptual Knowledge and Skills* is not sufficient for this Task Group to draw conclusions about the cognitive processes that contribute to these aspects of algebra learning; in many cases, sound studies are simply nonexistent.

a. Algebraic Equations

The majority of cognitive and learning research on algebra examines the processes underlying the solution of linear equations and the sources of problem-solving error. Some of the studies also include more complex equations (e.g., quadratic), but not enough research is available to discuss findings on these types of equations separately. The Task Group does, however, note a few common sources of problem-solving errors when students attempt to solve more complex algebraic equations.

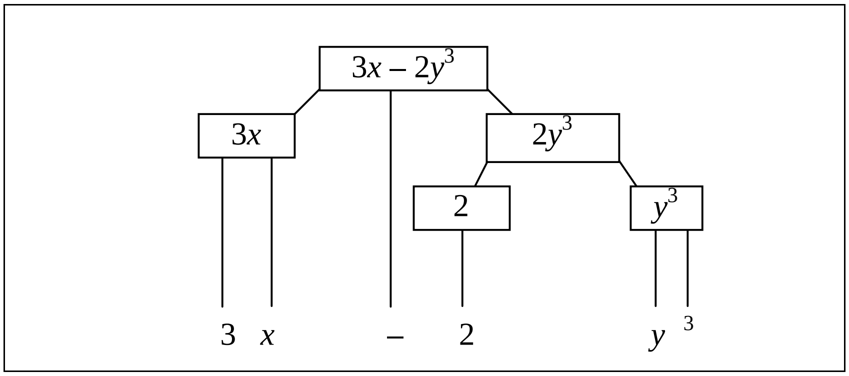
Paths of Acquisition

Conceptual and declarative knowledge. Studies of skilled adults and high school students who have taken several mathematics courses reveal that the processing of algebraic expressions is guided by an underlying syntax or system of implicit rules that guides the parsing and processing of the expressions (Jansen et al., 2003, 2007; Kirshner, 1989; Ranney, 1987). The learning of this syntax is not completely analogous to learning the syntax and grammar of natural language, because learning the syntax of algebra is strongly influenced by schooling. Learning of algebraic syntax is determined, in part, by earlier learned arithmetic rules, such as the order of operations; use of the commutative, associative and distributive properties; and by knowing the mathematical meaning of symbols, such as parentheses or summation signs, that note subexpressions within the equation.

Following Jansen et al. (2007), the parsing tree in Figure 2 illustrates the basic process followed by mathematicians when they solve algebraic equations, as revealed by an experimental method (Restricted Focus Viewer) that restricts the amount of information that can be viewed at one time and tracks the pattern with which components of the equation are processed. They typically scan the equation from left to right, but the variables, numbers, and exponents are not processed as individual symbols but rather as meaningful chunks, each of which is decomposed in turn. The processing is also influenced by core symbols that define chunks, including brackets, parentheses, horizontal bars in division, summation notations (∑), and so forth. For instance, mathematicians initially scan the following rational subexpressions from top to bottom, not strictly from left to right. Without an understanding of the mathematical meaning of these subexpressions, a person who is unfamiliar with algebra may view the subtraction sign and division lines as a continuous horizontal line that would be scanned from left to right and then top to bottom.

$$\frac{5x - 2}{2y + 7} - \frac{3y^2 - 1}{4}$$

Figure 2: Processing Linear Equations



Note: Experts scan algebraic equations in terms of meaningful chunks of information. For this expression, “$3x$” and “$2y^3$” are processed as chunks, and “$2y^3$” is then decomposed into the coefficient “2” or the variable, “y^3.”

These methods indicate that the solving of algebraic equations rests, in part, on learning the basic rules of arithmetic, mathematical meaning of core symbols, and eventually the automatic parsing of equations on the basis of this knowledge. Evidence for automaticity comes from the finding that skilled problem solvers scan and process basic sub-expressions in these equations in a fraction of a second (e.g., Jansen et al., 2007). Comparisons of novices and skilled problem solvers reveal that this fast and efficient processing is possible because the skilled problem solvers have formed long-term memory representations of the basic structure of algebraic equations and the sequences of procedural steps that can be used to solve them (Sweller & Cooper, 1985).

A small-scale ($n = 33$) experimental study of college students’ algebraic rule learning (e.g., when multiplying variables with exponents, add the exponents, $y^3\ x\ y^7 = y^{10}$) revealed substantial benefits to cumulative practice. This group practiced already learned rules with newly introduced rules and was contrasted with groups that only practiced rules individually or received follow-up reviews and practice of individual rules (Mayfield & Chase, 2002). In comparison to the two other conditions, cumulative practice resulted in better short-term and long-term retention of individual rules and a better ability to apply rules to solve problems that involved the integration of multiple rules. One potential reason for the advantage of cumulative practice is that it provides a context for comparing, contrasting, and eventually discriminating between rules that might otherwise be used inappropriately (e.g., confusing the rule for $(y^3)^7$ with the rule for $y^3 \times y^7$).

One method used to experimentally demonstrate the existence of such long-term memory representations is to compare the memory spans of experienced students and novices for meaningful and meaningless equations. In these studies, increasing skill is associated with longer memory spans for mathematically meaningful expressions but not for meaningless expressions with the same number of characters. When given 90 seconds to remember expressions such as $6y + 5\ (2x - 7)$, 11th-graders with several years of high school mathematics could remember strings of 10 to 12 symbols, whereas they remembered $4\frac{1}{2}$ to $5\frac{1}{2}$ symbols when the same strings were presented in a meaningless way; e.g., 5 (2*x* + – 7 6*y*. Memory span for meaningful but not meaningless expressions increased with number of mathematics classes and with practice (Sweller & Cooper, 1985).

Students who are first learning algebra and adults who are not skilled in mathematics do not have long-term memory representations of basic forms of linear equations or the sequences of procedural steps that can be used to solve these equations. The absence of or failure to access these long-term memory representations does not necessarily preclude the solving of linear equations, as long as the individual understands the general arithmetical and algebraic concepts and rules needed to solve the problem. Unfortunately, there are often substantive gaps in this knowledge. The result is that many students make mistakes. Problem solving is sometimes complicated by the execution of mathematically correct, but unnecessary, procedures. As an example, when presented with $a(y + 3z) - x = 4a$ and asked to solve for x, many students will unnecessarily expand $a(y + 3z)$ [i.e., $ay + 3az$]; the problem can still be solved but now requires several added steps.

Birenbaum et al. (1993) used a promising strategy for identifying sources of common errors such as these. A diagnostic test in which individual problems varied systematically in terms of the knowledge needed for correct solution was administered to eighth- and ninth-grade students in Israel. The problems ranged from relatively simple (e.g., $3 + x = 6 + 3 \times 2$) to those with more complex subexpressions [e.g., $6 + 4(x - 2) = 18$]. The pattern of correct and incorrect solutions across problems allowed for the identification of the most likely sources of error. The most common errors occurred because many students failed to correctly divide when terms included a coefficient and a variable (e.g., $9x$), and had difficulty applying the commutative and distributive properties [e.g., $4(x - 3)$]. Other common errors resulted from a failure to correctly order the operations, and to correctly add and subtract numbers on both sides of the equation, especially signed numbers (e.g., $5x - 4$). Using the same methods, Birenbaum and Tatsuoka (1993) found that many Israeli 10th-graders did not recall the laws of exponents. The two most common errors resulted from failure to recall that $X^0 = 1$ and that $(X^m)^n = X^{mn}$. Incorrect factoring was also a common source of error.

Similar types of errors have been found in the United States and other countries. In these studies, moving terms from the left to the right side of an equation was a common point at which errors occurred (Anderson, Reder, & Lebiere, 1996; Cooper & Sweller, 1987; Lewis, 1981). In keeping with the division errors found by Birenbaum et al. (1993), for the problem, $\frac{[2(x+6)]}{y} = z$ (solve for x), one type of error involves moving y from the left to the right, rather than multiplying both sides of the equation by y. With this error, the right side of the equation reads $\frac{z}{y}$, rather than zy. These types of errors often reflect a poor conceptual understanding of the syntax of algebraic expressions.

A poor understanding of the concept of mathematical equality and the meaning of the "=" is common for elementary school children in the United States, and continues for many children into the learning of algebra. Many elementary school children believe that the equal sign is simply a signal to execute an arithmetic operation. On typical problems such as 3 + 4 + 5 = __, this misinterpretation does not cause any difficulty. However, on less typical problems (at least in U.S. mathematics textbooks), such as 3 + 4 + 5 = __ + 5, it causes most third- and fourth-graders either to just add the numbers to the left of the equal sign, and answer "12," or to add all numbers on both sides of it, and answer "17" (Alibali & Goldin-Meadow, 1993).

Knuth et al. (2006) extended this research to 177 U.S. middle school children. They assessed children's understanding of the equal sign as expressed in arithmetic (e.g., the meaning of "=" in 4 + 8 = __) and how this knowledge of equality was related to their ability to solve simple linear equations; e.g., $5x – 5 = 30$. As found with third- and fourth-graders, many of the eighth-graders in this study interpreted the "=" as indicating the outcome of an arithmetic operation. Only 31% of the eighth-graders understood it as representing the equality of the terms on the left and right side of the equation. When solving the linear expressions, 33% of the eighth-graders used an algebraic strategy and these students always (100%) got the correct answer. The other two thirds of students used a "guess and check" strategy—as is often found for U.S. students (Cai, 2004; Johanning, 2004)—or some type of arithmetic strategy and frequently erred in solving the equation.

About 75% of the eighth-grade students who understood mathematical equality used algebra to solve linear equations, compared to less than 20% who understood "=" as a signal to perform an operation. The relation between understanding the concept of mathematical equality and skill at solving linear equations held, when standardized mathematics achievement scores and algebra course work were statistically controlled.

One potential source of U.S. students' poor understanding of the equal sign is the way in which problems are presented in textbooks. McNeil et al. (2006) provided a systematic examination of four commonly used textbooks series in middle school and found that the most frequent presentation of "=" was in the context of 'operate-equals-answer' format; e.g., $4 + 7 = 2x + 3 = 11$ (see also Seo & Ginsburg, 2003). Other studies have indicated that use of this format contributes to students' interpretation of "=" as operational rather than relational (Baroody & Ginsburg, 1983; McNeil & Alibali, 2005). A relational interpretation of "=" is most common for problems for which operations are needed on both sides of the equation (e.g., $4 + 5 = 11 – 2$; $3x + 5 = x + 15$). Yet, less than 5% of problems in middle school textbooks in the United States use this format, reaching a maximum of 9% of problems in eighth-grade textbooks.

Although it has not been empirically assessed, it is possible that the tendency of simple arithmetic problems to be presented vertically in U.S. textbooks may make the transition to left to right horizontal processing of algebraic expressions more difficult than it needs to be. This is a readily testable hypothesis and, if correct, can be easily remedied with the presentation of simple arithmetic problems in a horizontal format beginning with first-grade textbooks.

Procedural bugs. Errors in the solving of algebraic equations are sometimes classified as procedural bugs in a way analogous to the "buggy rules" noted earlier for subtraction (Birenbaum, Kelly, Tatsuoka, & Gutvirtz, 1994; Schoenfeld, 1985; Sleeman, 1984; Sleeman, Kelly, Martinak, Ward, & Moore, 1989; Wenger, 1987). These errors can occur due to overgeneralized use of procedures that are correct for some problems or from a misunderstanding of the procedure itself. Schoenfeld described a number of these types of procedural errors, a few of which are illustrated in Figure 3.

Figure 3: Algebraic Bugs

Expression	Buggy/Incorrect Translation	Potential Source of Confusion
$(X + Y)^2$	$X^2 + Y^2$	$2(X + Y) = 2X + 2Y$
$\sqrt{(X + Y)}$	$\sqrt{X} + \sqrt{Y}$	$\sqrt{(XY)} = \sqrt{X}\sqrt{Y}$
$\frac{X}{Y+Z}$	$\frac{X}{Y} + \frac{X}{Z}$	$\frac{Y+Z}{X} = \frac{Y}{X} + \frac{Z}{X}$

Unfortunately the nature of these bugs often differs from one student to the next and often for the same student from one equation to the next (Birenbaum et al., 1994; Sleeman et al., 1989). The problem of stability arises because the same equation can elicit several different types of bugs, and many students make errors on the same problem from one time to the next for different reasons. Although many bugs do not occur with enough consistency to inform specific classroom practices, a few bugs may be consistent across and within students. Preliminary studies by Sleeman et al. suggest that remediation that focuses on these specific bugs can reduce their frequency. Follow-up studies—perhaps using the classification methods described by Birenbaum and colleagues (1993, 1994; Birenbaum & Tatsuoka, 1993), focusing on identifying the sources of error underlying stable bugs or classes of bugs and assessing the effectiveness of instructional strategies in correcting them—are needed.

b. Word Problems

The Task Group's review of algebraic word problems includes studies of college students, due to a shortage of studies of middle and high school students' performance on such problems. Results for these college samples are likely to underestimate the difficulty of solving word problems for high school students. In a few places, the Task Group includes research on multistep arithmetical word problems, because the core processes and sources of error appear to be similar for arithmetical and algebraic word problems. The Task Group also notes that the focus of some of this research is on problem-solving processes and not the learning of specific algebraic content. A review of the these studies is, nonetheless, needed because of the wide use of word problems in the mathematics curriculum, because the application of algebraic skills (e.g., in physics classrooms) is often in the context of word problems, and because student difficulty with solving word problems was identified as an area of concern in the National Survey of Algebra Teachers (Hoffer et al., 2007, Table 3).

Paths of Acquisition

Problem translation and solution. Mayer (1982) proposed that the solution of algebraic word problems requires two general sets of processes: problem translation and problem solution. Problem translation involves transforming the verbal statement of the problem into a set of algebraic equations. It determines how the student forms a mental representation of the problem. The generation of the representation starts with an understanding of the text within which the problem is embedded (Kintsch & Greeno, 1985). Text comprehension involves understanding not only the meaning and mathematical implication of specific words (e.g., "speed" implies a rate problem), but also the structure of the entire problem.

In an analysis of word problems presented in algebra textbooks, Mayer (1981) found that most problems included four types of statements: assignment statements, relational statements, questions, and relevant facts. Assignment statements, not surprisingly, involve assigning a particular numerical value to some variable. Relational statements specify a single relationship between two variables. Questions involve the requested solution (e.g., "What is X?"). Relevant facts involve any other type of information that might be useful for solving the problem. Problem translation involves taking each of these forms of information and using them to develop corresponding algebraic equations. The translation of assignment statements, questions, and relevant facts does not pose much of a problem for most high school and college students (Lewis & Mayer, 1987; Mayer, 1982; Wenger, 1987). However, discriminating relevant from irrelevant information (Low, Over, Doolan, & Michell, 1994) and determining if the problem is solvable (Rehder, 1999) are potential sources of difficulty for many students. Translation errors most frequently occur during the processing of relational statements.

An example is provided by a simple problem: "There are six times as many students as professors at this university" (Clement, 1982, p. 17). Clement presented this problem to freshman engineering students at a major state university and asked them to write an equation that represented the relation between the number of students (s) and the number of professors (p). Thirty-seven percent of the engineering students committed an error on this problem, typically $6s = p$. This type of error is fairly common (Hinsley et al., 1977) for at least two reasons. The first is that the syntax, or structure, of the relational statement suggests a direct (though incorrect) translation into an algebraic expression. So "six times ... students" is literally translated into $6s$. Second, many students appear to interpret relational statements as requesting static comparisons. In this example, $6s$ is used to represent the group of students and p to represent the group of professors. In other words, for many students the "=" does not represent the actual equality of $6p$ and s, but rather simply separates the two groups. Students who correctly translate this relational statement understand that s and p represent variables, not static groups. These students understand that to make the number of professors equal to the number of students, some type of operation has to be performed; the smaller quantity, p, has to be increased so as to make it equal to the larger quantity, s. This translation leads to the correct algebraic expression, $6p = s$. The finding that types of errors are common in college students who intend to major in engineering implies that mistranslations of relational statements are likely to be widespread.

In a large-scale study that included 8th- to 10th-grade students, MacGregor and Stacey (1993) demonstrated that it is not simply the syntax of the relational wording that makes the translation process a common point of error. Errors occurred even for problems in which the statement could be directly translated into an equation; e.g., "*z* is equal to the sum of 3 and *y*." For one problem—"I have $*x* and you have $*y*. I have $6 more than you. Which of following must be true?"—students were asked to choose from among five alternatives. Across grades, only 34% to 38% of the 8th- to 10th-graders chose the correct, $x = y + 6$, equation. There was no predominantly incorrect response across the four potential mistranslations, although mistranslations that involved addition or subtraction (13% to 22% of choices; e.g., $6 + x = y$) were more common than mistranslations that involved multiplication (3% to 15%, e.g., $6x = y$) (p. 222). Capraro and Joffrion (2006) also found a variety of translation errors for a sample of 668 middle school students, as did Sebrechts, Enright, Bennett, and Martin (1996) when undergraduates solved algebraic word problems from the quantitative section of the Graduate Record Exam (GRE). The pattern indicates there are many ways to mistranslate the same word problem, just as there are many potential procedural bugs when solving algebraic equations.

At the same time, relational information conveyed in a word problem can sometimes aid problem solving if this relational information is consistent with students' previous out-of-classroom experiences and if these experiences and the corresponding situational representation can be used to create non-algebraic solution strategies (Bassok, Chase, & Martin, 1998; Koedinger & Nathan, 2004; Martin & Bassok, 2005). Bassok et al. found that the majority of word problems presented in one U.S. textbook series across first to eighth grade used story situations that were consistent with everyday activities or with everyday uses of objects described in the problems. Koedinger and Nathan discovered that these types of word problems are sometimes easier to solve (i.e., lower error rate) than corresponding linear equations. An analysis of solution strategies and error rates revealed that this was due to the frequent use of non-algebraic, arithmetic-based strategies for solving the word problems; these might involve a guess and test approach whereby the presented quantities are added, multiplied, etc. until an answer is obtained. Although these high school students were more successful with use of these non-algebraic strategies, the question of whether this contributes to their learning of formal algebraic representations of problem situations remains to be determined (Koedinger & Nathan).

Hembree's (1992) large-scale meta-analysis of students' ability to solve mathematical word problems from first grade to college level also reveals contextual effects. Abstract problems were more difficult to solve than concrete problems (mean $r = -.14$; mean across studies) but the largest effect was for familiarity ($r = .40$). Familiarity was defined in such a way that it included familiarity with classes of word-problem (e.g., interest, compare, distance problems) or familiarity with the cover context (e.g., baseball, travel). Evidence for the importance of familiarity of problem class comes from the finding that contexts that were based on the students' personal interests ($r = .04$) or preferences ($r = -.04$) were not related to problem-solving skill; that is, it was familiarity with solving the class of problem (e.g., rate) and not students' personal interests.

The second general set of processes suggested by Mayer (1982) as necessary for the solution of algebraic word problems or problem solution refers to the actual use of algebraic or arithmetical procedures to solve the resulting equations. The same potential sources of error described for solving of linear equations can occur during this stage of solving word problems.

Schema development. Hinsley et al. (1977) showed that successful translation of algebraic word problems, as well as the solution of algebraic equations and many other problem types, is guided by *schemas*—these include the syntax of equations. Sweller and Cooper (1985) provided a useful definition: "Schemas are defined as mental constructs that allow patterns or configurations to be recognized as belonging to a previously learned category and which specify what moves are appropriate for that category" (p. 60).

In short, a schema is a long-term memory representation that makes possible fast and automatic recognition of key elements of an equation or word problem, enables the classification of the problem into a conceptual group (e.g., velocity problems, interest problems), and has a linked system of procedures that can be used to solve the problem (e.g., Larkin, McDermott, Simon, & Simon, 1980). It should be noted that this cognitive science approach to schema development differs from Piaget's less precisely defined concept of schema.

Morales et al.'s (1985) study of third-, and a combined group of fifth- and sixth-grade children's conceptual understanding and ability to solve arithmetic word problems illustrates the usefulness of the concept of schema. The children's conceptual knowledge was inferred based on how they sorted word problems into categories. The question was whether the sorts were based on conceptual similarity (e.g., combine versus change problems), (Carpenter & Moser, 1984; Riley, Greeno, & Heller, 1983) or on unimportant (surface) similarities in how the problems were worded (e.g., both about baseball). For both grade levels, more than two-thirds of the errors were conceptually based—e.g., using a procedure appropriate for some class of problem but not the current problem—rather than due to computational error. More important, the categories formed by third-graders were more strongly influenced by the surface structure of the problems than by any underlying conceptual similarities, whereas the categorizations of the fifth- and sixth-graders were more strongly influenced by conceptual similarities. Third-graders who tended to organize the problems on the basis of conceptual categories, rather than surface structure, were much more accurate at solving the problems than were their peers who focused on surface features; the fifth- and sixth-graders did not make enough errors to conduct this type of analysis. The emerging ability to categorize word problems based on is underlying concepts (e.g., whether the problem asks for quantities to be combined or compared) and the corresponding reduction in problem-solving errors is consistent with development of category-specific schemas.

Sweller and colleagues have demonstrated that one way in which schema development can occur with both algebraic equations and word problems is through the use of worked examples (e.g., Cooper & Sweller, 1987; Sweller & Cooper, 1985). Worked examples provide students with a sequence of steps that can be used to solve these problems. The students then solve a series of related problems that are in the same category (e.g., interest problems) and involve the same or a very similar series of problem-solving steps. Studies by Reed and colleagues (Reed & Bolstad, 1991; Reed, Willis, & Guarino, 1994) reveal that, at least for word problems, worked examples that include an explanation of procedures (e.g., rate as related to work per unit of time or distance per unit of time) and several examples are more effective than simply providing students with the procedural steps.

The associated experimental and quasi-experimental studies have shown that the use of well-developed worked examples has several advantages over more conventional teacher instruction followed by worksheet practice (Carroll, 1994; Cooper & Sweller, 1987; Sweller & Cooper, 1985; Zhu & Simon, 1987). It is likely to have similar advantages over unguided discovery learning. Because of the many potential bugs that students can commit when solving algebraic equations and the common translation errors for word problems, teacher presentation of a problem or two followed by worksheet practice or homework can result in students' repeatedly committing (and therefore practicing) these bugs or translation errors.

In comparison to conventional practice, students provided with worked examples solve problem in the same class (e.g., distance) faster and with fewer errors (Carroll, 1994). The benefits of worked examples for solving different classes of problems (i.e., far transfer) are mixed. Some studies have revealed no skill transfer from one class of worked examples to another (Sweller & Cooper, 1985), but other studies have found positive transfer (Cooper & Sweller, 1987). Cooper and Sweller provided preliminary evidence consistent with the hypothesis that worked examples promote the learning of schemas and the memorization of embedded procedures. A tentative conclusion is that use of worked examples promotes the automatization and transfer of procedures used across classes of problems but does not appear to promote the transfer of the schema, that is, the specific sequence with which embedded procedures are used to solve the different classes of problem.

These same studies also have implications for unguided discovery. By definition, students using discovery approaches to learn algebra are novices and similar in some ways to students given conventional worksheets for problem-solving practice; students in both situations typically devise their own problem-solving strategies. Unlike students provided with worked examples, students engaging in conventional practice attempt to solve problems using the general problem-solving means-ends heuristic (i.e., working backward from the goal) (Newell & Simon, 1972), and often fall back on arithmetical rather than algebraic representations of the problem (Koedinger & Nathan, 2004). Even students provided with worked examples for one class of algebra problem revert to the means-ends heuristic when asked to solve problems for which a problem-solving schema has not yet been learned.

Although use of the means-ends heuristic can be an effective approach in many problem-solving situations, it requires considerable working memory resources and results in an attentional focus on the problem-solving goal and not on learning the sequence of problem-solving steps (Sweller, 1989). An attentional focus on the problem-solving goal appears to interfere with learning the sequence of these steps; that is, learning the underlying schema and connecting the goal to the sequence of steps needed to attain it (Cooper & Sweller, 1987; Sweller et al., 1983).

These and other results suggest that the effectiveness of worked examples appears to be due, at least in part, to a reduction in working memory demands that accompany use of means-ends problem solving. The elimination of these working memory demands allows attention to be focused on learning the sequence of steps that can be used to solve the class of problem (e.g., velocity) illustrated in the worked examples (Sweller, 1989). Worked examples can also provide a means of practicing embedded procedures.

Although it has not been as extensively studied in the context of mathematics learning, research in other areas reveal limits on the effectiveness of worked problems. Worked problems are most effective during the initial stages of learning and lose their advantage over other methods, such as exploration, as the learners' level of competence in the domain increases (Kalyuga et al., 2001; Tuovinen & Sweller, 1999).

The above noted limitations in terms of transfer for worked examples were demonstrated by Blessing and Ross (1996). These researchers found that undergraduates who had attended a high school for mathematically and scientifically gifted students, Illinois Mathematics and Science Academy, were skilled at translating algebraic word problems into appropriate equations. Their skill was, in part, related to fast and automatic access to problem solving schemas (e.g., distance = rate × time) associated with common word problems (e.g., those involving motion, interest, etc.) and the situations presented in these problems (e.g., an investor receiving dividends). However, when the "cover story," or the way the problem was presented, was modified, but the underlying algebra needed to solve the problem was left unchanged, these students committed more errors. In a series of studies that included undergraduates, and 9th- and 10th-grade mathematics honors students, Bassok (1990) found that transfer from one problem type (e.g., banking) to another (e.g., manufacturing) occurred when students recognized that the problems were asking the same basic question, such as questions about the rate of change. This spontaneous transfer did not occur for all students and largely disappeared if the problem contexts were too different, even if the underlying similarity of the problems was not changed.

Reed and colleagues (Reed, 1987; Reed, Dempster, & Ettinger, 1985) found the same pattern for undergraduates. They also found that if students' recognized conceptual similarities between problems they were much more likely to draw the analogy and use the same problem-solving procedures; for example, mixture problems that involve determining percentage of acid in a solution are conceptually the same as alloy problems (e.g., percentage of tin in bronze)

In other words, retrieval of the problem-solving schema is tightly tied to the ways in which the corresponding class of problem (e.g., distance, interest) is typically presented in word problems. Modification of this cover story can result in failure to retrieve the appropriate procedural sequence or retrieval of the wrong sequence. Hembree's (1992) meta-analysis sheds some light on student attributes that might promote transfer across classes of word problem. Hembree found that students who were skilled problem solvers also were skilled at analogical reasoning (r = .56) and at drawing inferences (r = .49). Other student-level traits, such as creativity (r = .22) and critical thinking (r = .37) were less closely related to successful problem solving. The importance of analogical and inferential reasoning is consistent with transfer effects in other areas (e.g., Holyoak & Thagard, 1997), and Reed's (1987) studies of algebraic word problems.

Obstacles to mastery

Stumbling blocks to the mastery of algebraic equations and the ability to translate word problems into algebraic expressions and equations are multifold and reflected in the many sources of problem-solving errors described in the preceding sections. *In addition, a common obstacle to ability to solve algebraic equations is inadequate preparation in arithmetic.*

In keeping with this conclusion, Hembree's (1992) meta-analysis revealed that for ninth-graders, the best predictors of the ability to solve word problems were computational skills (r = .51) and knowledge of mathematical concepts (r = .56). Other predictors were intelligence (r = .44), reading ability (r = .44), and vocabulary (r = .26).

The deficiency in basic arithmetic skills includes poor knowledge of the properties of arithmetic (e.g., order of operations; commutative, associative, and distributive property; the laws of exponents) and committing arithmetic errors, especially the manipulation of signed numbers (e.g., -5) and rational expressions. Students who struggle with algebraic equations also make factoring errors and use algebraic procedures incorrectly (i.e., commit bugs). In comparison to skilled problem solvers, poor problem solvers do not process algebraic equations by breaking them into mathematically meaningful subexpressions, and often do not even understand the significance of symbols that signal the existence of a subexpression, such as parentheses. Many do not even have a good understanding of mathematical equality or the "=" sign. Translation of word problems, especially relational information, into appropriate algebraic expressions and the discrimination of relevant and irrelevant information are consistent sources of student difficulty.

At a cognitive level, problem-solving errors and learning the syntax of algebraic expressions and algebraic schemas are influenced by working memory (Ayres, 2001, 2006; Cooney & Swanson, 1990; Lee, Ng, Ng, & Lim, 2004; Pawley, Ayres, Cooper, & Sweller, 2005). Working memory limitations also make the processing of relational sentences in word problems and the discrimination of relevant from relevant information especially difficult (Cooney & Swanson). As noted for whole number arithmetic, the commitment of procedures, rules, and often-used facts to long-term memory will reduce the working memory demands associated with solving the problem, thus freeing resources for processing less familiar problem features. As described above, well-designed worked examples may be effective in allowing students to focus working memory resources on learning classes of algebraic problem and sequences of problem-solving steps. At the same time, worked examples that include redundant or extraneous information may increase the working memory demands of processing these examples and thereby make them less effective (Pawley et al.). Pawley et al. found that redundant information for one student may, however, be helpful for another; thus, the effects of including redundant or irrelevant information appears to vary with the working memory capacity and mathematical competency of the student.

In a small-scale experimental study, Kalyuga and Sweller (2004) demonstrated that the potential cost of including irrelevant information can be addressed with use of faded worked examples. Here, the amount of information provided is reduced as students' skill level increases. Use of these learner-adapted worked examples resulted in moderate gains (d = .46) in the ability to solve linear equations. Use of other learner-adapted systems, especially cognitive tutors, is also associated with improved skills in geometry and algebra, if the tutor is well integrated with classroom instruction and the overall curriculum (Koedinger, Anderson, Hadley, & Mark, 1997; Ritter et al., 2007). In a large-scale quasi-experimental study, Koedinger et al. demonstrated substantial gains (d = 1.2) in the ability to translate algebraic word problems into equations. More modest gains were found for scores on two standardized algebra tests (d's = .30).

In addition to working memory, accuracy at solving various forms of mathematics word problems, such as those found on the SAT, is also related to spatial abilities across samples ranging from gifted middle school students to college students (Casey, Nuttall, Pezaris, & Benbow, 1995; Casey, Nuttall, & Pezaris, 1997; Geary, Saults, Liu, & Hoard, 2000; Johnson, 1984). The relation between spatial abilities and problem-solving accuracy may be due to the skilled use of visuospatial diagrams [see Larkin and Simon (1987) for general discussion of the utility of diagrams] or representations of the core relationships described in the problem, particularly the translation of relational information. Providing diagrams or instruction on the use of diagrams reduces errors rates when college students solve multi-step arithmetical word problem (Johnson; Lewis, 1989; Lewis & Mayer, 1987).

In Hembree's (1992) meta-analysis, the use of diagrams was more strongly related to the ability of fourth- and seventh-graders to solve word problems (r = .54) than was the use of other heuristics. From second grade to college, direct instruction on use of diagrams was much more effective for promoting their correct use than was practice alone (d = 1.16). Findings from a recent study of Japanese (n = 291) and New Zealand (n = 323) algebra students are also consistent with the usefulness of diagrams, at least for some types of problems (Uesaka, Manalo, & Ichikawa, 2007). These results are promising. However, the benefits and limitations of diagrams for facilitating the solving of different types of word problems remain to be determined and are more strongly related to instructional issues than to learning of specific algebraic content.

The ability to solve word problems is also related to reading ability and nonverbal reasoning ability, above and beyond the influence of working memory (Lee et al., 2004). It is also very likely that other factors reviewed earlier, including motivation, self-efficacy, anxiety, and so forth contribute to skill development in algebra (e.g., Casey et al., 1997). Although cause-effect relations cannot be determined, Hembree (1992) found that skill at solving word problems was related to positive attitudes towards mathematics (r = .23) and problem solving (r = .20), self-confidence in mathematics (r = .35), and self-esteem (r = .27). These correlations are consistently lower than those found between measures of mathematical preparation (e.g., computational skills) and cognitive factors (e.g., use of diagrams) and skill at solving word problems. Studies that simultaneously assess all of these constructs are needed to fully understand their relative contributions. As an illustration, one small-scale (n = 42) correlational study simultaneously assessed several of these constructs as related to achievement gains in Algebra II (Jones & Byrnes, 2006). The analyses allowed for estimates of the unique contribution of multiple constructs and classroom-related behaviors. Completion of homework (*ß* = .36) was associated with higher end of class achievement test scores. Higher preclass knowledge of algebra (*ß* = .32) and self-regulation (*ß* = .33; e.g., ability to organize and self-check work) were also predictive of higher postclass scores. Frustration was associated with lower scores (*ß* = -.26).

Research on learning in general (described in the General Principles: From Cognitive Processes to Learning Outcomes section in this report) indicates a benefit for practice that is distributed across time, as contrasted with the same amount of practice massed in a single session. Pashler, Rohrer, Cepeda, and Carpenter (2007) provide a recent review, including discussion of one area of mathematics. Initial experimental studies with mathematics, specifically teaching probability, are consistent with the more general literature. As with other content areas, distributed and massed practice reveal no difference after 1 week (70 to 75% correct), but after 4 weeks, the distributed practice group correctly solved 64% of the permutation problems as compared to 32% for the group trained with massed practice (Rohrer & Taylor, 2006).

A unique study of the longer-term benefits of distributed practice was provided by Bahrick and Hall (1991). In this study, algebra and geometry tests were carefully constructed based on textbooks and New York State Regents Examinations from 1945 to 1985. These exams were administered to about one thousand 19- to 84-year-olds. Information was obtained on high school and college course work, grades, and standardized test scores (for a sub-sample), as well as mathematics-related occupations (e.g., math teacher) and other activities that would involve rehearsal of algebra or geometry after completion of formal schooling. The retention interval for algebra began with the last algebra course taken in high school or college; for geometry, it began with the completion of plane geometry in high school. These data allowed for estimates of the degree of retention over a 50-year period as a function of these variables.

Overall, there was a steady decline in algebraic skills once the last course was taken. Over a 50-year interval between their last mathematics course to the time of the study assessment, about two-thirds of concepts and procedures typically taught in Algebra I was lost. Students who received an A in Algebra I, but took no other mathematics courses, retained more than students who received a B and these students in turn retained more than C students. The rate of decline in algebra skills was similar across groups; across a 50-year interval, the performance of all of these groups remained above that of a control group who had not taken high school algebra. The best predictor of long-term retention of competencies in algebra was the number of mathematics courses taken beyond Algebra I. Students taking college calculus showed a 20% decline in algebra performance over the 50 years, controlling for occupational and other potential confounds. Students taking a course beyond calculus showed no decline in algebra skills during this interval. A similar pattern emerged for content typically covered in Algebra II, but the effects of additional course work were less pronounced for geometry. The course work results suggest that the distributed review and integration—which was likely to have occurred more consistently for algebra than geometry—of the material across years contributes to the retention of the material throughout adulthood.

c. Conclusions and Recommendations

Too many students in high school algebra classes are woefully unprepared for learning even the basics of algebra. The types of errors these students make when attempting to solve algebraic equations reveal they do not have a firm understanding of many basic principles of arithmetic (e.g., commutativity, distributivity), and many do not even understand the concept of equality. Many students have difficulty grasping the syntax or structure of algebraic expressions and do not understand procedures for transforming equations (e.g., adding or subtracting the same value from both sides of the equation) or why transformations are done the way they are. These and other difficulties are compounded as equations become more complex and when students attempt to solve word problems.

With respect to policy, the situation is not likely to improve substantively without concerted and sustained federal efforts to make focused changes in teaching and curricula from elementary school forward, and efforts to change the ways in which teachers and future researchers are trained. There are many gaps in our current understanding of how students learn algebra and the preparation that is needed by the time they enter the algebra classroom. Funding to encourage scientists to enter research in this area is needed and to encourage the formation of research teams that will translate basic science findings into the design of instructional interventions to be assessed for effectiveness in the classroom.

Classroom

Teachers should not assume that all students understand even basic concepts, such as equality. Many students will not have a sufficient understanding of the commutative and distributive properties, exponents, and so forth to take full advantage of instruction in algebra.

Many students will likely need extensive practice at basic transformations of algebraic equations and explanation as to why the transformations are done the way they are; for instance, to maintain mathematical equality across the two sides of the equation. Common errors, as illustrated in Figure 3, may provide an opportunity to discuss and remediate overgeneralizations or misconceptions.

The combination of explanation of problem-solving steps combined with associated concepts is critically important for students to effectively solve word problems. For both equations and word problems, it is important that students correctly solve problems before given seatwork or homework. If students are making mistakes, then there may be a risk they will continue to make these errors and thus practice them during seatwork or homework.

Training

Teachers. Teachers should understand how students learn to solve equations and word problems and causes of common errors and conceptual misunderstandings. This training will better prepare them for dealing with the deficiencies students bring to the classroom and for anticipating and responding to procedural and conceptual errors during instruction.

Future researchers. To implement the recommendations that follow, the next generation of researchers to study algebra learning will need multidisciplinary training in mathematics, experimental cognitive psychology, and education. This can be achieved through interdisciplinary doctoral programs or at a federal level postdoctoral fellowships that involve work across these disciplines.

Curriculum

There are aspects of many, if not all, current textbook series in the United States that contribute to the poor preparation and background of algebra students. Modifying textbooks so that operations (arithmetical and algebraic) are presented on both sides of the equation, not just the typical operate-equals-answer format, is just one example of how textbooks can be improved.

The use of worked-out examples that include conceptual explanation, procedural steps, and multiple examples holds promise for teaching students to solve common classes of problems.

Retention of algebraic skills into adulthood requires repeated exposure that is distributed over time. This occurs as core procedures and concepts are encountered across grades. In much of mathematics, distributed practice should naturally occur as students progress to more complex topics. However, if basic skills are not well learned and understood, the natural progression to complex topics is impeded. This is because students will continue to make (and potentially practice) mistakes. As an example, procedures for transforming simple linear equations are embedded in more complex equations and thereby practiced as students solve them. The practice will not be effective, however, if students incorrectly transform basic equations, as they often do.

Research

***Basic*.** The development of assessment measures that teachers can use to identify core deficiencies in arithmetic (whole number, fractions, and decimals), and likely sources of procedural and conceptual errors in algebra are needed. The early work of Birenbaum and colleagues appears promising in this regard.

Research that explicitly explores the relation between conceptual understanding and procedural skills in solving algebraic equations is needed. Research on how student's solve linear equations and where and why they make mistakes needs to be extended to more complex equations and other Major Topics of School Algebra identified by the Conceptual Skills and Knowledge Task Group.

The issue of transfer, that is, the ability to use skills learned to solve one type or class of problem to solve another type or class of problem, needs considerable attention. Of particular importance is determining the parameters that impede or facilitate transfer, as illustrated by the work of Reed and Sweller.

Research on instructional methods that will reduce the working memory demands associated with learning algebra is needed. Although there are individual differences in working memory capacity, aspects of instruction (e.g., using faded worked examples) may be modifiable in ways that reduce working memory demands. Instructional or curricular changes that reduce working memory demands appear to provide students with an enhanced potential to learn the procedure or concept that is the focus of instruction.

Longitudinal research is needed to identify the early (e.g., kindergarten, first-grade) predictors of later success in algebra.

Classroom. A mechanism for fostering translation of basic research findings into potential classroom practices and for scientifically assessing their effectiveness in the classroom is needed. Cognitive tutors for algebra illustrate how this can be achieved. Equally important, mechanisms for reducing the lag time between basic findings and assessment in classroom settings need to be developed.

E. Differences Among Individuals and Groups

1. Sex Differences

For large, nationally representative samples, the average mathematics scores of boys and girls are very similar; when differences are found, they are small and typically favor boys (Appendix C). However, there are consistently more boys than girls at the low and high ends of mathematical performance (Hedges et al., 1995). The overrepresentation of boys at the high end of mathematical performance has garnered considerable media attention and debate, but it has obscured the fact that average differences are small, if they are found at all, and has been a distraction from the goal of improving the mathematical competencies of both boys and girls.

An overview of sex differences in overall performance across a variety of national and international data sets is presented in Appendix C. Mean differences often favor boys but are small, with effect sizes ranging from -01 to .16. In adulthood, men have a small advantage on measures of quantitative literacy, but this gap has narrowed since 1992 (d = .21 in 1993, d = .11 in 2003). These results are consistent with similar analyses (Hedges & Nowell, 1995) and with meta-analyses that include smaller-scale studies (Hyde, Fennema, & Lamon, 1990). The magnitude of the gap may have diminished, but any such changes have not been consistent across grades or tests (Nowell & Hedges, 1998). More consistent sex differences have been found for some measures and for more select samples. As an example and as recently reviewed by Halpern et al. (2007), the male advantage (d ~ .40) on the SAT mathematics test has been remarkably stable during the past 40 years.

Differences are also consistently found at the low and high ends of performance, with more boys than girls at these extremes (Hedges & Nowell, 1995; Strand & Deary, 2006). In a large-scale prospective study (see section on Learning Disabilities later in this report), Barbaresi et al. (2005) found that about two boys for every girl met one or several diagnostic criteria for a learning disability in mathematics sometime before the end of high school. The ratio of boys to girls at the high end tends to increase as the cutoff becomes more selective. Across multiple national studies, Nowell and Hedges (1998) found the ratio of boys to girls in the top 10% of mathematics scores ranged from 1.3:1 to 2.5:1. In these same studies, the ratio of boys to girls in the top 1% ranged from 2.6:1 to 5.7:1. Differences at the extremes begin to emerge in elementary school (Mills, Ablard, & Stumpf, 1993) and possibly before kindergarten (Robinson, Abbott, Berninger, & Busse, 1996), and in past decades has been quite large in mathematically talented adolescents (Benbow & Stanley, 1983).

Because the mean differences on mathematics measures are not large, and because several recent books and reviews have discussed potential mechanisms underlying differences at the high end of the distribution (Ceci & Williams, 2007; Gallagher & Kaufman, 2005; Halpern et al., 2007), the Task Group does not provide an extensive review of these mechanisms. The Task Group notes that the differences in the ratio of boys to girls and men to women at the high end of mathematical performance is likely to be related to a combination of factors, including stereotypes and beliefs regarding the mathematical abilities of boys and girls; the advantage of boys and men in some forms of spatial cognition (these differences can be reduced with practice on spatial tasks; Terlecki, Newcombe, & Little, in press); greater interest of boys and men in abstract, theoretical occupations and activities; and, the typically greater variability of boys and men in many cognitive domains (for a review of the evidence see Halpern et al.). Additional studies that simultaneously assess all of these potential mechanisms are needed to determine the relative importance of each of them.

2. Race and Ethnicity

One explicit charge to the National Mathematics Advisory Panel is to determine the processes by which students from diverse backgrounds learn mathematics. It is widely documented that black and Hispanic students perform substantially less well in our nation's schools than their white and Asian counterparts. These achievement and attainment gaps are found across a host of schooling indexes, including grade point average; performance on district, state, and national achievement tests; rigorous course-taking; as well as, across behavioral indicators such as school drop-out, suspension and referral rates, and differential placements in special education, and programs for the talented and gifted.

a. The Achievement Gap

As documented in Appendix D and elsewhere, the mathematics performance gap is found from preschool to college (Ryan & Ryan, 2005), and across the full range of mathematical content areas. Even early on it tends to be manifested more on measures of mathematical concepts than on measures of mathematical computation (U.S. Department of Education, 2006; Hall, Davis, Bolen, & Chea, 1999).

It is instructive to examine mathematics performance differences for high schools that serve white, black and Hispanic students together. Byrnes (2003) has done so by analyzing NAEP outcomes. Results from this national data base show that, overall, these mixed race schools (and that had at least one student scoring above the 80th percentile in mathematics), enrolled 79%, 13%, and 8% white, black and Hispanic students respectively. Yet, among the students who scored at or above the 80th percentile in mathematics, 94% were white, whereas only 3% were black and 3% were Latino. Representing these numbers somewhat differently, 26% of the white students enrolled in these schools performed at or above the 80th percentile, as compared to only 7% of their black and Hispanic peers. White students were almost four times more likely than black and Hispanic students to reach this performance level.

Hughes (2003) found mathematics performance differences when comparing third-grade black and white students attending schools in a generally affluent school district. Specifically, differences were found even in the midst of a wealth of material and human resources available to black and white students. Elsewhere, Schmidt (2003) showed a black-white difference in performance on the TIMSS even when controlling for socioeconomic status (SES), and Nettles (2000) has reported a 100-150 point difference on the SAT that holds up across all income levels.

Defying easy explanation, in particular, are data from the most recent NAEP tests. Using average main NAEP mathematics scores for eighth-graders broken down by race and parents' highest level of education, it was found that 23 scale points separated black and white test scores for students whose parents did not finish high school. Yet, white scores were 37 scale points higher than black scores for students whose parents graduated from college. The pattern is similar for white-Hispanic test score differences. The gap favoring white students was 9 scale points for students whose parents did not finish high school but 22 scale points for children of college graduates. For 12th-graders, the white-black difference was 16 scale points for students of parents who did not finish high school; this difference jumped to 37 scale points for students whose parents were college graduates. The respective Hispanic-white differences were 8 and 22 scale points.

It seems that whatever explanations are offered for these patterns, they cannot simply be reduced to a focus on social standing or SES (to the extent that parents' education level is a marker for SES). The findings defy this straightforward explanation.

Attempts to close these achievement gaps should be done in ways that raise achievement for all students, while simultaneously raising levels at a steeper rate for black and Hispanic students.

b. Potential Sources of the Achievement Gap

In this section, the Task Group reviews research literature on potential explanations of why mathematics performance is comparatively low for black and Hispanic students, and potential approaches for raising their mathematics achievement levels.

Socioeconomic status (SES)

The conventional explanations for poor math performance for black and Hispanic students center on inadequate social experiences and learning opportunities linked to low socioeconomic status. Because black and Hispanic children are disproportionately poor, and because poor children perform less well, this then identifies the root cause of such performance deficiencies.

SES is a generic construct that has had many definitions over the years, including family income, parental education, and occupational prestige, among others. As documented in Appendix E, whether SES is defined in terms of parental education, poverty level, parental income, or a composite index, there is a consistent association between SES and mathematics achievement. The mechanisms linking these broad constructs to mathematical learning and achievement are not well understood, nor are the relationships among SES, ethnicity, and mathematics learning.

With respect to the latter issue, the findings are inconsistent. Stevenson, Chen, and Utal (1990) found that white-Hispanic-black differences on mathematics curriculum tests essentially disappeared when controlling for parental education and income level among fifth-grade but not third-grade students; yet, differences remained for reading scores across both grade levels. They also found that black and Hispanic mothers rated their children's performance in mathematics as more important than did white mothers. Schultz (1993) found that SES was a major predictor of mathematics performance for fourth- to sixth-grade black and Hispanic students from urban school districts.

Stewart (2006) focused on results obtained across multiple administrations of the National Education Longitudinal Study of 1988 (NELS:88) data set and found that among black secondary level students, the presence of household educational resources common in higher SES households, such as books, encyclopedias, and computers, predicted combined mathematics and science performance, in the 12th-grade. For these students, neither family income nor parental educational level was directly related to mathematics and science achievement. Hall et al. (1999) found that fifth- to eighth-graders' performance on the mathematics concepts and computations sections of the California Achievement Test was correlated with parental background (a measure which included but was not limited to highest level of formal education and highest math course taken) for white students but not for black students.

In another analysis of the NELS:88 data, Thomas (1999) found that both home-based and school-based factors predicted performance outcomes across ethnic groups. When controlling for school-based and home-based factors, the mathematics performance gaps across white, black, and Hispanic students diminished substantially. A similar result was obtained in a study by Byrnes (2003). Drawing on the NAEP for 12th-graders, classroom experiences and learning opportunity factors accounted for more of the variance in mathematics scores across white, black, and Hispanic students than did SES. After statistically controlling for differences in parental background and school-based factors, the performance gap among these groups was substantially reduced.

Are learning processes among ethnic groups similar or different?

The weight of evidence supports the conclusion that learning processes are more similar than different across ethnic groups. This is not to say that there are no differences in how children from different ethnic groups approach the learning of mathematics, but rather that there are many similarities.

Thomas (1999), for example, found that the configuration of variables that predict mathematics achievement for white, black, Hispanic, and Asian 10th-graders are generally the same. Stevens, Olivarez, Lan, and Tallent-Runnels (2004) also found that the same constellation of predictors of mathematics achievement generally held for white and Hispanic high school students. This result was essentially duplicated by Stevens and his colleagues (Stevens et al., 2006) in a study of Hispanic and white students from 4th through 10th grade.

In a more process-oriented study, Malloy and Jones (1998) found that the spontaneous approaches to mathematics problem solving that emerged in a sample of middle-class black eighth-graders were highly similar to those found in studies of white students. Similar results have been reported by Kuhn and Pease (2006) and Rhymer, Henington, Skinner, and Looby (1999).

c. Potential Cognitive and Social Influences

The literature focusing on cognitive and social influences on the mathematics learning and performance of black and Hispanic students does not have a sufficient number of experimental studies to provide definitive results. Much of the research in this area is correlational, but many studies have nonetheless incorporated sophisticated multivariate analyses that can be used to control for potential confounding variables and to provide at least weak tests of potential “causal pathways.”

Many of these studies have drawn on national secondary data sets rather than on primary data. The advantages are large samples with results that can be generalized across the nation. The disadvantages include greater reliance on self-report data and constructs based on these data that are often formulated in a post hoc fashion, and thus may not measure the potential mechanism as precisely as is possible in an experimental study. Further, many of the studies have not formulated hypotheses about specific social or cognitive mechanisms, nor about whether there are racial or ethnic differences on mechanisms that can be changed in ways that help to close the performance gap.

Nevertheless, in recent years several hypothesized conceptions and processes have shown promise with respect to explaining and potentially narrowing ethnic differences in mathematics performance. Prominent among these are 1) stereotype threat; 2) cognitive load; 3) engagement, effort, and efficacy; 4) strategy use; 5) constructive and supportive academic interactions; 6) collaborative learning; and 7) culturally and socially meaningful learning contexts.

Stereotype threat

In the last decade, there has been increasing research attention given to the concept of stereotype threat as a contributing factor to group differences under certain specified conditions. This conception, first offered by Steele (1992), and then elaborated by Steele (1997), and Steele and Aronson (1995, 1998), hypothesizes that groups can be subjected to societal stereotypes that stigmatize their ability to perform in certain domains. For historical and sociological reasons, blacks have been viewed in the United States as having low intellectual ability and women as having low mathematical ability. These perceptions are particularly vexing because often attached to them is the presumption that the diminished ability is inherent and thus an unalterable characteristic of the group. When placed in a relevant performance setting, members of the stigmatized group are vulnerable to performing below their potential because of anxiety about upholding the negative stereotype.

Relation to ethnic differences in mathematics performance

Stereotype threat is a promising conception that has offered plausible explanations for certain group differences in academic performance. Unlike many other hypotheses in this area, stereotype threat has been investigated using experimental methods, giving greater confidence in results that confirm this hypothesis. Yet, there are limitations that temper claims that this concept can account for ethnic differences in mathematics performance among school-aged children and youths.

First, much of the research on stereotype threat and mathematics performance has focused on gender differences rather than on race or ethnicity differences. The study of stereotype threat in black and Hispanic samples has focused on general academic ability or intelligence; hence, the outcome measures for ethnic minority samples have typically been more general academic or test-performance related and not mathematics learning per se.

Second, the preponderance of the most rigorously executed research on stereotype threat has been done with college students. This is an important factor because the effects of stereotype threat are predicted to be more evident among group members who have a great investment in doing well and are typically high performers to begin with. In other words, a preoccupation with performance under stereotype-threat conditions is only predicted to affect performance when students are concerned about doing well on the task; students who are not invested in learning mathematics may not be influenced by any stereotype that involves mathematics.

As a result, there is not a sufficient research base testing the potential influence of stereotype threat in school-aged populations or focusing on mathematics performance of black and Hispanic students. Theoretically (as explained in the next section), it is unclear whether stereotype threat for mathematics can speak to the performance outcomes of black and Hispanic students who are not substantially invested in doing well in academic contexts. Nevertheless, studies addressing this issue are urgently needed.

Potential mechanisms

A recent study illustrates mechanisms that may link stereotype threat to performance outcomes. Keller (2007) investigated mathematics performance in a sample of 108 secondary level students in Germany (race was not specified, but they are presumably largely white). The students were randomly assigned to a stereotype threat or a no threat condition. Mathematics tasks were either difficult or easy. Those assigned to the threat condition were told in advance that for the mathematics tasks they were about to perform, gender differences in achievement had been found. The students in the no-threat condition were told gender differences had not been obtained. Further, the extent of identification with doing well in mathematics was assessed for all participants. Girls who value doing well in math and who were placed in the threat condition had larger decreases in mathematics task performance, from a pre-established baseline, when they worked on more difficult items. For difficult items, girls who did not value doing well in math performed better under the threat condition than under the nonthreat condition. There were no effects for the easy items. A similar result with a college student sample was obtained in an experiment by Beilock et al. (2007).

Ryan and Ryan (2005) offered a conceptual model for the processes underlying how stereotype threat influences quality of academic performance. When the conditions of stereotype threat are present—for individuals in which the domain is of importance to them and a negative stereotype exists—reminding the individual of the stereotype results in performance avoidance goals. These in turn result in heightened anxiety and lowered self-efficacy. As with mathematics anxiety, heightened anxiety under these conditions can result in thoughts about competence intruding into working memory, which functionally lowers this core capacity. Poor self-efficacy can result in diminished effort when problems become difficult. Although no study to date has tested the full model proposed by Ryan and Ryan, recent research has confirmed that each of these processes individually is linked to lowered performance outcomes in the face of stereotype threat.

Consider the work of Smith, Sansone, and White (2007) involving a sample of white college females. They found that in the presence of a salient stereotype threat, participants who were high on achievement motivation were more likely to spontaneously adopt performance avoidance goals when working on a mathematics task than were students who were not high in achievement motivation. Schmader and Johns (2003) provided evidence consistent with the hypothesis that stereotype threat interferes with mathematics performance by reducing individuals' working memory capacity. In this investigation, white men did better than white women on a mathematics task in a stereotype threat condition, and this difference was associated with reduced working memory resources for the women. No gender differences on the mathematics task or a working memory measure were found in a nonthreat control condition. Additional research revealed essentially the same pattern for Hispanic students.

For a sample of undergraduate women, Beilock et al. (2007) extended the work of Schmader and Johns (2003) by using a mathematics task where the level of working memory demands could be manipulated. Women were assigned to either a threat or nonthreat condition and asked to solve high- and low-demand problems. For women in the threat condition, performance was particularly poor for high-demand problems. These women reported worries about the task and had thoughts about confirming the stereotype during problem solving; women in the nonthreat condition did not report these concerns. The authors reasoned that these thoughts and worries functionally reduced working memory capacity which resulted in worse performance on high-demand problems. These results confirm the hypothesis that threat can result in intrusive thoughts about confirming the stereotype—thoughts that in turn lower working memory capacity and thereby lower performance.

Ryan and Ryan (2005) also hypothesized that anxiety could influence performance under conditions of stereotype threat, and there is some supporting evidence. The work of Osborne (2007) is notable in this regard. His research was also done with college students, but race of participants was not specified. Men and women were randomly assigned to either a threat or a nonthreat condition. For women, when using indexes of heightened anxiety, there were lowered levels of skin temperature, elevated levels of skin conductance, and heightened levels of diastolic blood pressure under the threat condition. No gender differences in physiological reactance occurred under the nonthreat condition. Moreover, women performed worse than men on the mathematics measure in the threat condition but not in the nonthreat condition. Ben-Zeev, Fein, and Inzlicht (2005) also found evidence for heightened arousal levels in women under conditions of stereotype threat.

Curiously few investigations have tested ways to alleviate the adverse influences of stereotype threat on performance. In one of the few studies that have done so, Beilock et al. (2007) found that extended practice on the difficult mathematics problems, which should make solving these problems more automatic and less dependent on working memory, eliminated the decrease in performance associated with stereotype threat.

A study by Good, Aronson, and Inzlicht (2003) is unique in that they attempted to enhance performance for stereotyped groups through a systematic intervention for school-aged children. This was a field experiment employing a sample of predominantly low-income, predominantly ethnic-minority seventh-graders; 67% Hispanic, 13% black, and 20% white. For the treatment condition, these students were mentored across an academic year by college students who encouraged them to regard intelligence as pliable rather than fixed and/or to attribute academic difficulties in the seventh grade to the uniqueness of the academic setting; but mentors also explained that academic performances can be improved over time. A control group of students was provided information linked to an antidrug campaign. The outcome measure was performance on a statewide standardized test of mathematics and reading achievement. The results revealed that girls' performance was substantially better in mathematics under the treatment condition than under the control condition. Boys performed essentially the same across conditions, with the exception of marginally significant ($p < .06$) better performance in the treatment condition (i.e., mentored) than the control condition. For reading, there was an overall main effect (across gender) for condition such that treatment students did better than control students.

These are striking results, but in this investigation, stereotype threat was not directly manipulated. The findings are encouraging in that academic performance was significantly improved in groups that often are stereotyped as doing poorly on academic measures. Because of the design of the study, however, it is not known if the improved performance was due to alleviation of vulnerability to stereotype threat or to other factors such as increased effort.

Cognitive load

As the Task Group described in previous sections, there is considerable evidence that when the working memory system is overloaded, performance in many domains including mathematics suffers. Putting in place procedures to reduce this load can enhance performance. The Task Group has documented how task practice leads to more automatic processing and thus reduces the working memory demands of the task. In the previous section, it was reported that practice at a task reduced vulnerability to stereotype threat in a sample of college women. Interventions that reduce cognitive load should improve the performance of all students. It would seem to follow that interventions which improve working memory functioning for low-achieving black and Hispanic students have high potential value.

Engagement, effort, and self-efficacy

In the earlier section in this report on Social, Affective and Motivational Influences on Learning, the Task Group reviewed work that indicated in general the positive influences that engagement, effort, and self-efficacy can have on mathematics performance. In reviewing research more specifically targeted to mathematics learning and performance of black and Hispanic students, the evidence strongly suggests that to the extent that such processes are positively manifested, mathematics performance can be improved. Findings in support of this conclusion have been documented across the full kindergarten to 12th-grade spectrum. These factors are more likely to be linked directly, rather than indirectly (e.g., as indexed by SES), to mathematics performance, and account for much more variance in mathematics outcomes than do global family background factors. Moreover, these processes are substantially malleable and can be changed in learning and classroom settings.

On the other hand, evidence suggests that important processes, such as effort devoted to school performance, are comparatively low for black and Hispanic students in traditional learning and classroom settings. The typical research investigating processes such as engagement, effort, and efficacy in black and Hispanic populations has not made use of experimental paradigms. These types of studies need to be conducted to determine how and why these processes influence mathematics learning and performance in ethnic minority populations, and how they can be improved in these populations.

In recent years, research has documented that general motivation level is functionally linked to mathematics outcomes for black and Hispanic students. A recent study by Borman and Overman (2004) is a case in point. They set out to determine the factors that differentiate between academically successful and unsuccessful black, Hispanic, and white students from low-income backgrounds. They examined such students' trajectory from third-grade to sixth-grade performance using the Comprehensive Test of Basic Skills, Fourth Edition (CTBS/4) math scores from the *Prospects* national data set. This was a congressionally mandated study conducted between 1991 and 1994 as part of the federal evaluation of Title I at the elementary school level. The focus was students who performed comparably in the third grade but whose performance diverged substantially in the sixth grade. Students whose scores increased substantially were labeled *resilient* and those whose scores declined were termed *nonresilient*. The percentile ranks for the two groups were 39th and 38th respectively in third grade. In the sixth grade, the percentile ranks were 59th and 11th, respectively, for the resilient versus the nonresilient group. Students were polled each of the four years of the investigation on certain beliefs, attitudes, and practices pertaining to their schooling experiences, and for each factor, average ratings were calculated. One factor that distinguished the resilient from the nonresilient children was having a positive attitude toward school.

In the previously cited Stewart (2006) study, the one factor that stood out as a predictor of combined mathematics and science achievement for the black students was general motivation level. This measure included items such as the importance of getting good grades and satisfaction from doing well in school. A similar result was obtained by Byrnes (2003) with 12th-grade black and Hispanic students, as well as white students. In a recent study, Balfanz and Byrnes (2006) found that self-reported effort emerged as a significant predictor of yearly gains in mathematics performance for black and Hispanic middle school students from an "urban background;" the gains were in terms of whether the students' performance exceeded what would have been expected by average yearly grade-equivalent increments. This outcome, by implication, suggests that interventions such as the one described earlier (Blackwell et al., 2007) which focus on the importance and malleability of effort, have the potential to help reduce achievement differences in mathematics across racial and ethnic groups.

Sirin and Rogers-Sirin (2004) found that student engagement in school was among the two strongest correlates (among a host of variables) of combined math and English grades in a sample of middle class adolescent black students. Borman and Overman (2004) also found that student engagement differentiated between the academically successful and nonsuccessful students. In this investigation, student engagement was not a self-reported measured but instead was indexed by the extent to which teachers agreed that a student conveyed attitudes and manifested behaviors indicative of an interest in school work and a desire to learn.

It is of interest that Borman and Overman (2004) also reported significant race differences for predictor variables. It was found that black students overall had substantially lower student engagement scores than did their white and Hispanic counterparts. However, the previously described experimental study by Blackwell et al. (2007) indicates that engagement scores can be raised for low-income minority students through certain targeted interventions. They deployed an intervention strategy similar to that used by Good et al. (2003). For a description of Blackwell et al., see the *Goals and Beliefs About Learning* section in this report.

Self-efficacy has also been found to be an important correlate of mathematics achievement. In the Borman and Overman (2004) study, self-efficacy differentiated between resilient and nonresilient students. Elsewhere, Stevens et al. (2006) reported that across 4th to 10th grade self-efficacy was a significant correlate of math achievement for Hispanic and white students (SES level was not reported). Similar findings have been obtained in many other recent studies; Navarro, Flores, and Worthington (2007) for Mexican-American 8th-graders; Long, Monoi, Harper, Knoblauch, and Murphy (2007) for black low-income 8th- and 9th-graders; Stevens et al. (2004) for Hispanic and white 9th- and 10th-graders (41% of the sample were from low-income backgrounds); Byrnes (2003) for white, black, and Hispanic 12th-graders.

Two studies have found that Hispanic students have lower-levels of self-efficacy, on average, than their counterparts from other ethnic groups. In the Borman and Overman (2004) study, Hispanic students had lower self-efficacy scores than did black or white students (d = .27), and in the Stevens et al. (2004) study, Mexican American students had lower mathematics self-efficacy than their white school counterparts (d = .25).

These results, however, do not directly address the questions of the antecedents of self-efficacy and the factor(s) that can increase self-efficacy. At least two studies speak to these issues for ethnic minority populations. For a sample of black high school students, Gutman (2006) found that exposure to mastery goals in the classroom were associated with increased mathematics self-efficacy, as well as to higher mathematics grades. Similarly, students who espoused mastery goals had higher mathematics self-efficacy and higher mathematics course grades.

In a related study, Fuchs et al. (1998) produced noteworthy results through an intervention experiment designed to heighten students' mastery goal orientations. For the relevant part of this investigation, participants were second- to fourth-graders who began the school year at or near the bottom of their classes in mathematics performance; 78% of these participants were black. The dependent variable was performance on a curriculum-based mathematics test at the end of the school year. Instruction focused on fostering mastery-oriented beliefs through targeted activities across a full 17–18 weeks of the academic year. Over the course of the study, students were also provided opportunities to receive assessment feedback. Students were randomly assigned to one of three conditions: 1) mastery-focused plus assessment feedback, 2) assessment feedback only, and 3) a standard classroom instruction control condition. It was found that the mastery plus assessment treatment led to the highest end-of-year mathematics test scores, followed by the assessment only condition, whose participants in turn had higher scores than those receiving only the standard classroom instruction (for the mastery versus control difference, d = .94; for the mastery versus assessment feedback only difference, d = .42; and for the assessment feedback versus control difference, d = .43).

Strategy use

Studies of explicit instruction of problem-solving strategies indicate it is a potentially useful intervention for improving the mathematics achievement in racial and ethnic minority populations. Although strategy use has generally been understood to foster greater academic performance (e.g., Pressley and Woloshyn, 1995), much of this work has centered on reading performance. Of the many studies that have focused on mathematics, only a few have focused squarely on racial and ethnic minority populations.

In a study of the correlates of mathematics performance, Schultz (1993) found that for black and Hispanic fourth- through sixth-graders, higher self-reported academic motivation (for which self-regulatory strategies figured prominently) was associated with higher mathematics achievement test scores. Malloy and Jones (1998) found that in comparing successful and unsuccessful mathematics problem solvers among their sample of black eighth-graders, the more successful students were more likely to use a mix of strategies and more often verified their procedures than their less successful peers. The less successful students often guessed. Examples of the strategies employed by the successful students were drawing diagrams, looking for patterns, or systematic guessing and checking. Among the verification procedures employed were rereading problems, checking calculations, or re-doing the problems.

Fuson, Smith, and Lo Cicero (1997) conducted a classroom based year-long intervention with first-grade Hispanic students from low-income backgrounds to determine if explicitly teaching certain strategies would improve their mathematics outcomes. Specifically, the children were taught to think of two-digit numbers as quantities of 10s and 1s. By year's end, these children could add and subtract two digit numbers with regrouping on par with similarly aged children in eastern Asian nations.

Learning opportunities and constructive, supportive academic interactions

At the heart of Walberg's (1984) productivity model is the assumption that students will learn more if they are given more opportunities, more contact, and more exposure to settings where they can actually learn what is demanded, expected, or required of them. Correlational and quasi-experimental evidence supports this claim. There is also evidence that the broader settings in which the learning occurs can be important. Specifically, socially supportive learning contexts are tied to enhanced academic performance (Patrick, Kaplan, & Ryan, 2007), and there is accumulating evidence that these contexts are particularly effective for black and Hispanic.

With respect to learning opportunities, Byrnes' (2003) earlier described study is especially telling. Byrnes used several opportunities to learn variables, along with key attitudinal variables. Among these variables were number of algebra and calculus courses taken; use of worksheets; and student attitudinal factors such as self-efficacy in relation to and liking of (taken together as a composite variable) mathematics, perceived utility of mathematics, and the perception that mathematics is more than just memorization. When comparing white versus black and Hispanic students who scored at or above the 80th percentile in mathematics performance on the NAEP tests, there were no differences across these variables.

However when Byrnes (2003) compared black and Hispanic students who scored above the 80th percentile to black and Hispanic students who scored below this level, notable differences were found. Eighty-five percent of the minority students who scored above the 80th percentile had taken courses beyond Algebra I, whereas only 47% of minority students who scored below the 80th percentile took these courses. Twenty-nine percent of those scoring above the 80th percentile had worksheets at least once a week versus 59% of those below the 80th percentile. Sixty-nine percent of those above the 80th percentile expressed self-efficacy for/ liking of mathematics, as compared to only 35% of those scoring below the 80th percentile. Moreover, 75% of those scoring above the 80th percentile agreed that mathematics is more than just memorization, but this was found for only 25% of lower-scoring students. In contrast, perceived utility of mathematics was not a differentiating factor in these comparisons. For that matter, it was not a predictor of mathematics outcomes in this study. Yet, courses beyond algebra, worksheet use, and math memorization were all significant predictors (self-efficacy and liking as significant predictors were discussed in a previous section). While these are correlational data where cause and effect cannot be determined, the study nevertheless reveals significant difference within black and Hispanic students in attitudes towards and views of mathematics.

With respect to constructive and supportive social interactions, a qualitative study by Brand, Glasson, and Green (2006) deserves mention. They conducted in-depth interviews with five black students (four high school seniors and one college freshman) who were participating in a program designed to encourage them to become teachers. This is a highly selective program, in which students who finish high school are guaranteed four-year scholarships to college. Among other things, students in the study were asked to describe their experiences in mathematics class. One central theme across students, in terms of school success, was having meaningful interactions with their teachers. This was taken to mean experiences that included having teachers who validated their capabilities, were accessible and approachable, were supportive, and held high expectations for them.

These qualitative insights are consistent with empirical data from other investigations. Mooney and Thornton (1999) polled black and white seventh-graders from a range of SES backgrounds regarding their attributions for success in school. Although the relative endorsement of the various attribution types was the same within race (for example, effort was most favored by both black and white students), cross-race comparisons revealed some important differences. White students, more so than their black counterparts, attributed success to a student's own abilities. Black students, in contrast and to a much greater extent than white students, attributed success to rapport with their teachers. Also worth noting is a study by Casteel (1997) that asked the question, "Whom do you most want to please with your class work?" Of the more than 1,600 black and white middle school respondents from diverse SES backgrounds, 71% of all black respondents answered "my teacher," whereas only 30% of the white respondents answered in this way. The more common response from the white students was "my parents." This pattern of results suggests that for many black students, they do not just learn *from* their teachers, but also they learn *for* their teachers.

In a study focusing on low-income black students in 1st to 12th grades, Tucker and colleagues (2002) found that higher levels of classroom engagement were found when students reported that teachers were caring and interested in their doing well in school, and showed a personal interest in them. This teacher variable was the strongest predictor of student engagement in the study. In fact, the path analysis indicated that student engagement in class was directly related to this teacher factor; this pattern of findings had not been found in previous studies of white students. Other aspects of teacher behavior such as teacher structure—the extent teachers have fair and consistent consequences in response to student behavior, or provide clear feedback—influenced engagement only indirectly.

Other studies demonstrate the connection between interpersonal academic context and mathematics performance. In the Borman and Overman (2004) study, another variable that differentiated resilient from nonresilient elementary students in their mathematics test performance was positive teacher-student interactions in the classroom.

In a study of 12th-grade black students, Stewart (2006) found that a positive perception of the school environment (i.e., the perception that students get along well with teachers, have caring teachers, and teachers provide praise for good efforts) was a significant predictor of mathematics and science achievement. Elsewhere, Balfanz & Byrne (2006) report that the greater the number of "supportive classrooms" middle school black and Latino students participated in over time, the more likely the math performance gap would be closed between them and other racial/ethnic groups.

These results are consistent with claims concerning importance of supportive social contexts, especially support provided by teachers, for the mathematics achievement of black and Hispanic students. However, definitive results await use of experimental tests of potential causal mechanism. One possibility is that these social contexts result in greater engagement and increased effort in the classroom and through this better mathematics achievement. Another possibility is that these contexts reduce stereotype threat effects, namely cognitive overload, increased anxiety, or the promotion of performance avoidance goals. Perhaps students from certain social or cultural backgrounds have been socialized such that they are more responsive to the combined power of the school and classroom context.

Research, however, has been gathered in recent years to suggest that the interpersonal relationships that do occur in classrooms serving low-income African American and Hispanic students are often not supportive and may result in disengagement. Ferguson's (2003) research review and analyses are relevant to this point. He has gathered evidence that teacher expectations of future performance for black students are regularly more negative than they are for their white counterparts. Further, teacher perceptions and future expectations may affect future mathematics performance of black students, both positively and negatively, to a greater degree than that of white students. On the basis of data he had reported in 1998, Ferguson found that the relative influence of the teacher was nearly three times larger for black than white students in elementary school, whether the outcome was mathematics grades or mathematics achievement scores. In the Ferguson (1998) article, the data of interest examined the extent to which teacher perceptions of students' "performance, talent, and effort" measured in the fall semester predicted students' math achievement scores and math grades the following spring term (p. 286). The corresponding effect sizes for the prediction were .14 and .37 for white and black students, respectively, on the math achievement test, and .20 and .56, respectively, for mathematics grades.

A recent meta-analysis provides further evidence concerning teacher expectations (Tenenbaum & Ruck, 2007). Their review covered research done between 1968 and 2003. The majority of these reviewed studies focused on elementary students only (approximately 60%). The remainder included students at the secondary or university level, students across school levels, or in a few cases, unspecified sample characteristics. They found that teachers had more positive expectations for white than for black ($d = .25$) or Hispanic students ($d = .46$). Moreover, teachers directed more positive speech in the form of praise, affirmations, and positive feedback toward white than minority children. White students also received more product- and process-based questions, and therefore black and Hispanic students had fewer overall opportunities to respond academically in their classrooms. At the same time, the review did not reveal differences in the amount of negative speech directed at white, black, or Hispanic children. A study by Hauser-Cram, Sirin, and Stipek (2003) adds another dimension to this line of inquiry. They found that elementary school teachers held lower expectations for the future mathematics success of their current students to the extent that they perceived social and educational value differences between themselves and a student's parents. Although the finding was marginally significant ($p < .06$), elementary school teachers also perceived the difference between themselves and parents to be larger for black parents than for white parents.

Collaborative learning

The available evidence suggests that when properly structured, generally speaking, collaboration for learning can have a positive influence on mathematics performance and may be relatively important for minority students, particular those from low-income backgrounds. This finding appears to be especially robust at the elementary school level. Research for or against the effectiveness of collaborative learning at the middle and high school level is generally absent from the literature.

Perhaps the best source to assess the effects of collaborative learning on mathematics outcomes in elementary school comes from a meta-analytic review conducted by Rohrbeck, Fantuzzo, Ginsburg-Block, and Miller (2003). They set certain conditions for inclusion of studies in their review. Among these criteria, there had to be ethnic group comparisons, explicit peer assistance with interdependent reward contingencies, and independent accountability or evaluation procedures. The latter two conditions were necessary because the extant literature on peer-assisted learning indicates these conditions are crucial to positive outcomes. Other qualifiers were that all studies had to appear in peer-reviewed journals, and had to have used experimental or quasi-experimental designs. Moreover, the interventions had to be classroom-based and occur for more than 1 week. Ninety studies published between the years 1966 and 2000 met these criteria.

Overall, peer-assisted learning led to greater mathematics performance outcomes than did individual or competitively structured learning. But, the magnitude of these effects varied. Larger effects were found for: 1) urban versus rural and suburban settings, 2) low-SES versus middle and higher SES, and 3) minority status (black and Hispanic) versus majority (white) status. The effect sizes were .44 and .23 for urban and suburban/rural locations, respectively. In the case of SES, the mean effect size was .45 when more than 50% of the sample was low SES, and .32 when less than 50 % of the sample size was low SES. For minority status, the mean effect size was .51 when more than 50 % of the sample was minority status (black and/or Hispanic) and .23 when less than 50 % of the sample size was of minority status. The largest effect size was obtained for samples consisting of primarily black and/or Hispanic students, and the magnitude of the effect of collaborative learning on mathematics performance was largest when contrasting these ethnic minority children with their white counterparts.

To illustrate the type of research assessed in this meta-analysis, consider a study conducted by Ginsburg-Block and Fantuzzo (1997). These researchers contrasted a reciprocal peer-tutoring dyad condition with a condition where students worked individually. The dyads met 2 times a week across a 10 week intervention period. The sample consisted of fourth- to sixth-grade black students from low-income backgrounds. For the reciprocal peer-tutoring condition, the two students alternated between tutor and tutee. As the tutee answers test questions or performs a given task, the tutor prompts, provides feedback, and offers evaluative comments. The dyad work toward a common goal, that is, their reinforcement was contingent on the performance of both students. In this study, participants in the reciprocal peer-tutoring condition had higher mathematics classroom performance outcomes than those in the practice control condition; and they received higher ratings of teacher-observed task-relevant behaviors during mathematics lessons. It was also found that these reported engagement levels were positively related to scores on a mathematics curriculum-based computation test.

Socially and culturally meaningful learning contexts

One final area with promising research is with respect to socially and culturally meaningful learning contexts. The goal is to better link what happens in school to experiences, values, and practices that are salient in the lives of black and Hispanic students (Perry, Steele, & Hillard, 2003; Ladson-Billings, 1997; Moll, Amanti, & Neff, 2005; Sternberg, 2006). Much of the actual scholarship done to establish such links has typically not brought systematic empirical research to bear, and what empirical research that has been done, has most typically been linked to reading rather than math performance. There are a few exceptions, but even these have not used experimental methods or explored underlying processes that directly link social and cultural processes to academic outcomes. Yet, results from these few recent investigations indicate that this is an area of investigation that merits further study.

A recent randomized field experiment by Cohen, Garcia, Apfel, and Master (2006) addressed the usefulness of linking school performance to matters of personal relevance for black students. The participants were black and white seventh-graders from middle- to lower-middle-class backgrounds, and they all attended the same school. The authors describe their experimental treatment as a *self-affirmation* intervention. Early in the school year, students randomly assigned to this condition selected one or more of their most important values and then wrote brief paragraphs in which they justified why these values were chosen. The exercise was presented to the students as a normal lesson and took approximately 15 minutes to complete. After this exercise was completed, the teacher resumed the focal subject lesson. Students who had been randomly assigned to the "control" condition were asked to select one or more of the least personally important value(s) and write about why these values might be important to someone else. The same procedural protocol was followed for the control condition. Teachers themselves were blind to which students participated in what condition. Two parallel studies were conducted, separated by one year. In the first, students completed the exercise once; in the second, students completed the exercise twice in the fall semester. In the first, participants wrote about only one value; in the second they could choose up to three.

For both studies, the first semester course grades of the black students in the treatment condition were significantly higher than those obtained for the black students who participated in the control condition. No treatment effect was obtained for the white students in either study. The black students in the treatment condition did even better than their black control group counterparts in other courses for which the treatment did not occur. For this investigation, the actual course subject in which the treatment was provided was not specified. But, given that the authors stated that the subject was not one linked to gender stereotype, it is very likely these were not mathematics classrooms.

Although the Task Group noted earlier in the report concerns about sociocultural claims regarding learning, and specifically that many claims have not been scientifically evaluated, there are several studies from the sociocultural perspective that might provide insights for more fully interpreting some of the results described earlier.

Another relevant approach has been to focus on cultural values or themes that may be more prominent in certain populations than in others and that may enhance learning and performance outcomes for these populations. One such theme is *communalism* (Boykin, 1986; Boykin & Ellison, 1995), which has been hypothesized as being particularly prominent for many people of African descent, including African Americans. To be sure, there is no claim that all black people are communal or that communalism is a fixed trait of a given person or group of people. Rather, if this theme is more salient in the communities of blacks, then the corresponding social expectations may influence how children interpret and perform in school settings (Boykin & Allen, 2004).

Communalism as a cultural theme implies that a premium is placed on collaborative interdependence. If this is salient for many black Americans, then this could be one factor potentially contributing to the receptiveness to collaborative learning for black students described in the previous section. Moreover, if communalism is a culturally meaningful theme, then performance enhancements in group settings would occur even in the absence of individual incentives to perform well in collaborative settings, as found with reciprocal peer tutoring. This hypothesis has been tested in several experimental investigations, but only in a handful that have used mathematics achievement as an outcome variable.

One such experiment was conducted by Hurley, Boykin and Allen (2005). In this study, fifth-grade black children from low-income backgrounds were given opportunities to learn effective strategies for solving mathematics estimation problems, and then to examine their subsequent performance on a mathematics estimation test. Students were randomly assigned to one of two conditions. In one condition, and after completing a 15-item mathematics estimation pretest that used grade-appropriate multiplication problems, children were given a 20-minute practice exercise in which they had to complete a workbook to help them become more facile with mathematics estimation. During the learning/practice phase, these students were encouraged to work alone and prompted to exercise their individual effort and autonomy. They were also offered a reward if their posttest performance reached a certain criterion level. This was the individual learning condition. The other children were assigned to the communal condition. After the pretest, these children were formed into groups of three and given a prompt during the learning phase. The prompt emphasized the importance of working together for the good of the group so that everyone in the group could benefit and learn that it is important to help each other. These children were not offered a reward for good performance. They were told to work together but not told how to work together. It was reasoned that if interdependence were a salient theme for them, they would not need external incentives to do well, nor would they require explicit instructions on how to work together. Participants in both conditions worked on the follow-up 15-item mathematics estimation posttest on an individual basis. Results revealed that performance on the posttest was superior for those who had worked in the communal learning condition ($d = .56$).

This study had certain limitations, not the least of which was that the intervention only lasted for 20 minutes. However, results reported in a recent doctoral dissertation (Coleman, 2003) tentatively suggested that these effects can extend across a 4-week intervention done in conjunction with the actual classroom teaching of a fractions unit to third- and fourth-grade low-income black students.

An intriguing study with an international comparison is also worth mentioning. Huntsinger, Jose, Fong-Ruey, and Wei-Di (1997) examined cultural differences in early mathematics learning among European Americans, second-generation Chinese Americans, and Chinese students residing in Taiwan. They sought to determine if there were differences in family practices related to mathematics and if any family differences were related to children's mathematics outcomes in school. The focus was on children at the preschool and kindergarten levels. Families from all three comparison groups were from middle-class/professional backgrounds. It was found that Chinese Americans and Chinese families in Taiwan structured more daily time for homework or music practice and encouraged their children to participate in mathematics-related activities more so than did the white parents. Chinese American parents, and to a lesser extent the Chinese parents in Taiwan, engaged in more direct, formal teaching of mathematics to their children than did white parents. Moreover, Chinese American and Chinese children in Taiwan performed better than white children on the Test of Early Mathematics Ability (TEMA-2).

It was found that children who received more formal teaching at home and spent more time doing homework had higher mathematics test scores. Certainly this was a fairly small-scale study, with narrow SES backgrounds of the participants. Furthermore, it is not clear if the differences were actually due to cultural value factors, per se. But the implications for the importance of family organization of children's activities, as it relates to mathematics outcomes, is relevant to the Task Group's review of group differences in the mathematics competencies children bring to school.

3. Learning Disabilities

At least 5% of students will experience a significant mathematics learning disability (MLD) before completing high school, and many more children will show learning difficulties in specific mathematical content areas. Intervention studies are in the early stages and should be a focus of future research efforts. Further research also is needed to identify the sources of MLD and learning difficulties in the areas of fractions, geometry, and algebra.

The issues of diagnostic criteria and the percentage of children with an MLD remain to be fully resolved. Change in the stringency of the diagnostic criteria (e.g., cutoff on a mathematics achievement test) used to diagnose MLD can significantly influence the pattern of identified deficits and explains some differences in results across studies (Murphy, Mazzocco, Hanich, & Early, 2007). Nevertheless, progress has been made in the past decade. Using a population-based birth cohort sample that provided medical, academic, and other information on 5,718 individuals from birth to age 19 years, Barbaresi et al. (2005) assessed the incidence of MLD using different diagnostic criteria. On the basis of the two criteria that involved at least a one standard-deviation difference between an intelligence quotient (IQ) score and a math achievement score, 6% to 10% of children showed evidence of MLD before they completed high school (the potential relations among IQ, mathematics learning, and MLD are not yet known and thus control of IQ is important). An additional 6% of children were diagnosed as MLD using a more lenient criterion. The two discrepancy-based criteria yielded estimates similar to the 5% to 8% of children estimated as having MLD in previous studies (Badian, 1983; Kosc, 1974; Gross-Tsur, Manor, & Shalev, 1996; Ostad, 1998; Shalev, Manor, & Gross-Tsur, 2005). In one of these studies, Shalev and colleagues identified 5% of 3,029 5th-graders as having MLD and found that 40% of these children remained at or below the 5th percentile in math achievement in 11th grade. Almost all of the remaining children were in the lowest quartile in math achievement, despite average IQ scores, and most would have been diagnosed as MLD using at least one of the Barbaresi et al. criteria. The pattern across studies suggests that 5% to 10% of children will meet at least one relatively strict criterion for MLD before reaching adulthood and at least another 5% might be diagnosed as MLD using more lenient criteria.

These large-scale studies are important for identifying the percentage of children who likely have some form of MLD. Although they do not provide detailed information on the nature of the underlying deficits in mathematics learning or in the cognitive mechanisms (e.g., working memory) that contribute to these deficits they are nonetheless informative regarding the early deficits of children with MLD and illustrate the usefulness of this approach for studying learning disabilities in other areas of mathematics.

The cognitive and neuropsychological studies have revealed several sources of the poor early learning of arithmetic by children with MLD or other low-achieving children (Geary, 2004; Jordan et al., 2003; Ostad, 1998). The first involves delayed adoption of efficient counting procedures for problem solving and is manifest as frequent reliance on finger counting, infrequent use of the counting-on procedure, and frequent counting errors (Geary, 1990). The reliance on finger counting and the frequent counting errors are related to below-average working memory capacity. The delayed adoption of counting-on is related to a poor conceptual understanding of some counting concepts (Geary et al., 2004) and may also reflect a poor understanding of number and quantity per se (Butterworth & Reigosa, 2007). Many children with MLD eventually develop normal procedural competencies for solving simple arithmetic problems, although they usually do so several years after their peers.

A second source of the low achievement of these children involves difficulties in the learning or retrieving of basic facts (Jordan & Montani, 1997; Russell & Ginsburg, 1984). This is not to say these children never correctly retrieve answers, but rather that they correctly retrieve basic facts less often; at times, they also generate different pattern of retrieval errors. Although not conclusive, evidence to date suggests two potential sources of these difficulties. The first involves the formation of long-term memory representations of basic facts, and the second involve interference during the retrieval process (Barrouillet, Fayol, & Lathuliére, 1997; Geary, Hamson, & Hoard, 2000); interference is related to attentional and inhibitory control mechanisms of the central executive component of working memory. Whatever the source, short-term longitudinal and cross-sectional studies suggest that the difficulty in learning or retrieving basic facts is more persistent than the procedural delay (Jordan et al., 2003).

The central executive component of working memory has also been implicated in the procedural delays of children with MLD (e.g., Geary et al., 2007; McLean & Hitch, 1999; Swanson, 1993; Swanson & Sachse-Lee, 2001), and their deficits in this core cognitive competency will almost certainly result in delayed learning in novel and complex mathematical topics. The two other core components of working memory—the phonological loop and visuospatial sketch pad—may also contribute to MLD but in more circumscribed ways.

Butterworth and colleagues, however, have proposed that a poor “number sense” is the core deficit for children with MLD (Butterworth & Reigosa, 2007; Landerl et al., 2003). Number sense is defined in terms of the competencies that are evident in infants and young children and do not require formal schooling. These would involve, as an example, the ability to quickly subitize, or determine with a quick glance without counting that the quantity represented by ●● is less than that represented by ●●●. Deficits in these very fundamental areas would impede the learning of arithmetic in school. There is evidence consistent with the view that children with MLD have deficits in such areas, independent of any deficits in the central executive (e.g., Koontz & Berch, 1996; Jordan et al., 2003; Landerl et al.).

The studies of number sense deficits tend to include children often with average cognitive ability but with lower achievement, as evidenced by test scores, than is typical for research on MLD. It is possible that there are multiple forms of MLD. Central executive deficits would result in a broad range of deficits in mathematics and other areas. Deficits in the number sense system—potentially involving the intraparietal sulcus (see section on Brain Sciences and Mathematics Learning)—would be associated primarily with difficulties understanding quantity and, of course, with all of the mathematics dependent on this knowledge.

Much less is known about MLDs in relation to learning other areas of arithmetic, and very little is known about the specific deficits associated with learning fractions, estimation, geometry, or algebra. The work that has been conducted suggests that children with MLD, and often more general learning disabilities, have difficulties with arithmetic algorithms (Russell & Ginsburg, 1984), quantitative estimation (Hanich, Jordan, Kaplan, & Dick, 2001), rationale numbers (Mazzocco & Devlin, in press), and with algebraic equations and word problems (Hutchinson, 1993; Ives, 2007). Further studies about learning disabilities and learning processes in these and related areas of mathematics are needed, as are studies of the underlying cognitive mechanisms (e.g., central executive component of working memory and basic number knowledge) and brain systems (e.g., areas of the prefrontal cortex that support working memory, and areas of the parietal cortex that support number-related processes and representations; see section on Brain Sciences and Mathematical Learning).

The Task Group also notes that many students with MLD have comorbid reading disabilities or attentional difficulties. Whereas it is known that children with such multiple deficits have more difficulty learning in many areas of mathematics than do children with MLD and no other deficits, the sources of the comorbidity are not well understood.

4. Gifted Students

There are only a few cognitive studies of the sources of the accelerated learning of mathematically gifted students, but those that have been conducted suggest an enhanced ability to remember and process numerical and spatial information. Quasi-experimental and longitudinal studies consistently reveal that accelerated and demanding instruction is needed for these students to reach their full potential in mathematics.

In most academic domains, gifted children achieve the same academic milestones as their more typical peers but do so at an earlier age (for reviews and discussion see Benbow & Lubinski, 1996; Siegler & Kotovsky, 1986). On the basis of this general pattern, intellectually or mathematically gifted children are predicted to learn arithmetic, fractions, algebra, and other areas of mathematics at an earlier age and in many cases with less exposure than other children. There are only a handful of cognitive studies of the processes that might underlie this accelerated learning in mathematics, and even in these studies, the criteria used to define giftedness has varied considerably (Dark & Benbow, 1990, 1991; Geary & Brown, 1991; Mills et al., 1993; Robinson et al., 1996; Swanson, 2006). Nonetheless, the results of these studies suggest an enhanced ability to retrieve spatial and numerical (but not verbal) information from long-term memory and an enhanced ability to manipulate these forms of information in working memory; the extent to which these advantages are learned, inherent, or some combination is not known. Cognitive and developmental studies of children who show promise in the learning of mathematics are clearly needed to better understand the sources of their advantage and to better facilitate their long-term mathematical development.

Even in the absence of detailed studies of cognitive processes, other forms of research on academically and mathematically gifted children and adolescents—as defined by performance on achievement and aptitude tests—reveal that acceleration, alone or in combination with curriculum differentiation, is a best practice for serving the academic needs of these students (Colangelo, Assouline, & Gross, 2004; Southern, Jones, & Stanley, 1993). It is an educational option that is most strongly supported by research (Benbow, 1991; Benbow & Stanley, 1996; Colangelo et al.; Kulik & Kulik, 1984). The underlying principle for educating gifted youth is "appropriate developmental placement," or providing students with educational opportunities tailored to their rates of learning and level of competence (Benbow & Stanley; Colangelo et al.). In the words of Stanley (2000), the idea is to teach students "only what they don't already know" (p. 216). Although multiple studies have been conducted on a variety of accelerative options, the Task Group can summarize the results easily: When differences are found, they favor accelerated programs over traditional instruction, regardless of the mode of acceleration (e.g., Swiatek & Benbow, 1991a, 1991b; The benefits of accelerated instruction remain evident, even 50 years later (Cronbach, 1996). Moreover, students who receive accelerated instruction in math are more likely to be pursuing science, technology, engineering, and math (STEM) careers in their mid-30s (Lubinski, Benbow, Shea, Eftekhari-Sanjani, & Halvorson, 2001; Swiatek & Benbow, 1991a, 1991b). In addition, most students express satisfaction with their acceleration in both the short term and long term (Richardson & Benbow, 1990; Swiatek & Benbow, 1992).

5. Conclusions and Recommendations

Research efforts are needed in areas that assess the effectiveness of interventions designed to: 1) reduce the vulnerability of black and Hispanic students to negative stereotypes about their academic abilities, 2) functionally improve working memory capacity, and 3) provide explicit instruction on how to use strategies for effective and efficient problem solving.

More experimental work is needed to specify the underlying processes that link task engagement and self-efficacy, and the mathematics outcomes for black and Hispanic students. Urgently needed are a scaling-up and experimental evaluation of the interventions that have been found to be effective in enhancing engagement and self-efficacy for black and Hispanic students.

Intervention studies of students with MLD are in the early stages and should be a focus of future research efforts. Further research also is needed to identify the sources of MLD and learning difficulties in the areas of fractions, geometry, and algebra.

F. Brain Sciences and Mathematics Learning

Brain sciences research has the potential to contribute to knowledge of mathematical learning and eventually educational practices. Nevertheless, attempts to connect research in the brain sciences to classroom teaching and student learning in mathematics are premature. Instructional programs in mathematics that claim to be based on brain sciences research remain to be validated.

Although it is sometimes suggested that brain research should provide the scientific foundation for children's education in mathematics and in other academic areas, it is too early to directly apply findings from studies of brain processes during mathematical reasoning to classroom teaching and learning. Yet promising research emerging from the field of cognitive neuroscience is permitting investigators to begin forging links between neurobiological functions and mathematical cognition.

Most research making use of brain imaging and related techniques has focused on basic mental representations of number and quantity (Chochon, Cohen, van de Moortele, & Dehaene, 1999; Dehaene et al., 1999; Göbel, Calabria, Farné, & Rossetti, 2006; Halgren, Boujon, Clarke, Wang, & Chauvel, 2002; Kadosh et al., 2005; Pinel, Piazza, Le Bihan, & Dehaene, 2004; Temple & Posner, 1998; Vuilleumier, Ortigue, & Brugger, 2004; Zorzi et al., 2002), with a few studies exploring problem solving in arithmetic (Gruber, Indefrey, Steinmetz, & Kleinschmidt, 2001; Rickard et al., 2000; Rivera et al., 2005) and simple algebra (Anderson, Qin, Sohn, Stenger, & Carter, 2003; Qin et al., 2003; Qin et al., 2004). In most of these studies, researchers have contrasted the brain regions activated when children (or adolescents) and adults solve the same arithmetic or algebra problems (Kawashima et al., 2004; Qin et al., 2003; Qin et al., 2004; Rivera et al.); mapped changes in neural activity associated with practice at arithmetic (Delazer et al., 2003; Pauli et al., 1994); and differentiated the brain regions involved in arithmetic fact retrieval from those recruited for executing complex calculation procedures, such as regrouping in addition (Kong et al., 2005). In other studies, researchers have compared brain activity when the same quantities are presented in different notations (e.g., 8 versus eight; Kadosh, Kadosh, Kaas, Henik, & Goebel, 2007; Piazza, Pinel, Bihan, & Dehaene, 2007).

There is of course some variation across studies in the brain areas engaged when solving different types of mathematical problems—due to differences in experimental procedures and specific math problems presented across studies—but there are also intriguing consistencies. It has been repeatedly found that comparisons of number magnitudes (Pinel et al., 2004; Temple & Posner, 1998), quantitative estimation (Dehaene et al., 1999), use of a mental number line (Vuilleumier et al., 2004; Zorzi et al., 2002), and problem solving in arithmetic and algebra (Chochon et al., 1999; Qin et al., 2003; Rivera et al., 2005) activate several areas of the parietal cortex, including the bilateral intraparietal sulcus and angular gyrus. The intraparietal sulcus is also active when non-human animals engage in numerical activities (Sawamura, Shima, & Tanji, 2002; Thompson, Mayers, Robertson, & Patterson, 1970) and it has been proposed that a segment of this sulcus, particularly in the left hemisphere, may support an inherent number representational system (Dehaene et al., 2003). The evidence bearing on this last proposal, however, is mixed (Piazza et al., 2007; Shuman & Kanwisher, 2004; Simon & Rivera, 2007).

In all, researchers have used brain imaging and related methods to study the brain regions activated when children and adults solve arithmetic and simple algebra problems (Qin et al., 2003; Qin et al., 2004; Rivera et al., 2005), when the same individuals solve arithmetic problems at earlier and later points in learning (Delazer et al., 2003; Delazer et al., 2005), when individuals solve simple or more complex arithmetic problems (Dehaene et al., 1999; Kong et al., 2005), or when people solve arithmetic problems that involve different operations (Isheback et al., 2006). During the early phases of learning in childhood, numerical and arithmetical estimation and arithmetical problem solving generally engage the intraparietal sulcus of both hemispheres (Dehaene et al., 2003), as well as areas of the prefrontal cortex that support aspects of attentional control and working memory manipulations (Delazer et al., 2003; Menon, Rivera, White, Glover, & Reiss, 2000; Pauli et al., 1994). The execution of arithmetical procedures, such as regrouping in complex arithmetic, is also dependent on these prefrontal regions (Kong et al.). The evidence to date indicates that practice of simple (e.g., 2 × 5) and more complex (e.g., 23 × 5) arithmetic results in changes in recruitment of the brain regions supporting these competencies; that is, on easier problems, there is a decreased involvement of the prefrontal and perhaps intraparietal regions and increased engagement of the angular gyrus, especially in the left hemisphere (Delazer et al., 2003; Pauli et al.; Rivera et al.; but see Rickard et al., 2000). There is not a sufficient number of studies with children of various ages and grades to draw strong conclusions about schooling and mathematical development, but the research that has been conducted thus far suggests a similar pattern, that is, decreased involvement of the prefrontal/working memory regions and increased involvement of the angular gyrus with increasing grade level and mathematical experience (Rivera et al.).

This summary is an incomplete picture of schooling- and practice-related changes in brain functioning during mathematical learning. For example, Rivera et al.’s (2005) study also implicates other brain regions—such as the hippocampus which supports the formation of declarative memories—involved in the learning of basic arithmetic facts; Qin et al.’s (2003, 2004) studies suggest the parietal cortex in the adolescent brain may be more responsive than the same regions in the adult brain when individuals are learning to solve simple algebraic equations; Sohn et al.’s (2004) study suggests differences in the brain regions that contribute to success at solving algebraic word problems and algebraic equations; and, Ischebeck et al.’s (2006) results suggest that there may be differences in the network of posterior brain regions engaged during the learning of different arithmetical operations. The progress to date indicates that when combined with insights provided by cognitive research, brain imaging and related methodologies can provide unique and essential information on how children and adults learn mathematics. In coming years, these technologies will almost certainly help answer core questions associated with mathematical learning, such as the sources of learning disabilities and the effects of different forms of instruction on the acquisition of declarative, conceptual, and procedural competencies.

1. Conclusions and Recommendations

Brain sciences research has a unique potential for contributing to knowledge of mathematical learning and cognition, and eventually educational practices. Nevertheless, attempts to connect research in the brain sciences to classroom teaching and student learning in mathematics should not be made until instructional programs in mathematics based on brain sciences research are created and validated.

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APPENDIX A: Literature Search Guidelines

The goal of the literature search was to identify experimental, cognitive, or related studies of children's mathematics learning in specific content areas (see key words). These involve measures of children's learning, problem solving, or understanding that are more precisely defined (e.g., trial-by-trial assessment of problem solving strategy) than is typically found with psychometric measures (e.g., achievement tests).

First search. This covered a designated set of core learning, cognition, and developmental journals: *American Educational Research Journal; Child Development; Cognition; Cognition and Instruction; Cognitive Development; Cognitive Psychology; Cognitive Science; Current Directions in Psychological Sciences; Developmental Psychology; Developmental Review; Journal of Cognition and Development; Journal of Education Psychology; Journal of Experimental Child Psychology; Journal of Experimental Psychology, Learning, Memory and Cognition; Journal of Experimental Psychology: General; Journal of Memory and Language; Journal of Personality and Social Psychology; Learning and Individual Differences; Mathematical Cognition; Memory and Cognition; Nature; Psychological Bulletin; Psychological Review; Psychological Science; Review of Educational Research; Science.*

Second search. This covered other English-language, peer-reviewed journals that primarily publish empirical studies and are indexed in PsychInfo and Web of Science (Social Sciences Citation Index).

Criteria for Inclusion

- Published in English.
- Participants are age 3 years to young adult.
- Published in a peer-reviewed empirical journal, or a review of empirical research in books or annual reviews.
- Experimental, quasi-experimental, or correlational methods.

APPENDIX B: Search Terms

Search Terms Used in Literature Review

Five Core topics:

Learning and Cognition of:

whole number arithmetic	estimation
fractions	algebra
	geometry

Specific Key Terms:

arithmetic	mathematical equality	arithmetic word problems	math LD (learning disability)
addition	mathematical inequality	algebra word problems	arithmetic LD
subtraction	ratio	fractions	dyscalculia
multiplication	equation		
division	number sense	algorithm	math race
base-10	ordinal	counting	math ethnicity
fraction	cardinal	distributive property	math sex
number		estimation	math gender
number line	variable	integers	math socioeconomic status
	set	magnitude comparison	math sociocultural background
commutativity	numerosity	math anxiety	math gifted
associativity	zero	mental arithmetic	
place-value	proportion	natural numbers	
perimeter	proportional reasoning	numeracy	
area	number comparison	part-whole relationships	
volume	exponents	problem-size effect	
linear equations	radical	rational numbers	
function		real numbers	
		regrouping	
		subitizing	
		transcoding	

APPENDIX C: Sex Differences

The following tables and figures summarize the data on math performance by gender using data available on national samples. Data from the Trends in Math and Science Survey (TIMSS) illustrate the math performance of fourth- and eighth-graders. Data from the National Assessment of Educational Progress (NAEP) Long-Term Trend study illustrate performance between groups over the last 30 years. Data from the High School and Beyond (HS&B:80), National Education Longitudinal Study of 1988 (NELS:88), and Education Longitudinal Study of 2002 (ELS:2002) illustrate the math performance of 10th-grade students. Data from the National Adult Literacy Survey (NALS) and the National Assessment of Adult Literacy (NAAL) survey illustrate the quantitative literacy of adults. Data from the Program for International Student Assessment (PISA) illustrate the mathematics literacy and problem-solving proficiency of 15-year-olds. To facilitate the interpretation of the various scores, a description of the test benchmarks and performance levels associated with each test is provided.

National Assessment of Educational Progress Long-Term Trends: Mathematics Scores

This section presents the long-term trends in NAEP mathematics scores. The goal is to describe the differences in performance between groups over the last 30 years and to describe how their scores have evolved over time. For each reporting group, results are presented in the form of the average scale score for intermittent years from 1978 to 2004 and the percent of students at each achievement level in 1978, 1999, and 2004.

Methodology

All data presented in this section were obtained from the NAEP Data Explorer.[2] The Data Explorer allows users to create tables of results by custom combinations of reporting variables. The results can be reported in terms of mean score, percentage of students at or above performance levels, and score percentile.

The Data Explorer also reports standard errors and can calculate the statistical significance of changes in a variable between years or between variables in the same year. The statistical significance of changes between variables over time (e.g., the score difference between girls and boys in 1978 versus the score difference between girls and boys in 2004) is taken either directly from the *NAEP 2004 Trends in Academic Progress* or estimated using the reported standard error provided by the Data Explorer. Only differences that are statistically significant beyond the 0.05 level are described in the text of this section.

[2] http://nces.ed.gov/nationsreportcard/naepdata.

Average Scale Scores and Performance Levels

The NAEP long-term trend assessments are scored on a 0–500 point scale, but all average scale score charts presented here are ranged from 180–340 for consistency and best visibility of score differences. Charts of average scale scores are reconstructed to resemble the gap charts in *NAEP 2004 Trends in Academic Progress*.

The following text was taken verbatim from the National Center for Education Statistics website, http://nces.ed.gov/nationsreportcard/ltt/performance-levels.asp in April 2007.

> More detailed information about what students know and can do in each subject area can be gained by examining their attainment of specific performance levels in each assessment year. This process of developing the performance-level descriptions is different from that used to develop achievement-level descriptions in the main NAEP reports.
>
> For each of the subject area scales, performance levels were set at 50-point increments from 150 through 350. The five performance levels—150, 200, 250, 300, and 350—were then described in terms of the knowledge and skills likely to be demonstrated by students who reached each level.
>
> A "scale anchoring" process was used to define what it means to score in each of these levels. NAEP's scale anchoring follows an empirical procedure whereby the scaled assessment results are analyzed to delineate sets of questions that discriminate between adjacent performance levels on the scales. To develop these descriptions, assessment questions were identified that students at a particular performance level were more likely to answer successfully than students at lower levels. The descriptions of what students know and can do at each level are based on these sets of questions.
>
> The guidelines used to select the questions were as follows: Students at a given level must have at least a specified probability of success with the questions (75 % for mathematics, 80 % for reading), while students at the next lower level have a much lower probability of success (that is, the difference in probabilities between adjacent levels must exceed 30 percent). For each curriculum area, subject-matter specialists examined these empirically selected question sets and used their professional judgment to characterize each level. The scale anchoring for mathematics trend reporting was based on the 1986 assessment.

The five performance levels are applicable at all three age groups, but only three performance levels are discussed for each age: levels 150, 200, and 250 for age 9; levels 200, 250, and 300 for age 13; and levels 250, 300, and 350 for age 17. These performance levels are the ones most likely to show significant change within an age across the assessment years and do not include the levels that nearly all or almost *no students attained at a particular age in each year.*

The following description of each mathematics performance level was copied from http://nces.ed.gov/nationsreportcard/ltt/math-descriptions.asp in April 2007.

Level 350: Multistep Problem Solving and Algebra

Students at this level can apply a range of reasoning skills to solve multistep problems. They can solve routine problems involving fractions and percents, recognize properties of basic geometric figures, and work with exponents and square roots. They can solve a variety of two-step problems using variables, identify equivalent algebraic expressions, and solve linear equations and inequalities. They are developing an understanding of functions and coordinate systems.

Level 300: Moderately Complex Procedures and Reasoning

Students at this level are developing an understanding of number systems. They can compute with decimals, simple fractions, and commonly encountered percents. They can identify geometric figures, measure lengths and angles, and calculate areas of rectangles. These students are also able to interpret simple inequalities, evaluate formulas, and solve simple linear equations. They can find averages, make decisions based on information drawn from graphs, and use logical reasoning to solve problems. They are developing the skills to operate with signed numbers, exponents, and square roots.

Level 250: Numerical Operations and Beginning Problem Solving

Students at this level have an initial understanding of the four basic operations. They are able to apply whole number addition and subtraction skills to one-step word problems and money situations. In multiplication, they can find the product of a two-digit and a one-digit number. They can also compare information from graphs and charts, and are developing an ability to analyze simple logical relations.

Level 200: Beginning Skills and Understandings

Students at this level have considerable understanding of two-digit numbers. They can add two-digit numbers but are still developing an ability to regroup in subtraction. They know some basic multiplication and division facts, recognize relations among coins, can read information from charts and graphs, and use simple measurement instruments. They are developing some reasoning skills.

Level 150: Simple Arithmetic Facts

Students at this level know some basic addition and subtraction facts, and most can add two-digit numbers without regrouping. They recognize simple situations in which addition and subtraction apply. They also are developing rudimentary classification skills.

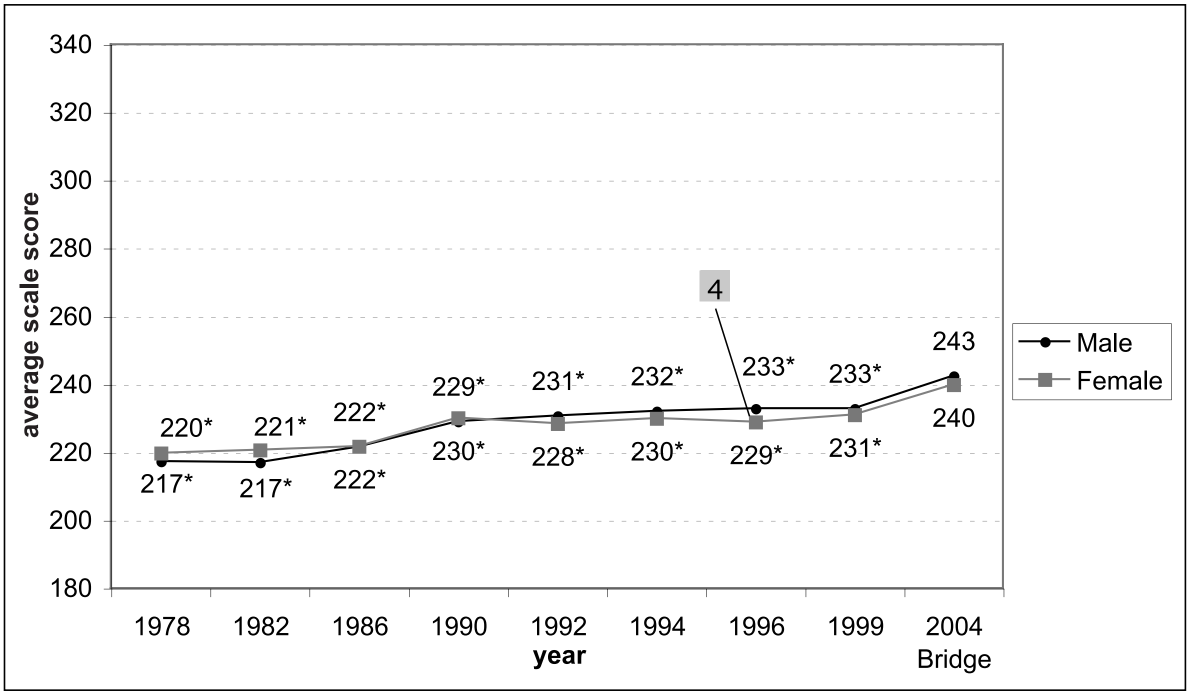
Table C-1: Number of Students in Each NAEP Reporting Group, by Age, Gender, Race/Ethnicity, and Parents' Level of Education: 1978, 1999, and 2004

	Age 9			Age 13			Age 17		
Reporting Group/Year	**1978**	**1999**	**2004**	**1978**	**1999**	**2004**	**1978**	**1999**	**2004**
Total	14800	6000	5200	24200	5900	5700	26800	3800	3800
Male	7400	2940	2548	12100	2950	2736	13132	1824	1824
Female	7400	3060	2652	12100	2950	2964	13668	1976	1976
White	11692	4200	3068	19360	4189	3648	22244	2736	2584
Black	2072	1080	728	3146	885	798	3216	570	456
Hispanic	740	480	988	1452	590	912	1072	380	532
Other			416			342			228
Parents' Level of Education									
Less than high school				2904	354	399	3484	266	342
Graduated high school				7986	1239	1083	8844	76	722
Some education after high school				3388	1003	855	4288	874	836
Graduated college				6292	2832	2679	8576	1824	1786
Unknown				3630	531	684	1340	1140	114

Source: Created by the Institute for Defense Analysis Science and Technology Policy Institute using the NAEP Data Explorer (http://nces.ed.gov/nationsreportcard/naepdata/).

Note: Level of education is parents' level of education and was not collected for 9-year-olds.

Figure C-1: Average NAEP Scale Scores by Gender, Age 9: Intermittent Years From 1978–2004



*Indicates score or gap is significantly different from 2004.
+ 1996 was an exception to general trend of no gender gap in scores at age 9.

Note: Data labels for male (above) and female (below). Between gender score differences (gaps) are shown in shaded boxes only for years in which the gap is statistically significant. Labeled gaps may not reflect labeled scores because of rounding.

“Bridge” refers to updates made to NAEP in 2004. The updates replaced outdated material and accommodated more students with disabilities. In order to maintain the long-term trend, test takers were randomly assigned to either the old test form, called the bridge assessment, or the modified test form. Results from the bridge assessment should be compared to results from assessments prior to 2004, while results from the modified assessment should be compared to assessments given after 2004.

Source: Created by the Institute for Defense Analysis Science and Technology Policy Institute using the NAEP Data Explorer (http://nces.ed.gov/nationsreportcard/naepdata/).

What Is This Indicator?

This indicator presents the average scale score for 9-year-old boys and girls for each assessment since 1978.

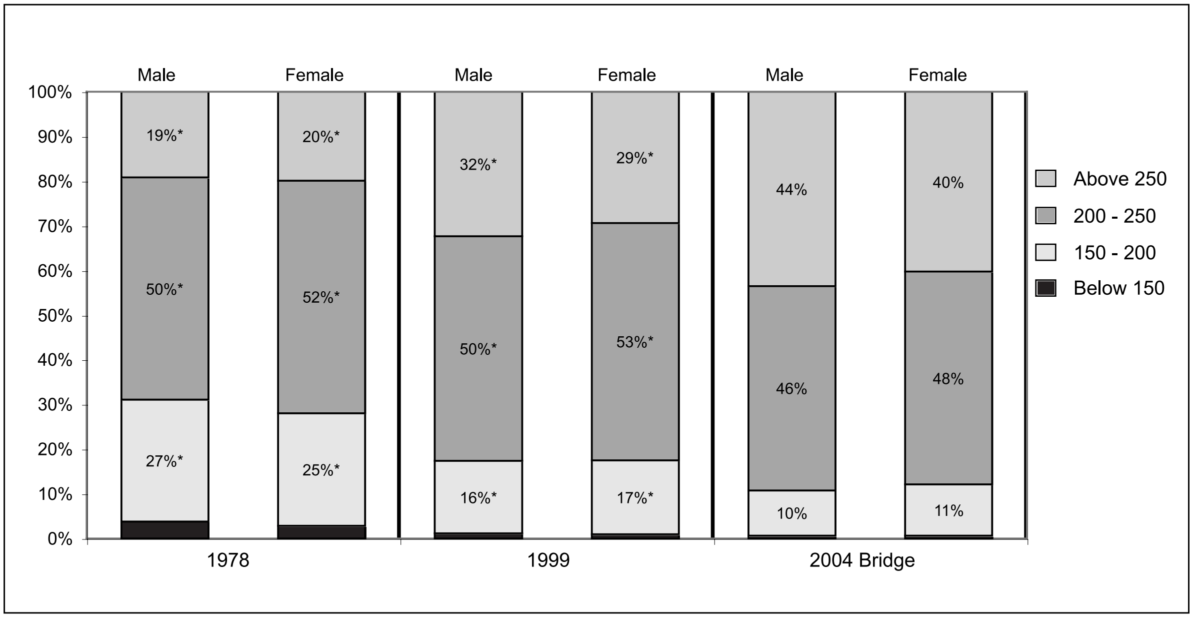
Discussion

- In 2004, the average score for both boys and girls was higher than in any previous assessment.
  - — The average score for 9-year-old boys increased by 10 points between 1999 and 2004, going from 233 in 1999 to 243 in 2004. The average score for boys in 2004 was a 23 point increase from the average score of 220 in 1978.

— The average score for 9-year-old girls increased by 9 points between 1999 and 2004, going from 231 in 1999 to 240 in 2004. The average score for girls in 2004 was a 23-point increase from the average score of 217 in 1978.

- In general, there was no gender gap at age 9. The difference in average score for 9-year-old boys and 9-year-old girls has not been significant in most years.
  — The one exception is 1996, when boys scored 4 points higher than girls on average.

Figure C-2: Percent at NAEP Performance Levels by Gender, 9-Year-Olds: 1978, 1999, and 2004



*Indicates percentage is significantly different from 2004. Differences between categories (genders) may not be statistically significant.

Note: “Bridge” refers to updates made to NAEP in 2004. The updates replaced outdated material and accommodated more students with disabilities. In order to maintain the long-term trend, test takers were randomly assigned to either the old test form, called the bridge assessment, or the modified test form. Results from the bridge assessment should be compared to results from assessments prior to 2004, while results from the modified assessment should be compared to assessments given after 2004.

Source: Created by the Institute for Defense Analysis Science and Technology Policy Institute using the NAEP Data Explorer (http://nces.ed.gov/nationsreportcard/naepdata/).

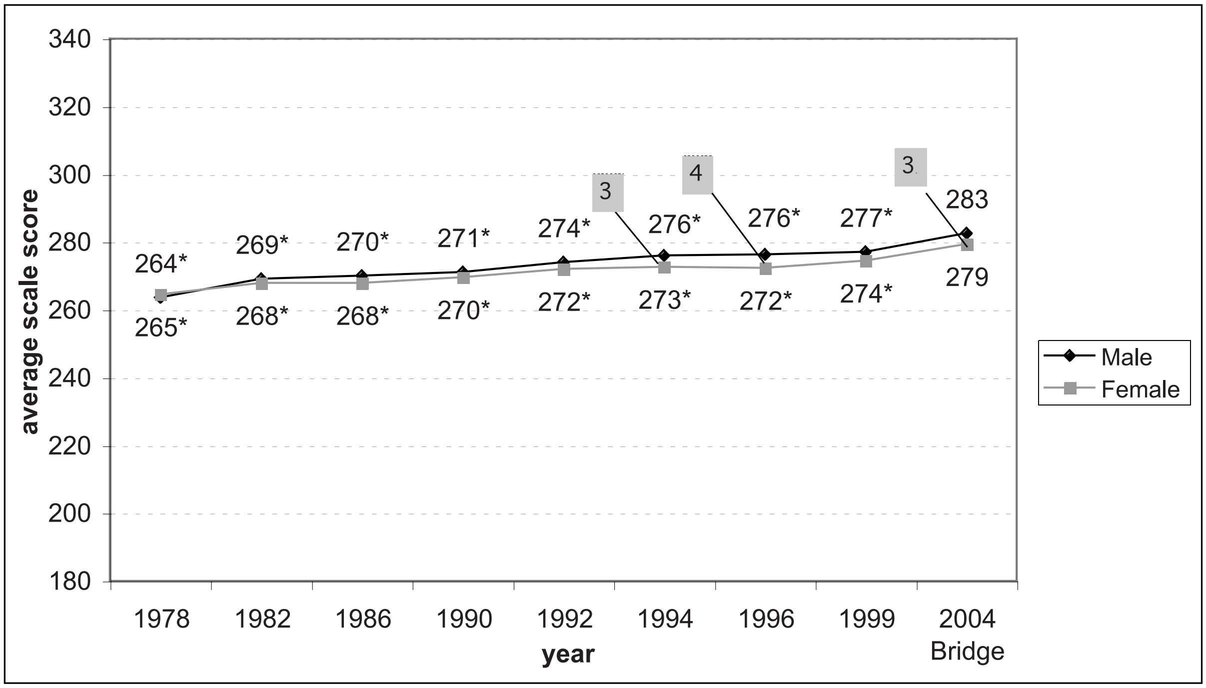
What Is This Indicator?

This indicator presents the percentage of 9-year-olds reaching each performance level by gender. The performance levels reported at age 9 are 150—Simple Arithmetic Facts, 200—Beginning Skills and Understandings, and 250—Numerical Operations and Beginning Problem Solving.

Discussion

- 9-year-old boys and girls reach similar performance levels. The differences in the percent of 9-year-old boys and 9-year-old girls reaching each performance level are not significant.
- The percentage of 9-year-olds reaching the highest achievement level for this age group, at or above 250, has doubled since 1978 and has increased from approximately 30% to 40% between 1999 and 2004.

Figure C-3: Average NAEP Scale Scores by Gender, Age 13: Intermittent Years From 1978–2004



*Indicates score or gap is significantly different from 2004.

Note: Data labels for male (above) and female (below). Between gender score differences (gaps) are shown in shaded boxes only for years in which the gap is statistically significant. Labeled gaps may not reflect labeled scores because of rounding.

"Bridge" refers to updates made to NAEP in 2004. The updates replaced outdated material and accommodated more students with disabilities. In order to maintain the long-term trend, test takers were randomly assigned to either the old test form, called the bridge assessment, or the modified test form. Results from the bridge assessment should be compared to results from assessments prior to 2004, while results from the modified assessment should be compared to assessments given after 2004.

Source: Created by the Institute for Defense Analysis Science and Technology Policy Institute using the NAEP Data Explorer (http://nces.ed.gov/nationsreportcard/naepdata/).

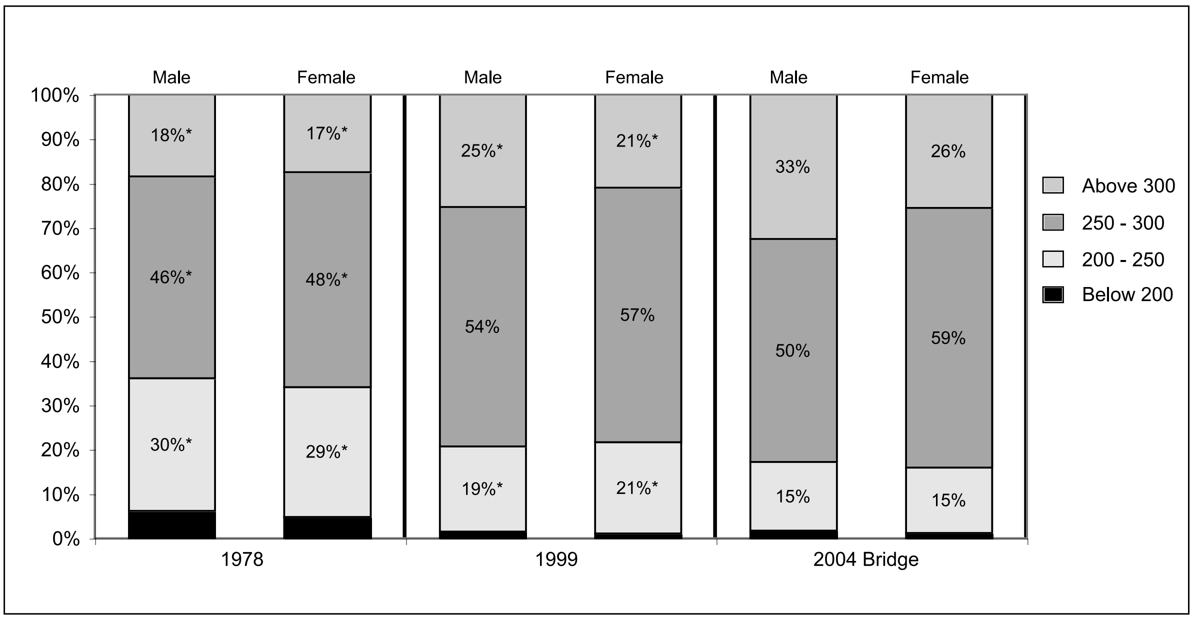
What Is This Indicator?

This indicator presents the average scale score for 13-year-old boys and girls for each assessment since 1978.

Discussion

- In 2004, the average score for both 13-year-old boys and 13-year-old girls was higher than in any previous assessment.
  - — The average score for 13-year-old boys increased by 6 points between 1999 and 2004, going from 277 in 1999 to 283 in 2004. The average score for boys in 2004 was a 19 point increase from the average score of 264 in 1978.
  - — The average score for 13-year-old girls increased by 5 points between 1999 and 2004, going from 274 in 1999 to 279 in 2004. The average score for girls in 2004 was a 14 point increase from the average score of 265 in 1978.
- In general, there was no consistent gender gap at age 13. The difference in average score for 13-year-old boys and 13-year-old girls has not been significant in most years.
  - — In 2004, 1996, and 1994 the average score for boys was 3 to 4 points higher than the average score for girls.

Figure C-4: Percent at NAEP Performance Levels by Gender, 13-Year-Olds: 1978, 1999, and 2004



*Indicates percentage is significantly different from 2004. Differences between categories (genders) may not be statistically significant.

Note: “Bridge” refers to updates made to NAEP in 2004. The updates replaced outdated material and accommodated more students with disabilities. In order to maintain the long-term trend, test takers were randomly assigned to either the old test form, called the bridge assessment, or the modified test form. Results from the bridge assessment should be compared to results from assessments prior to 2004, while results from the modified assessment should be compared to assessments given after 2004.

Source: Created by the Institute for Defense Analysis Science and Technology Policy Institute using the NAEP Data Explorer (http://nces.ed.gov/nationsreportcard/naepdata/).

What Is This Indicator?

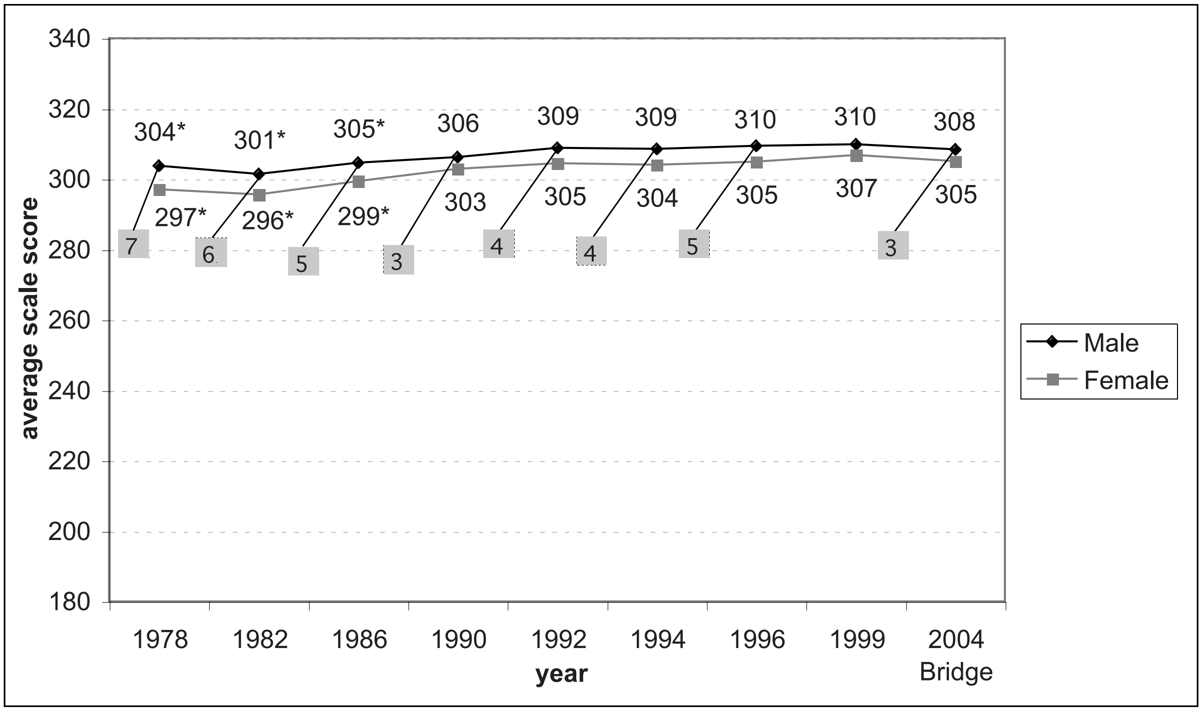
This indicator presents the percentage of 13-year-olds reaching each performance level by gender in 1978, 1999, and 2004. The performance levels reported at age 13 are 200—Beginning Skills and Understandings, 250—Numerical Operations and Beginning Problem Solving, and 300—Moderately Complex Procedures and Reasoning.

Discussion

- In 1999 and 2004, slightly more 13-year-old boys scored at or above 300 than did 13-year-old girls.
  - — In 1999 the gender gap at the 300 level was 4%, with 25% of boys and 21% of girls performing at or above 300.
  - — In 2004 the gender gap at the 300 level was 7%, with 33% of boys and 26% of girls performing at or above 300.
  - — The change in gender gap from 1999 to 2004 was not statistically significant.

- The percentages of boys and girls scoring at or above 300 have increased since 1999 and 1978.
  - — The percentage of 13-year-old boys at or above 300 was 33% in 2004, which was 7% higher than in 1999 and 14% higher than in 1978.
  - — The percentage of 13-year-old girls at of above 300 was 26% in 2004, which was 5% higher than in 1999 and 8% higher than in 1978.

Figure C-5: Average NAEP Scale Scores by Gender, Age 17: Intermittent Years From 1978–2004



*Indicates score or gap is significantly different from 2004.

Note: Data labels for male (above) and female (below). Between gender score differences (gaps) are shown in shaded boxes only for years in which the gap is statistically significant. Labeled gaps may not reflect labeled scores because of rounding.

“Bridge” refers to updates made to NAEP in 2004. The updates replaced outdated material and accommodated more students with disabilities. In order to maintain the long-term trend, test takers were randomly assigned to either the old test form, called the bridge assessment, or the modified test form. Results from the bridge assessment should be compared to results from assessments prior to 2004, while results from the modified assessment should be compared to assessments given after 2004.

Source: Created by the Institute for Defense Analysis Science and Technology Policy Institute using the NAEP Data Explorer (http://nces.ed.gov/nationsreportcard/naepdata/).

What Is This Indicator?

This indicator presents the average scale score for 17-year-old boys and girls for each assessment since 1978.

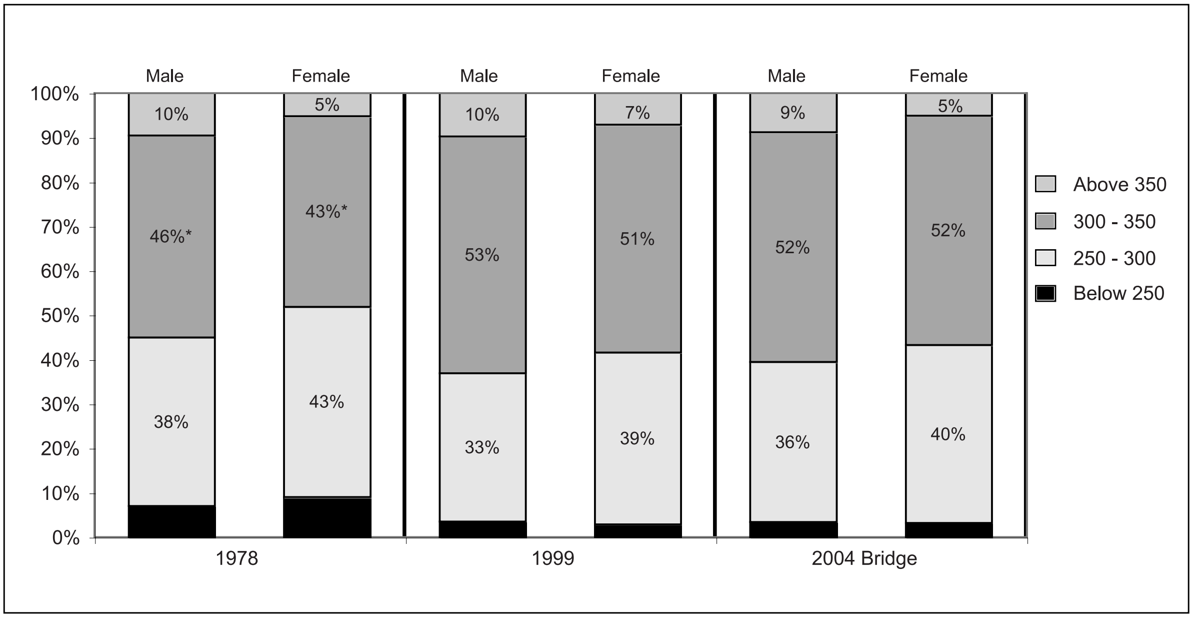
Discussion

- The average score for both girls and boys at age 17 has been flat since 1990, although average scores have increased slightly since 1978.
  - — The average score for 17-year-old boys increased by four points from 304 in 1978 to 308 in 2004.

— The average score for 17-year-old girls increased by eight points from 297 in 1978 to 205 in 2004.

- 17-year-old boys have consistently outscored 17-year-old girls on the long-term mathematics NAEP.
  — The gender gap for 17-year-olds in 2004 was three points and was not significantly different from previous years.

Figure C-6: Percent at NAEP Performance Levels by Gender, 17-Year-Olds: 1978, 1999, and 2004



*Indicates percentage is significantly different from 2004. Differences between categories (genders) may not be statistically significant.

Note: “Bridge” refers to updates made to NAEP in 2004. The updates replaced outdated material and accommodated more students with disabilities. In order to maintain the long-term trend, test takers were randomly assigned to either the old test form, called the bridge assessment, or the modified test form. Results from the bridge assessment should be compared to results from assessments prior to 2004, while results from the modified assessment should be compared to assessments given after 2004.

Source: Created by the Institute for Defense Analysis Science and Technology Policy Institute using the NAEP Data Explorer (http://nces.ed.gov/nationsreportcard/naepdata/).

What Is This Indicator?

This indicator presents the percentage of 17-year-olds reaching each performance level by gender.

The performance levels reported at age 17 are 250—Numerical Operations and Beginning Problem Solving, 300—Moderately Complex Procedures and Reasoning, and 350—Multistep Problem Solving and Algebra.

Discussion

- In 1978 and 2004, slightly more 17-year-old boys scored at or above 350 than did 17-year-old girls, but in 1999 gender differences were not significant.
  - — The gender gap in 1978 was 5%; 10% of boys and 5% of girls scored at or above 350.
  - — The gender gap in 2004 was 4%; 9% of boys and 5% of girls scored at or above 350.
- The percentages of 17-year-old boys and girls at each performance level have, for the most part, not changed significantly between assessments.
  - — The percentage of both girls and boys scoring at the 300 level was higher in 2004 than in 1978. The percentage of boys at the 300 level increased by 9% to 52%, and the percentage of girls at the 300 level increased by 12% to 52%.

Trends in Math and Science Survey: TIMSS

The TIMSS 2003 International Benchmarks of Mathematics Achievement are defined in Mullis et al. (2004, p. 63) as follows.

Grade 8

Advanced International Benchmark – 625

Students can organize information, make generalizations, solve non-routine problems, and draw and justify conclusions from data. They can compute percent change and apply their knowledge of numeric and algebraic concepts and relationships to solve problems. Students can solve simultaneous linear equations and model simple situations algebraically. They can apply their knowledge of measurement and geometry in complex problem situations. They can interpret data from a variety of tables and graphs, including interpolation and extrapolation.

High International Benchmark – 550

Students can apply their understanding and knowledge in a wide variety of relatively complex situations. They can order, relate, and compute fractions and decimals to solve word problems, operate with negative integers, and solve multi-step word problems involving proportions with whole numbers. Students can solve simple algebraic problems including evaluating expressions, solving simultaneous linear equations, and using a formula to determine the value of a variable. Students can find areas and volumes of simple geometric shapes and use knowledge of geometric properties to solve problems. They can solve probability problems and interpret data in a variety of graphs and tables.

Intermediate International Benchmark – 475

Students can apply basic mathematical knowledge in straightforward situations. They can add, subtract, or multiply to solve one-step word problems involving whole numbers and decimals. They can identify representations of common fractions and relative sizes of fractions. They understand simple algebraic relationships and solve linear equations with one variable. They demonstrate understanding of properties of triangles and basic geometric concepts including symmetry and rotation. They recognize basic notions of probability. They can read and interpret graphs, tables, maps, and scales.

Low International Benchmark – 400

Students have some basic mathematical knowledge. (Mullis et al., 2004, p. 62)

Grade 4

Advanced International Benchmark – 625

Students can apply their understanding and knowledge in a wide variety of relatively complex situations. They demonstrate a developing understanding of fractions and decimals, and the relationship between them. They can select appropriate information to solve multi-step word problems involving proportions. They can formulate or select a rule for a relationship. They show understanding of area and can use measurement concepts to solve a variety of problems. They show some understanding of rotation. They can organize, interpret, and represent data to solve problems.

High International Benchmark – 550

Student can apply their knowledge and understanding to solve problems. Students can solve multi-step word problems involving addition, multiplication, and division. They can use their understanding of place value and simple fractions to solve problems. They can identify a number sentence that represents situations. Students show understanding of three-dimensional objects, how shapes can make other shapes, and simple transformation in a plane. They demonstrate a variety of measurement skills and can interpret and use data in tables and graphs to solve problems.

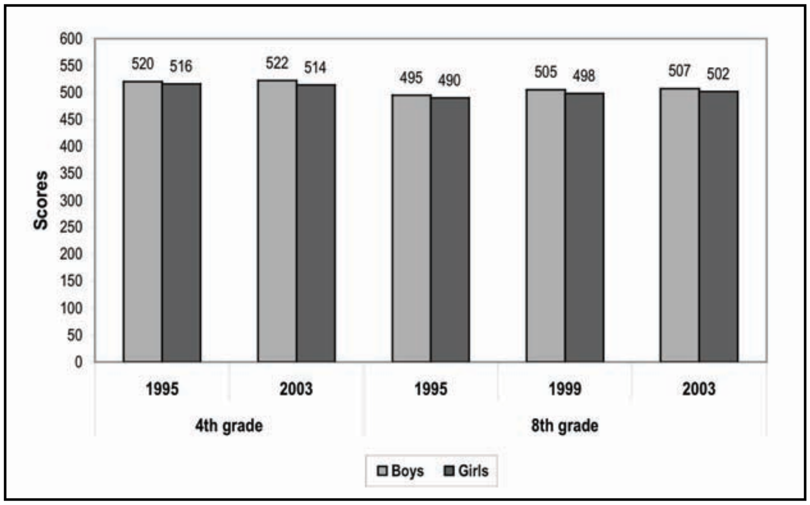
Intermediate International Benchmark – 475

Students can apply basic mathematical knowledge in straightforward situations. They can read, interpret, and use different representations of numbers. They can perform operations with three- and four-digit numbers and decimals. They can extend simple patterns. They are familiar with a range of two-dimensional shapes and read and interpret different representations of the same data.

Low International Benchmark – 400

Students have some basic mathematical knowledge. Students demonstrate an understanding of whole numbers and can do simple computations with them. They demonstrate familiarity with the basic properties of triangles and rectangles. They can read information from simple bar graphs.

Figure C-7: Average TIMSS Mathematical Scale Scores of U.S. 4th- and 8th-Graders, by Sex: Various Years From 1995–2003

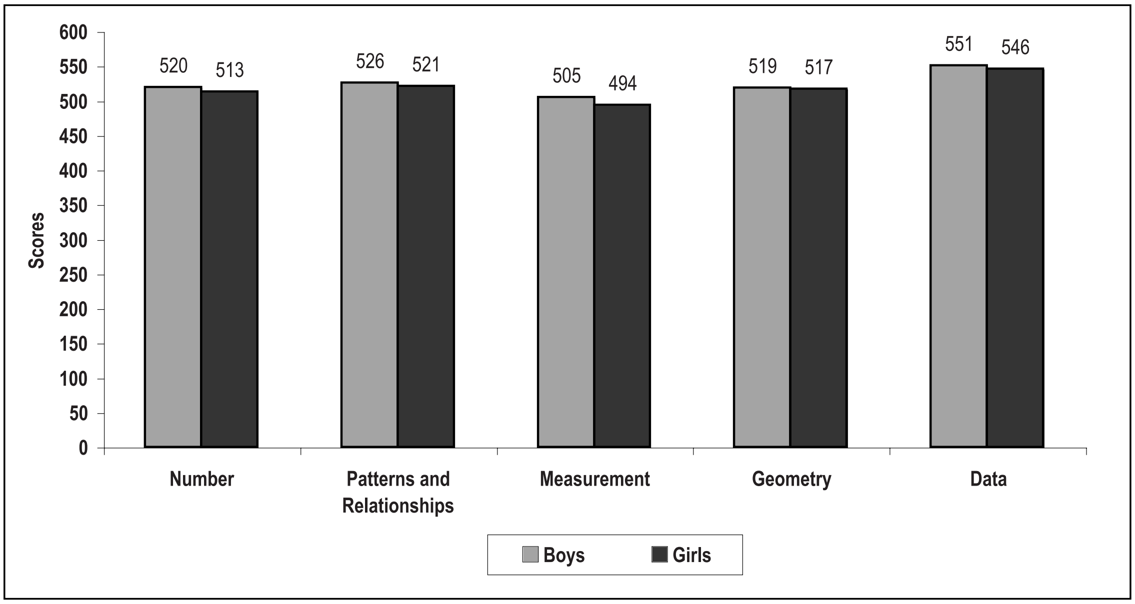


Note: TIMSS international benchmarks: Low 400, Intermediate 475, High 550, Advanced 625

Source: Gonzales et al. (2004), Figures 1 and 2.

Standardized mean difference TIMSS, gender					
	4th grade		8th grade		
	1995	*2003*	*1995*	*1999*	*2003*
Boys-Girls	0.05	0.11	0.06	0.08	0.06

Figure C-8: Average TIMSS Mathematical Scale Scores of U.S. 4th-Graders, by Sex, by Content Area: 2003

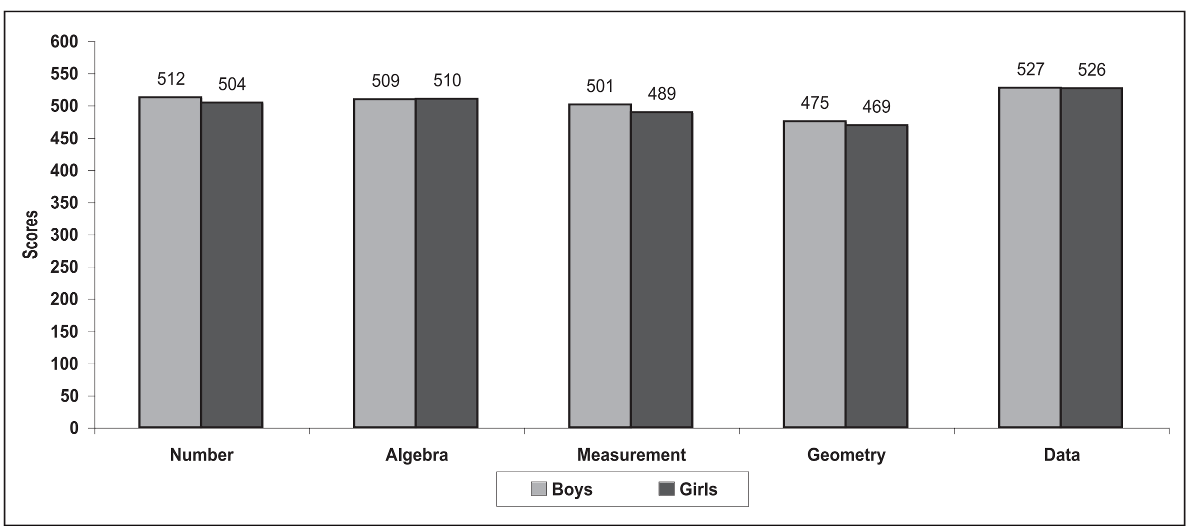


Note: TIMSS international benchmarks: Low 400, Intermediate 475, High 550, Advanced 625

Source: Mullis et al. (2003), Exhibit 3.3.

Standardized mean difference TIMSS, content areas 4th grade					
	Number	**Patterns and Relationships**	**Measurement**	**Geometry**	**Data**
Boys-Girls	0.09	0.06	0.14	0.02	0.06

Figure C-9: Average TIMSS Mathematical Scale Scores of U.S. 8th-Graders, by Sex, by Content Area: 2003



Note: TIMSS international benchmarks: Low 400, Intermediate 475, High 550, Advanced 625

Source: Mullis et al. (2003), Exhibit 3.3.

High School and Beyond of 1980: HS&B:80
National Education Longitudinal Study of 1988: NELS:88
Education Longitudinal Study of 2002: ELS:2002

The scores on the HS&B:80, NELS:88, and ELS:2002 are Item Response Theory (IRT) number-right scores on the NELS:88 1990 58-item scale. IRT estimates achievement based on patterns of correct, incorrect, and unanswered questions. "The IRT-estimated number-right score reflects an estimate of the number of these 58 items that an examinee would have answered correctly if he or she had taken all of the items that appeared on the multiform 1990 NELS:88 mathematics test. The score is the probability of a correct answer on each item, summed over the total mathematics 58-item pool" (Cahalan, Ingels, Burns, Planty, & Daniel, 2006, p.45). These scores are not directly translated into probability-of-proficiency scores. However, five probability-of-proficiency scores in mathematics were estimated for students using performance on clusters of four items each as follows:

Probability of Mastery, Mathematics Levels

1) Simple arithmetical operations on whole numbers, such as simple arithmetic expressions involving multiplication or division of integers;

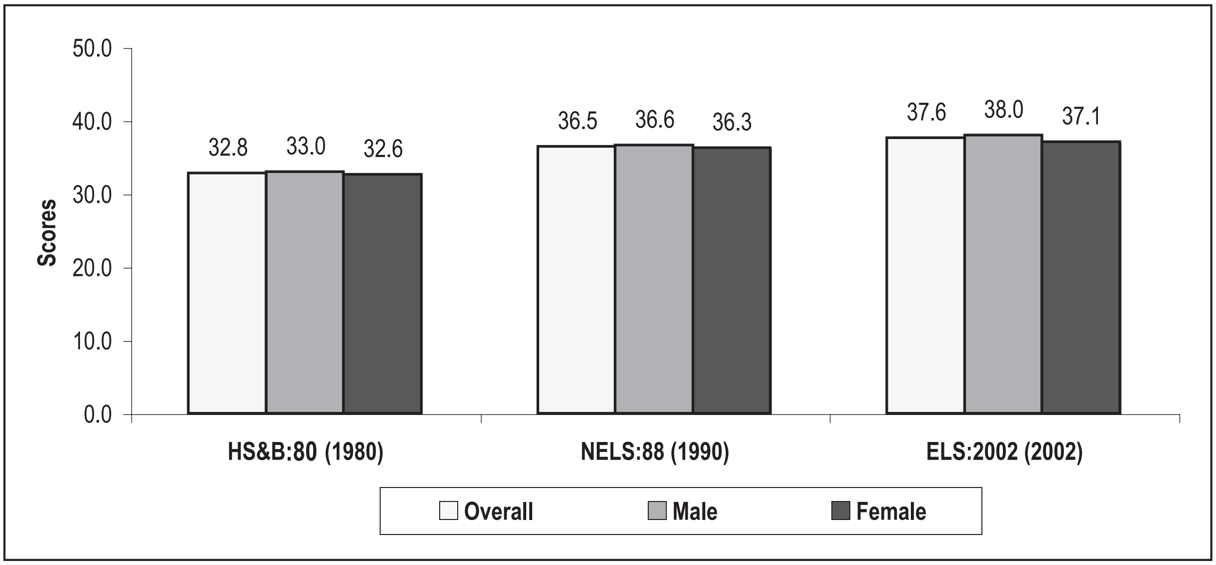
2) Simple operations with decimals, fractions, powers, and roots, such as comparing expressions, given information about exponents;

3) Simple problem solving, requiring the understanding of low-level mathematical concepts, such as simplifying an algebraic expression or comparing the length of line segments illustrated in a diagram;

4) Understanding of intermediate-level mathematical concepts and/or multistep solutions to word problems such as drawing an inference based on an algebraic expression or inequality; and

5) Complex multistep word problems and/or advanced mathematics material such as a two-step problem requiring evaluation of functions. (Cahalan et al., 2006, p. A-28)

Figure C-10: IRT—Estimated Average Math Score (10th-Grade), by Sex (HS&B:80, NELS:88, ELS:2002)



Note: IRT scale score is the estimated number right out of a total of 58.

Source: Cahalan et al. (2006), Tables 18 and 19.

Standardized mean difference sophomores, gender			
	HS&B (1980)	**NELS:88 (1990)**	**ELS:2002 (2002)**
Male-Female	0.03	0.02	0.08

Table C-2: Probability of 10th-Grade Proficiency in Mathematics, by Gender

	NELS:88 (1990)	ELS:2002 (2002)
Level 1		
Male	90.7	91.7
Female	90.8	91.6
Level 2		
Male	62.8	68.4
Female	63.3	65.7
Level 3		
Male	44.3	48.0
Female	42.8	44.7
Level 4		
Male	20.2	22.3
Female	17.8	18.5
Level 5		
Male	0.5	1.3
Female	0.3	0.6

Note: Proficiency levels – 1) Simple arithmetical operations with whole numbers; 2) Simple operations with decimals, fractions, powers, and roots; 3) Simple problem solving, requiring the understanding of low-level mathematical concepts; 4) Understanding of intermediate-level mathematical concepts and/or multistep solutions to word problems; and 5) Complex multistep word problems and/or advanced mathematics material.

Source: Cahalan et al., 2006, p. 57.

Program for International Student Assessment: PISA

Mathematics literacy can be classified by proficiency levels, based on scores on the PISA, as follows:

Below level 1 (*less than or equal to 357.77*)

Level 1 (*greater than 357.77 to 420.07*) At Level 1, students can answer questions involving familiar contexts where all relevant information is present and the questions are clearly defined. They are able to identify information and to carry out routine procedures according to direct instructions in explicit situations. They can perform actions that are obvious and follow immediately from the given stimuli.

Level 2 (*greater than 420.07 to 482.38*) At Level 2, students can interpret and recognize situations in contexts that require no more than direct inference. They can extract relevant information from a single source and make use of a single representational mode. Students at this level can employ basic algorithms, formula, procedures, or conventions. They are capable of direct reasoning and making literal interpretations of the results.

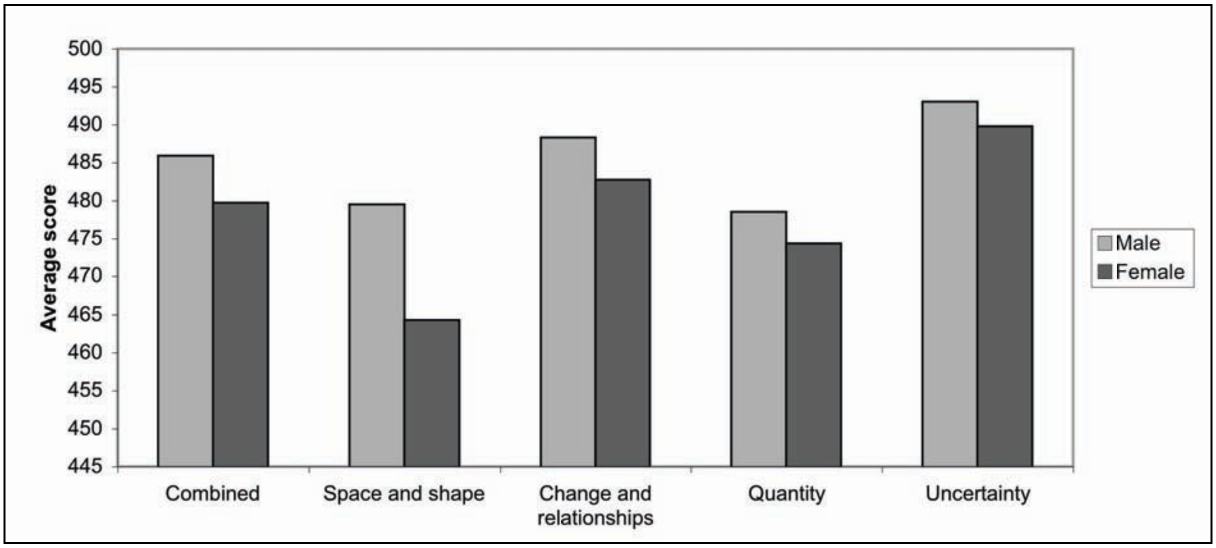
Level 3 (*greater than 482.38 to 544.68*) At Level 3, students can execute clearly described procedures, including those that require sequential decisions. They can select and apply simple problem solving strategies. Students at this level can interpret and use representations based on different information sources and reason directly from them. They can develop short communications reporting their interpretations, results, and reasoning.

Level 4 (*greater than 544.68 to 606.99*) At Level 4, students can work effectively with explicit models for complex concrete situations that may involve constraints or call for making assumptions. They can select and integrate different representations, including symbolic, linking them directly to aspects of real-world situations. Students at this level can utilize well developed skills and reason flexibly, with some insight, in these contexts. They can construct and communicate explanations and arguments based on their interpretations, arguments, and actions.

Level 5 (*greater than 606.99 to 669.3*) At Level 5, students can develop and work with models for complex situations, identifying constraints and specifying assumptions. They can select, compare, and evaluate appropriate problem solving strategies for dealing with complex problems related to these models. Students at this level can work strategically using broad, well-developed thinking and reasoning skills, appropriate linked representations, symbolic and formal characterizations, and insight pertaining to these situations. They can reflect on their actions, and formulate and communicate their interpretations and reasoning.

Level 6 (*greater than 669.3*) At Level 6, students can conceptualize, generalize, and utilize information based on their investigations and modeling of complex problem situations. They can link different information sources and representations and flexibly translate among them. Students at this level are capable of advanced mathematical thinking and reasoning. These students can apply this insight and understandings along with a mastery of symbolic and formal mathematical operations and relationships to develop new approaches and strategies for attacking novel situations. Students at this level can formulate and precisely communicate their actions and reflections regarding their findings, interpretations, arguments, and the appropriateness of these to the original situations (Lemke et al., 2005, p.18).

Figure C-11: Average Mathematics Literacy Scores of U.S. 15-Year-Olds, by Gender: 2003 PISA



Note: Level 1 (greater than 357.77 to 420.07), Level 2 (greater than 420.07 to 482.38), Level 3 (greater than 482.38 to 544.68), Level 4 (greater than 544.68 to 606.99), Level 5 (greater than 606.99 to 669.3), Level 6 (greater than 669.3)

Source: Lemke et al. (2005) Tables B-18 and B-20

Standardized mean difference, 15 year olds, gender*	
	PISA (2003)
Male-Female	0.07

*Standard deviations not provided for subscales

Table C-3: Percentage of U.S. 15-Year-Old Students Scoring at Each Proficiency Level, by Gender: 2003 PISA

	Below level 1	Level 1	Level 2	Level 3	Level 4	Level 5	Level 6
Male	10.5	14.7	23.2	23.1	16.9	8.9	2.8
Female	9.9	16.4	24.6	24.5	16.2	7.2	1.2
Overall	10.2	15.5	23.9	23.8	16.6	8.0	2.0

Source: Lemke et al., 2005, Tables B-19 and B-6

Table C-4: Comparison of U.S. and Organisation for Economic Co-operation and Development (OECD) Countries' Average Scores on 2003 PISA Math Literacy

	U.S. average	OECD average	Number of OECD countries scoring higher than U.S.
Combined	483	500	20
Space and shape	472	496	20
Change and relationships	486	499	18
Quantity	476	501	23
Uncertainty	491	502	16

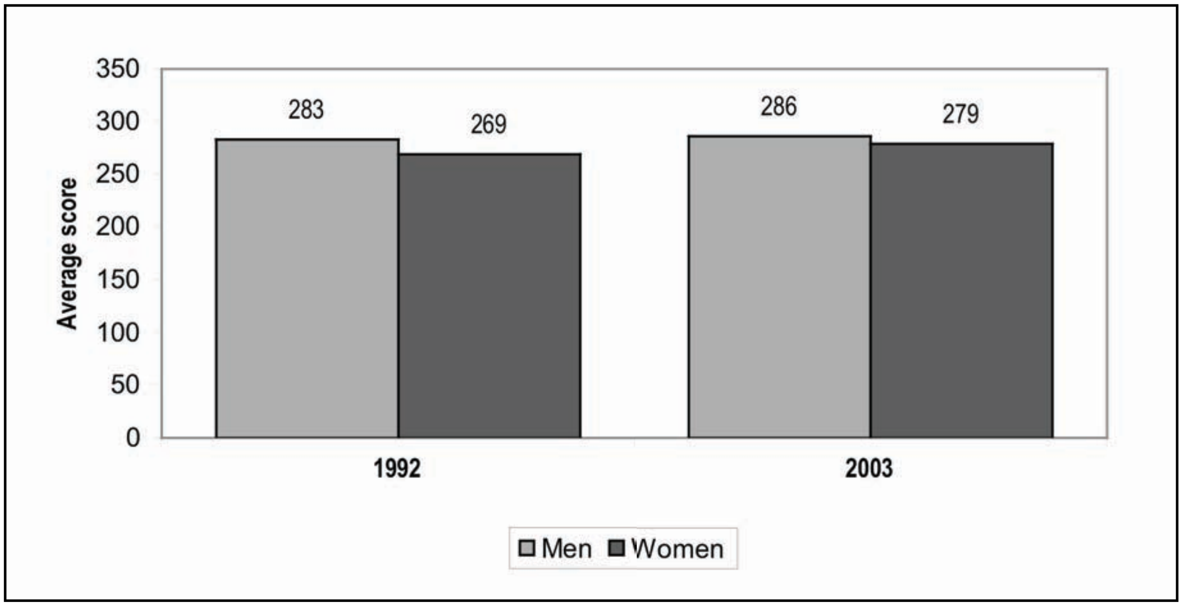
Source: Lemke et al., 2005, Table 2

National Adult Literacy Survey: NALS
National Assessment of Adult Literacy: NAAL

The Committee on Performance Levels for Adult Literacy set performance levels for quantitative literacy as Below Basic, Basic, Intermediate, and Proficient and defined them as follows, based on scores on NALS and NAAL:

> ***Below Basic (0–234)*** indicates no more than the most simple and concrete literacy skills.
>
> *Key abilities*—locating numbers and using them to perform simple *quantitative* operations (primarily addition) when the mathematical information is very concrete and familiar.
>
> ***Basic (235–289)*** indicates skills necessary to perform simple and everyday literacy activities.
>
> *Key abilities*—locating easily identifiable *quantitative* information and using it to solve simple, one-step problems when the arithmetic operation is specified or easily inferred.
>
> ***Intermediate (290–349)*** indicates skills necessary to perform moderately challenging literacy activities.
>
> *Key abilities*—locating less familiar *quantitative* information and using it to solve problems when the arithmetic operation is not specified or easily inferred.
>
> ***Proficient (350–500)*** indicates skills necessary to perform more complex and challenging literacy activities.
>
> *Key abilities*—locating more abstract quantitative information and using it to solve multistep problems when the arithmetic operations are not easily inferred and the problems are more complex (Kutner et al., 2006, p. 3).

Figure C-12: Average Quantitative Literacy Scores of Adults, by Sex: NALS 1992 and NAAL 2003



Note: Literacy levels: Below basic 0–234, Basic 235–289, Intermediate 290–349, Proficient 350–500

Source: Kutner, Greenberg, and Baer (2006), Figure 4.

Standardized mean difference adults, gender		
	1992	**2003**
Male-Female	0.21	0.11

Table C-5: Percentage of Adults in Each Quantitative Literacy Level, by Gender: NALS 1992 and NAAL 2003

		NALS, 1992	NAAL, 2003
Below basic	Male	24	21
	Female	28	22
Basic	Male	29	31
	Female	34	35
Intermediate	Male	31	33
	Female	28	32
Proficient	Male	17	16
	Female	9	11

Note: Below Basic (0–234) no more than the most simple and concrete literacy skills; Basic (235–289) skills necessary to perform simple and everyday literacy activities; Intermediate (290–349) skills necessary to perform moderately challenging literacy activities; Proficient (350–500) skills necessary to perform more complex and challenging literacy activities.

Source: Kutner et al., 2007, p. 14

APPENDIX D: Racial/Ethnic Differences

The following tables and figures summarize the data on math performance by Race/Ethnicity using data available on national samples. Data from the National Assessment of Educational Progress (NAEP) Long-Term Trend study illustrate performance between groups over the last 30 years. Data from the Trends in Math and Science Survey (TIMSS) illustrate the math performance of fourth- and eighth-graders. Data from the High School and Beyond (HS&B:80), National Education Longitudinal Study of 1988 (NELS:88), and Education Longitudinal Study of 2002 (ELS:2002) illustrate the math performance of 10th-grade students. Data from the National Adult Literacy Survey (NALS) and the National Assessment of Adult Literacy (NAAL) survey illustrate the quantitative literacy of adults. Data from the Program for International Student Assessment (PISA) illustrate the mathematics literacy and problem-solving proficiency of 15-year-olds. To facilitate the interpretation of the various scores, a description of the test benchmarks and performance levels associated with each test is provided.

National Assessment of Educational Progress Long-Term Trends: Mathematics Scores

This section presents the trends in long-term NAEP mathematics scores. The goal is to describe the differences in performance between groups over the last 30 years and to describe how their scores have evolved over time. For each reporting group, results are presented in the form of the average scale score for 1978–2004 and the percent of students at each achievement level in 1978, 1999, and 2004.

Methodology

All data presented in this section were obtained from the NAEP Data Explorer.[3] The Data Explorer allows users to create tables of results by custom combinations of reporting variables. The results can be reported in terms of mean score, percentage of students at or above performance levels, and score percentile.

The Data Explorer also reports standard errors and can calculate the statistical significance of changes in a variable between years or between variables in the same year. The statistical significance of changes between variables over time (e.g., the score difference between girls and boys in 1978 versus the score difference between girls and boys in 2004) is taken either directly from the *NAEP 2004 Trends in Academic Progress* or estimated using the reported standard error provided by the Data Explorer. Only differences that are statistically significant beyond the 0.05 level are described in the text of this section.

[3] http://nces.ed.gov/nationsreportcard/naepdata/.

Average Scale Scores and Performance Levels

The NAEP long-term trend assessments are scored on a 0–500 point scale, but all average scale score charts presented here are ranged from 180–340 for consistency and best visibility of score differences. Charts of average scale scores are reconstructed to resemble the gap charts in *NAEP 2004 Trends in Academic Progress*.

The following text was taken verbatim from the National Center for Education Statistics website, http://nces.ed.gov/nationsreportcard/ltt/performance-levels.asp in April 2007.

More detailed information about what students know and can do in each subject area can be gained by examining their attainment of specific performance levels in each assessment year. This process of developing the performance-level descriptions is different from that used to develop *achievement-level* descriptions in the main NAEP reports.

For each of the subject area scales, performance levels were set at 50-point increments from 150 through 350. The five performance levels—150, 200, 250, 300, and 350—were then described in terms of the knowledge and skills likely to be demonstrated by students who reached each level.

A “scale anchoring” process was used to define what it means to score in each of these levels. NAEP’s scale anchoring follows an empirical procedure whereby the scaled assessment results are analyzed to delineate sets of questions that discriminate between adjacent performance levels on the scales. To develop these descriptions, assessment questions were identified that students at a particular performance level were more likely to answer successfully than students at lower levels. The descriptions of what students know and can do at each level are based on these sets of questions.

The guidelines used to select the questions were as follows: Students at a given level must have at least a specified probability of success with the questions (75% for mathematics, 80 % for reading), while students at the next lower level have a much lower probability of success (that is, the difference in probabilities between adjacent levels must exceed 30%). For each curriculum area, subject-matter specialists examined these empirically selected question sets and used their professional judgment to characterize each level. The scale anchoring for mathematics trend reporting was based on the 1986 assessment.

The five performance levels are applicable at all three age groups, but only three performance levels are discussed for each age: levels 150, 200, and 250 for age 9; levels 200, 250, and 300 for age 13; and levels 250, 300, and 350 for age 17. These performance levels are the ones most likely to show significant change within an age across the assessment years and do not include the levels that nearly all or almost no students attained at a particular age in each year.

The following description of each mathematics performance level was copied from http://nces.ed.gov/nationsreportcard/ltt/math-descriptions.asp in April 2007.

Level 350: Multistep Problem Solving and Algebra

Students at this level can apply a range of reasoning skills to solve multistep problems. They can solve routine problems involving fractions and percents, recognize properties of basic geometric figures, and work with exponents and square roots. They can solve a variety of two-step problems using variables, identify equivalent algebraic expressions, and solve linear equations and inequalities. They are developing an understanding of functions and coordinate systems.

Level 300: Moderately Complex Procedures and Reasoning

Students at this level are developing an understanding of number systems. They can compute with decimals, simple fractions, and commonly encountered percents. They can identify geometric figures, measure lengths and angles, and calculate areas of rectangles. These students are also able to interpret simple inequalities, evaluate formulas, and solve simple linear equations. They can find averages, make decisions based on information drawn from graphs, and use logical reasoning to solve problems. They are developing the skills to operate with signed numbers, exponents, and square roots.

Level 250: Numerical Operations and Beginning Problem Solving

Students at this level have an initial understanding of the four basic operations. They are able to apply whole number addition and subtraction skills to one-step word problems and money situations. In multiplication, they can find the product of a two-digit and a one-digit number. They can also compare information from graphs and charts, and are developing an ability to analyze simple logical relations.

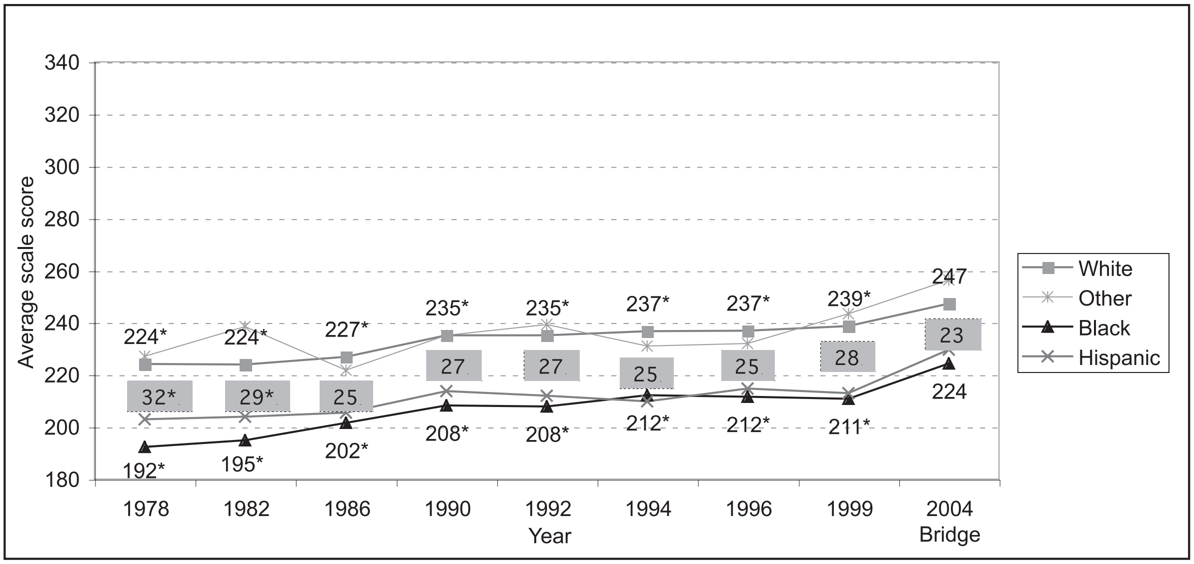
Level 200: Beginning Skills and Understandings

Students at this level have considerable understanding of two-digit numbers. They can add two-digit numbers but are still developing an ability to regroup in subtraction. They know some basic multiplication and division facts, recognize relations among coins, can read information from charts and graphs, and use simple measurement instruments. They are developing some reasoning skills.

Level 150: Simple Arithmetic Facts

Students at this level know some basic addition and subtraction facts, and most can add two-digit numbers without regrouping. They recognize simple situations in which addition and subtraction apply. They also are developing rudimentary classification skills.

Figure D-1: Average NAEP Scale Scores by Race/Ethnicity, Age 9: Intermittent Years From 1978–2004



*Indicates score or gap is significantly different from 2004.

Note: Data labels for white (above) and black (below). Between race/ethnicity score differences (gaps) are shown in shaded boxes only for years in which the gap is statistically significant. Labeled gaps may not reflect labeled scores because of rounding.

“Bridge” refers to updates made to NAEP in 2004. The updates replaced outdated material and accommodated more students with disabilities. In order to maintain the long-term trend, test takers were randomly assigned to either the old test form, called the bridge assessment, or the modified test form. Results from the bridge assessment should be compared to results from assessments prior to 2004, while results from the modified assessment should be compared to assessments given after 2004.

Source: Created by the Institute for Defense Analysis Science and Technology Policy Institute using the NAEP Data Explorer (http://nces.ed.gov/nationsreportcard/naepdata/).

What Is This Indicator?

This indicator presents the average scale score for 9-year-olds by race. “Other” includes Asian/Pacific Islander and American Indian/Alaska Native.

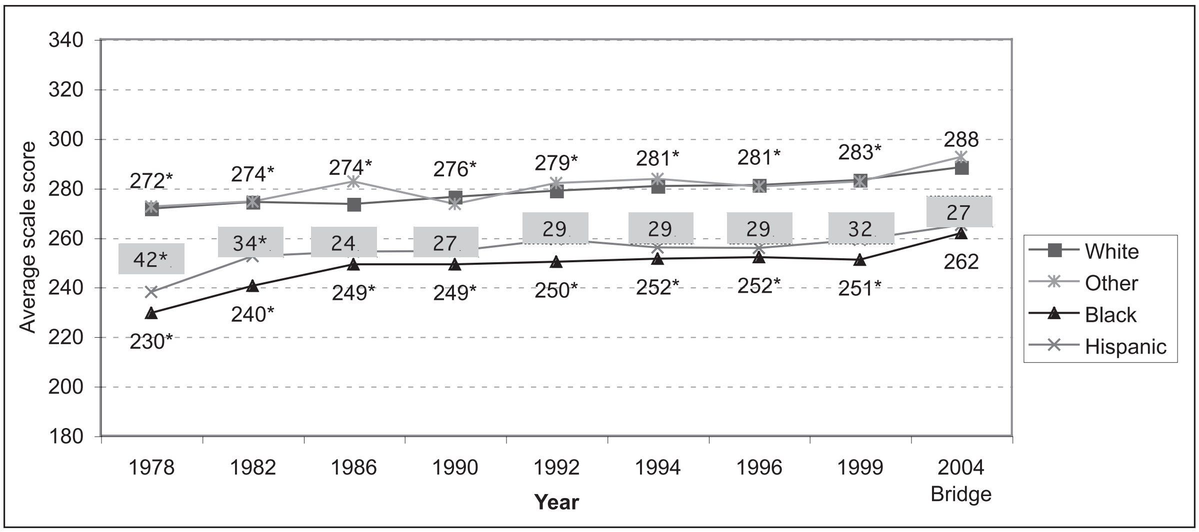
Discussion

- Whites and Other races significantly outscore blacks and Hispanics on the long-term mathematics NAEP at age 9.
- The average score for black students of 224 was 23 points lower than the average score for white students in 2004.
  - — The black-white gap has not changed significantly since 1986.
  - — The black-white gap has closed by 9 points since 1978.
- The average score for Hispanic students of 230 was 18 points lower than the average score for white students in 2004.
  - — The Hispanic-white gap has closed by 8 points since 1999.

— The Hispanic-white gap in 2004 was not significantly different from the gap in 1978.

- Average scores for all racial groups were higher in 2004 than in previous assessment years.
  - — The average score for black students was 224 in 2004, which was a 13 point increase from 1999 and a 32 point increase from 1978.
  - — The average score for Hispanic students was 230 in 2004, which was a 17 point increase from 1999 and a 27 point increase from 1978.
  - — The average score for white students was 247 in 2004, which was an 8 point increase from 1999 and a 23 point increase from 1978.
  - — The average score for Other students was 256 in 2004, which was a 13 point increase from 1999 and a 29 point increase from 1978.

Figure D-2: Average NAEP Scale Scores by Race/Ethnicity, Age 13: Intermittent Years From 1978–2004



*Indicates score or gap is significantly different from 2004.

Note: Data labels for white (above) and black (below). Between race/ethnicity score differences (gaps) are shown in shaded boxes only for years in which the gap is statistically significant. Labeled gaps may not reflect labeled scores because of rounding.

“Bridge” refers to updates made to NAEP in 2004. The updates replaced outdated material and accommodated more students with disabilities. In order to maintain the long-term trend, test takers were randomly assigned to either the old test form, called the bridge assessment, or the modified test form. Results from the bridge assessment should be compared to results from assessments prior to 2004, while results from the modified assessment should be compared to assessments given after 2004.

Source: Created by the Institute for Defense Analysis Science and Technology Policy Institute using the NAEP Data Explorer (http://nces.ed.gov/nationsreportcard/naepdata/).

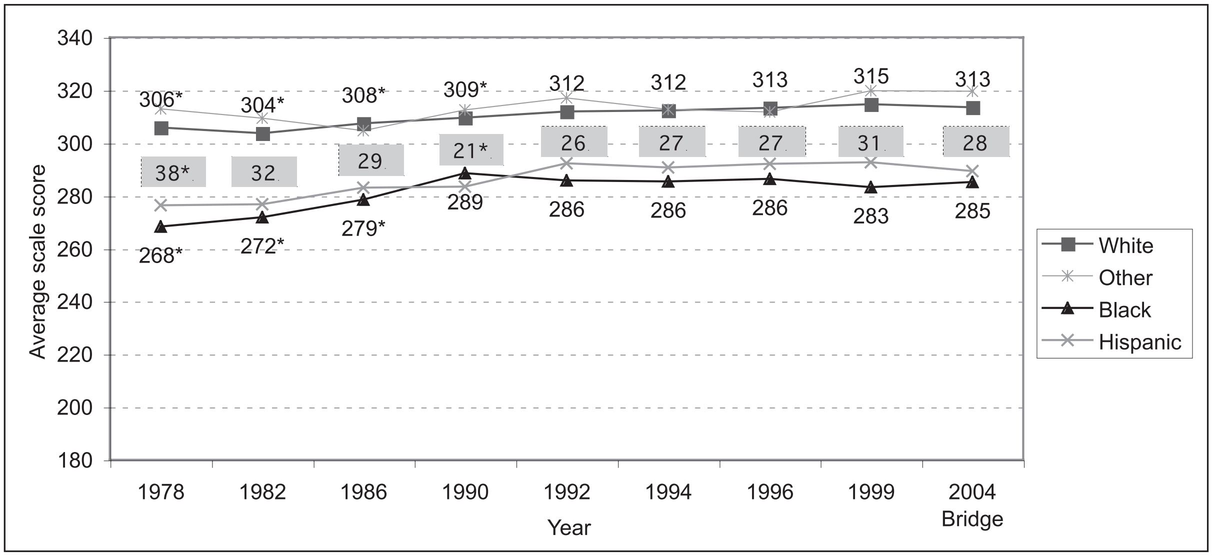
What Is This Indicator?

This indicator presents the average scale score for 13-year-olds by race. “Other” includes Asian/Pacific Islander and American Indian/Alaska Native.

Discussion

- Whites and Other races significantly outscore blacks and Hispanics on the long-term mathematics NAEP at age 13.
- The average score for black students was 27 points lower than the average score for white students in 2004.
  - — The black-white gap has not changed significantly since 1986.
  - — The black-white gap has closed by 15 points since 1978.
- The average score for Hispanic students was 23 points lower than the average score for white students in 2004.
  - — The Hispanic-white gap in 2004 was not significantly different from the gap in 1999.
  - — The Hispanic-white gap has closed by 11 points since 1978.
- Average scores for whites, blacks, and Hispanics were higher in 2004 than in previous assessment years.
  - — The average score for black 13-year-old students was 262 in 2004, which was an 11 point increase from 1999 and a 32 point increase from 1978.
  - — The average score for Hispanic students was 265 in 2004, which was a 6 point increase from 1999 and a 27 point increase from 1978.
  - — The average score for white students was 288 in 2004, which was a 5 point increase from 1999 and a 17 point increase from 1978.
  - — The average score for Other students was 292 in 2004, which was a 20 point increase from 1978, but not significantly different from 1999.

Figure D-3: Average NAEP Scale Scores by Race/Ethnicity, Age 17: Intermittent Years From 1978–2004



*Indicates score or gap is significantly different from 2004.

Note: Data labels for white (above) and black (below). Between race/ethnicity score differences (gaps) are shown in shaded boxes only for years in which the gap is statistically significant. Labeled gaps may not reflect labeled scores because of rounding.

"Bridge" refers to updates made to NAEP in 2004. The updates replaced outdated material and accommodated more students with disabilities. In order to maintain the long-term trend, test takers were randomly assigned to either the old test form, called the bridge assessment, or the modified test form. Results from the bridge assessment should be compared to results from assessments prior to 2004, while results from the modified assessment should be compared to assessments given after 2004.

Source: Created by the Institute for Defense Analysis Science and Technology Policy Institute using the NAEP Data Explorer (http://nces.ed.gov/nationsreportcard/naepdata/).

What Is This Indicator?

This indicator presents the average scale score for 17-year-olds by race. "Other" includes Asian/Pacific Islander and American Indian/Alaska Native.

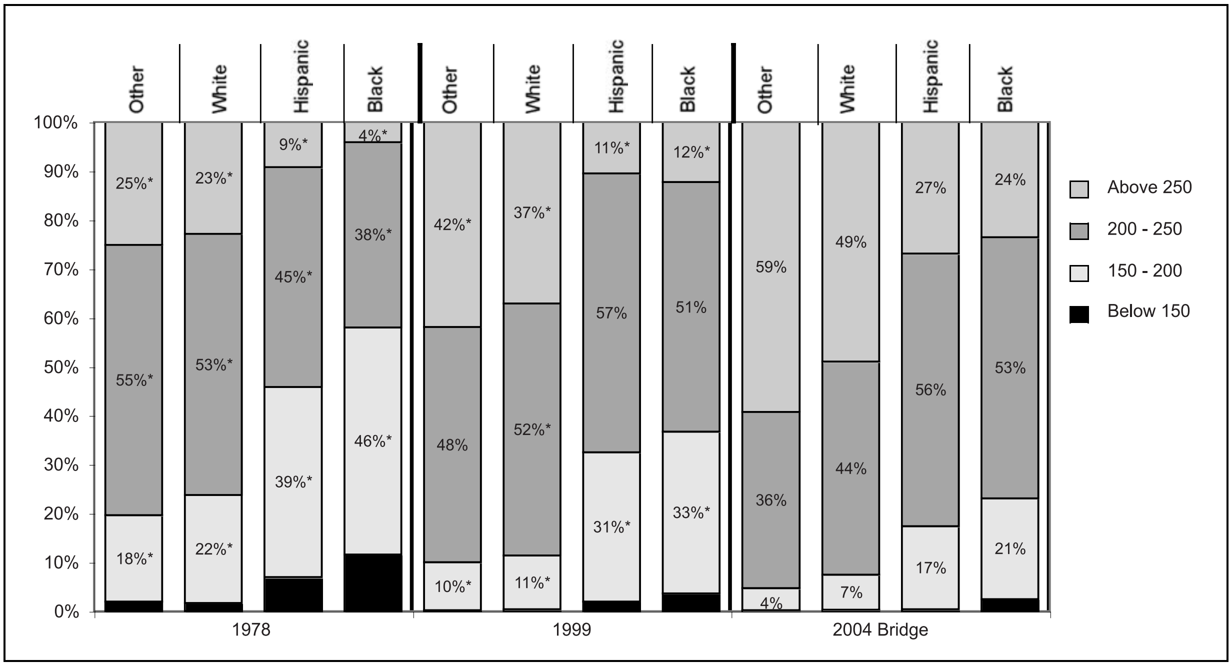
Discussion

- Whites and Other races significantly outscore blacks and Hispanics on the long-term mathematics NAEP at age 17.
- The average score for black students was 28 points lower than the average score for white students in 2004.
  - — The black-white gap has not changed significantly since 1992.
  - — The black-white gap has closed by 10 points since 1978.
- The average score for Hispanic students was 24 points lower than the average score for white students in 2004.
  - — The Hispanic-white gap in 2004 is not significantly different from the gap in 1999.

— The Hispanic-white gap in 2004 is not significantly different from the gap in 1978.

- The average scale scores for whites, blacks, and Hispanics have increased since 1978, but the average scale scores for all races have been flat since 1992.
  — The average score for black students was 285 in 2004, which was a 17 point increase from 1978.
  — The average score for Hispanic students was 289 in 2004, which was a 13 point increase from 1978.
  — The average score for white students was 313 in 2004, which was a 7 point increase from 1978.

Figure D-4: Percent at NAEP Performance Levels by Race/Ethnicity, Age 9: 1978, 1999, and 2004



*Indicates percentage is significantly different from 2004. Differences between categories (racial/ethnic groups) may not be statistically significant.

Note: "Bridge" refers to updates made to NAEP in 2004. The updates replaced outdated material and accommodated more students with disabilities. In order to maintain the long-term trend, test takers were randomly assigned to either the old test form, called the bridge assessment, or the modified test form. Results from the bridge assessment should be compared to results from assessments prior to 2004, while results from the modified assessment should be compared to assessments given after 2004.

Source: Created by the Institute for Defense Analysis Science and Technology Policy Institute using the NAEP Data Explorer (http://nces.ed.gov/nationsreportcard/naepdata/).

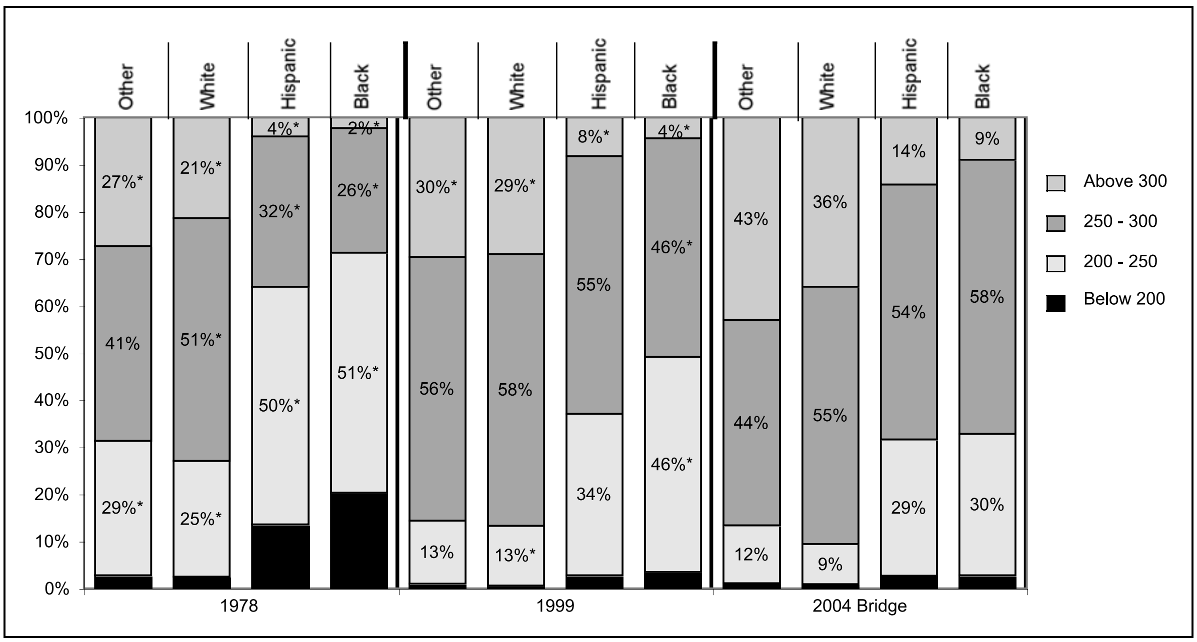
What Is This Indicator?

This indicator presents the percentage of 9-year-olds reaching each performance level by race. The performance levels reported at age 9 are 150—Simple Arithmetic Facts, 200—Beginning Skills and Understandings, and 250—Numerical Operations and Beginning Problem Solving.

Discussion

- More 9-year-old whites and Other races scored at or above the 250 level than blacks and Hispanics.
  - — Differences between whites and Other, and between blacks and Hispanics at the 250 level were generally not significant.
  - — 49% of white students scored at or above 250 in 2004, while only 24%, or half as many black 9-year-olds reached the 250 performance level in 2004.
- The percentages of 9-year-olds of all races at or above the 250 level have increased since the 1999 and the 1978 assessments.
  - — The percentage of black 9-year-olds scoring at or above 250 has increased by a factor of 6 since 1978 and doubled since 1999, going from 4% in 1978 to 12% in 1999 and 24% in 2004.
  - — Meanwhile, the percentage of white 9-year-olds scoring at or above 250 increased from 23% to 49% between 1978 and 2004.
- While blacks and Hispanics have seen large increases in the percent of 9-year-olds in the top performance level, the black-white gap at the 250 performance level has widened slightly since 1978, going from 19% in 1978 to 25% in 1999 and 2004.

Figure D-5: Percent at NAEP Performance Levels by Race/Ethnicity, Age 13: 1978, 1999, and 2004



*Indicates percentage is significantly different from 2004. Differences between categories (racial/ethnic groups) may not be statistically significant.

Note: “Bridge” refers to updates made to NAEP in 2004. The updates replaced outdated material and accommodated more students with disabilities. In order to maintain the long-term trend, test takers were randomly assigned to either the old test form, called the bridge assessment, or the modified test form. Results from the bridge assessment should be compared to results from assessments prior to 2004, while results from the modified assessment should be compared to assessments given after 2004.

Source: Created by the Institute for Defense Analysis Science and Technology Policy Institute using the NAEP Data Explorer (http://nces.ed.gov/nationsreportcard/naepdata/).

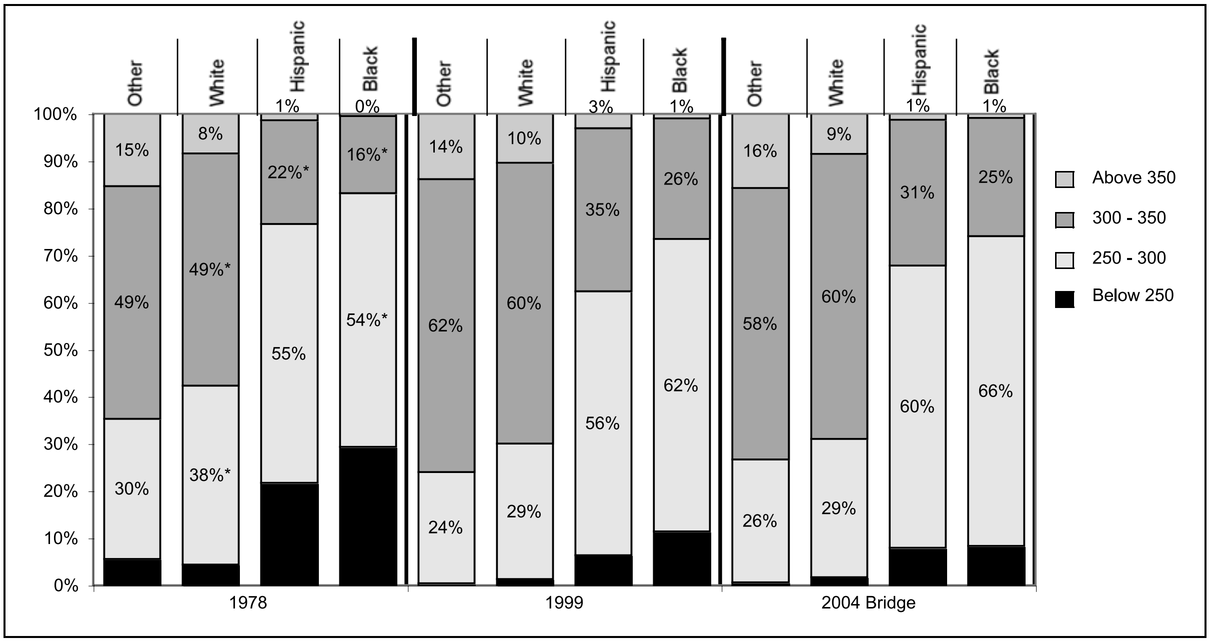
What Is This Indicator?

This indicator presents the percentage of 13-year-olds reaching each performance level by race. The performance levels reported at age 13 are 200—Beginning Skills and Understandings, 250—Numerical Operations and Beginning Problem Solving, and 300—Moderately Complex Procedures and Reasoning.

Discussion

- More 13-year-old whites and Other scored at or above the 300 level than blacks and Hispanics.
  - Differences between whites and Other and between blacks and Hispanics at the 300 level were generally not significant.
  - 36% of white students scored at or above 300 in 2004, while only 9% of black 9-year-olds reached the 300 performance level in 2004.
- The percentages of 13-year-olds of all races at or above the 300 level have increased since the 1999 and the 1978 assessments.
- The black-white gap for 13-year-olds at the 300 performance level has widened slightly since 1978, going from 19% in 1978 to 27% in 2004.
  - During this time period the percentage of black 13-year-olds at the 300 level only increased from 2% in 1978 to 9% in 2004.
  - Meanwhile, the percentage of white 13-year-olds at the 300 level increased from 21% in 1978 to 36% in 2004.

Figure D-6: Percent at NAEP Performance Levels by Race/Ethnicity, Age 17: 1978, 1999, and 2004



*Indicates percentage is significantly different from 2004. Differences between categories (racial/ethnic groups) may not be statistically significant.

Note: "Bridge" refers to updates made to NAEP in 2004. The updates replaced outdated material and accommodated more students with disabilities. In order to maintain the long-term trend, test takers were randomly assigned to either the old test form, called the bridge assessment, or the modified test form. Results from the bridge assessment should be compared to results from assessments prior to 2004, while results from the modified assessment should be compared to assessments given after 2004.

Source: Created by the Institute for Defense Analysis Science and Technology Policy Institute using the NAEP Data Explorer (http://nces.ed.gov/nationsreportcard/naepdata/).

What Is This Indicator?

This indicator presents the percentage of 17-year-olds reaching each performance level by gender. The performance levels reported at age 17 are 250—Numerical Operations and Beginning Problem Solving, 300—Moderately Complex Procedures and Reasoning, and 350—Multistep Problem Solving and Algebra.

Discussion

- The percentage of 17-year-olds at or above the 350 level has not changed over time, although the percentage at the 300 level has increased for all races since 1978.
- In 2004 and 1978, 8.5% of whites reached the 350 performance level, while less than one percent of black 17-year-olds scored at or above 350. In 1999, the difference was not statistically significant.
- Differences between whites and Other, and blacks and Hispanics are generally not statistically significant for 17-year-olds at the 350 level.

Trends in Math and Science Survey: TIMSS

The TIMSS 2003 International Benchmarks of Mathematics Achievement are defined in Mullis et al. (2004, p. 63) as follows:

Grade 8

Advanced International Benchmark – 625

Students can organize information, make generalizations, solve non-routine problems, and draw and justify conclusions from data. They can compute percent change and apply their knowledge of numeric and algebraic concepts, and relationships to solve problems. Students can solve simultaneous linear equations and model simple situations algebraically. They can apply their knowledge of measurement and geometry in complex problem situations. They can interpret data from a variety of tables and graphs, including interpolation and extrapolation.

High International Benchmark – 550

Students can apply their understanding and knowledge in a wide variety of relatively complex situations. They can order, relate, and compute fractions and decimals to solve word problems, operate with negative integers, and solve multi-step word problems involving proportions with whole numbers. Students can solve simple algebraic problems including evaluating expressions, solving simultaneous linear equations, and using a formula to determine the value of a variable. Students can find areas and volumes of simple geometric shapes and use knowledge of geometric properties to solve problems. They can solve probability problems and interpret data in a variety of graphs and tables.

Intermediate International Benchmark – 475

Students can apply basic mathematical knowledge in straightforward situations. They can add, subtract, or multiply to solve one-step word problems involving whole numbers and decimals. They can identify representations of common fractions and relative sizes of fractions. They understand simple algebraic relationships and solve linear equations with one variable. They demonstrate understanding of properties of triangles and basic geometric concepts including symmetry and rotation. They recognize basic notions of probability. They can read and interpret graphs, tables, maps, and scales.

Low International Benchmark – 400

Students have some basic mathematical knowledge. (Mullis et al., 2004, p.62)

Grade 4

Advanced International Benchmark – 625

Students can apply their understanding and knowledge in a wide variety of relatively complex situations. They demonstrate a developing understanding of fractions and decimals, and the relationship between them. They can select appropriate information to solve multistep word problems involving proportions. They can formulate or select a rule for a relationship. They show understanding of area and can use measurement concepts to solve a variety of problems. They show some understanding of rotation. They can organize, interpret, and represent data to solve problems.

High International Benchmark – 550

Student can apply their knowledge and understanding to solve problems. Students can solve multistep word problems involving addition, multiplication, and division. They can use their understanding of place value and simple fractions to solve problems. They can identify a number sentence that represents situations. Students show understanding of three-dimensional objects, how shapes can make other shapes, and simple transformation in a plane. They demonstrate a variety of measurement skills, and can interpret and use data in tables and graphs to solve problems.

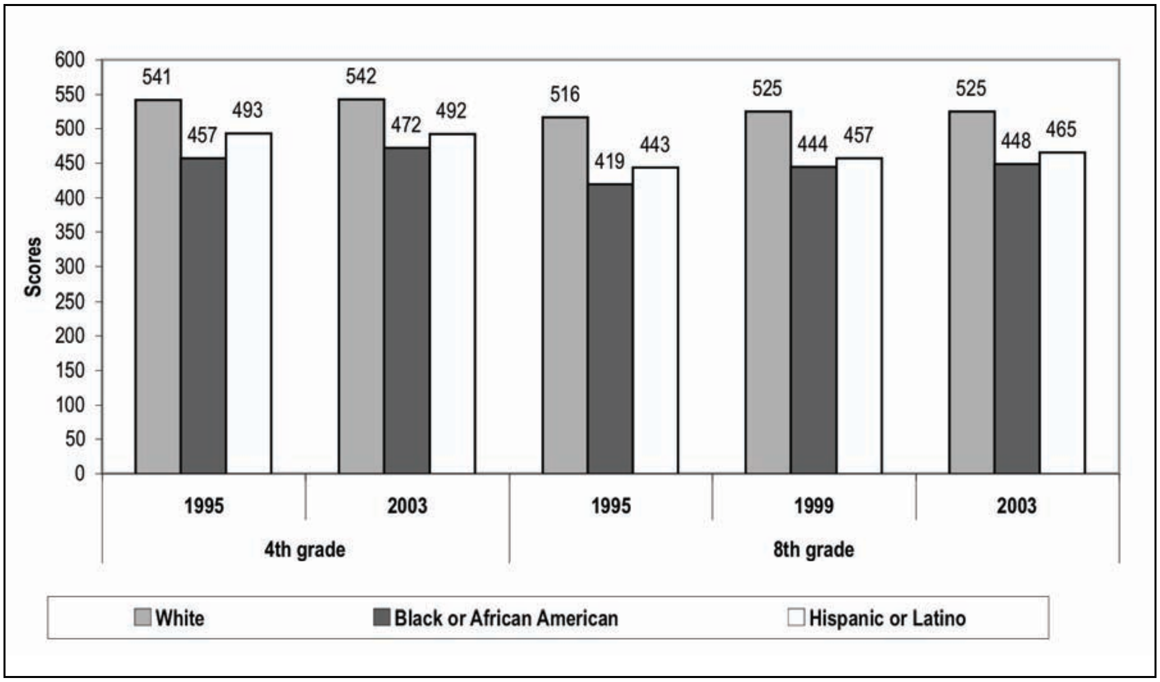
Intermediate International Benchmark – 475

Students can apply basic mathematical knowledge in straightforward situations. They can read, interpret, and use different representations of numbers. They can perform operations with three- and four-digit numbers and decimals. They can extend simple patterns. They are familiar with a range of two-dimensional shapes, and read and interpret different representations of the same data.

Low International Benchmark – 400

Students have some basic mathematical knowledge. Students demonstrate an understanding of whole numbers and can do simple computations with them. They demonstrate familiarity with the basic properties of triangles and rectangles. They can read information from simple bar graphs.

Figure D-7: Average TIMSS Mathematical Scale Scores of U.S. 4th- and 8th-Graders, by Race/Ethnicity: Various Years From 1995–2003



Note: TIMSS international benchmarks: Low 400, Intermediate 475, High 550, Advanced 625

Source: Gonzales et al. (2004), Figures 1 & 2.

Standardized mean difference TIMSS, race/ethnicity					
	4th grade		8th grade		
	1995	*2003*	*1995*	*1999*	*2003*
White-Black	1.01	0.92	1.08	0.92	0.96
White-Hispanic	0.57	0.66	0.81	0.77	0.75

High School and Beyond of 1980: HS&B:80
National Education Longitudinal Study of 1988: NELS:88
Education Longitudinal Study of 2002: ELS:2002

The scores on the HS&B:80, NELS:88, and ELS:2002 are Item Response Theory (IRT) scores on the NELS:88 1990 58-item scale. IRT estimates achievement based on patterns of correct, incorrect, and unanswered questions. "The IRT-estimated number-right score reflects an estimate of the number of these 58 items that an examinee would have answered correctly if he or she had taken all of the items that appeared on the multiform 1990 NELS:88 mathematics test. The score is the probability of a correct answer on each item, summed over the total mathematics 58-item pool" (Cahalan et al., 2006, p. 45). These scores are not directly translated into probability of proficiency scores. However, five probability of proficiency scores in mathematics were estimated for students using performance on clusters of four items each as follows:

Probability of Mastery, Mathematics Levels

1) Simple arithmetical operations on whole numbers, such as simple arithmetic expressions involving multiplication or division of integers;

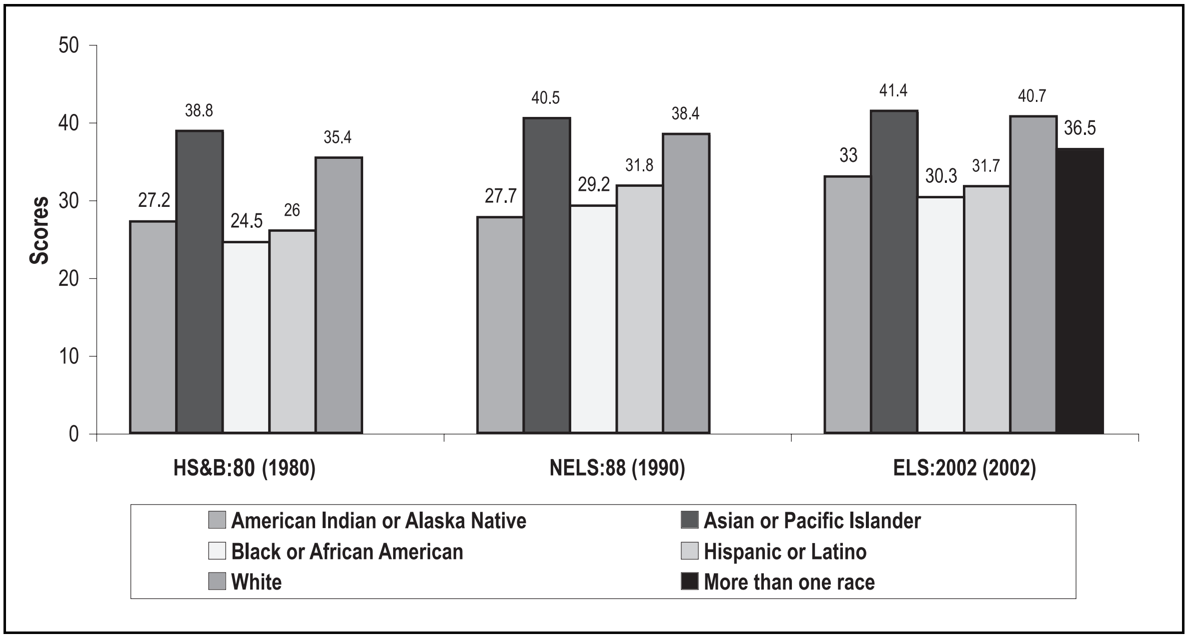
2) Simple operations with decimals, fractions, powers, and roots, such as comparing expressions, given information about exponents;

3) Simple problem solving, requiring the understanding of low-level mathematical concepts, such as simplifying an algebraic expression or comparing the length of line segments illustrated in a diagram;

4) Understanding of intermediate-level mathematical concepts and/or multistep solutions to word problems such as drawing an inference based on an algebraic expression or inequality; and

5) Complex multistep word problems and/or advanced mathematics material such as a two-step problem requiring evaluation of functions. (Cahalan et al., 2006, p. A-28)

Figure D-8: IRT—Estimated Average Math Score (10th-Grade), by Race/Ethnicity (HS&B:80, NELS:88, ELS:2002)



Note: IRT scale score is the estimated number right out of a total of 58.

Source: Cahalan et al. (2006), Tables 18 and 19.

Standardized mean difference sophomores, race/ethnicity			
	HS&B (1980)	**NELS:88 (1990)**	**ELS:2002 (2002)**
White-American Indian	0.71	0.94	0.78
White-Asian	-0.28	-0.18	-0.07
White-Black	1.01	0.82	1.03
White-Hispanic	0.84	0.58	0.84
White-Other			0.39

Table D-1: Probability of 10th-Grade Proficiency in Mathematics by Race/Ethnicity

	NELS:88 (1990)	ELS:2002 (2002)
Level 1		
Asian or Pacific Islander	93.7	95.2
Black or African American	80.8	83.8
Hispanic or Latino	85.0	83.7
White	93.3	95.5
Level 2		
Asian or Pacific Islander	73.7	77.6
Black or African American	38.4	42.3
Hispanic or Latino	44.9	46.9
White	69.6	77.9
Level 3		
Asian or Pacific Islander	57.8	60.2
Black or African American	18.7	19.4
Hispanic or Latino	24.4	25.5
White	50.1	57.9
Level 4		
Asian or Pacific Islander	29.6	31.7
Black or African American	5.2	4.7
Hispanic or Latino	8.0	8.8
White	22.5	27.0
Level 5		
Asian or Pacific Islander	1.2	4.0
Black or African American	less than 0.1	0.1
Hispanic or Latino	0.1	0.3
White	0.5	1.2

Note: Proficiency levels – 1) Simple arithmetical operations with whole numbers; 2) Simple operations with decimals, fractions, powers, and roots; 3) Simple problem solving, requiring the understanding of low-level mathematical concepts; 4) Understanding of intermediate-level mathematical concepts and/or multistep solutions to word problems; and 5) Complex multistep word problems and/or advanced mathematics material.

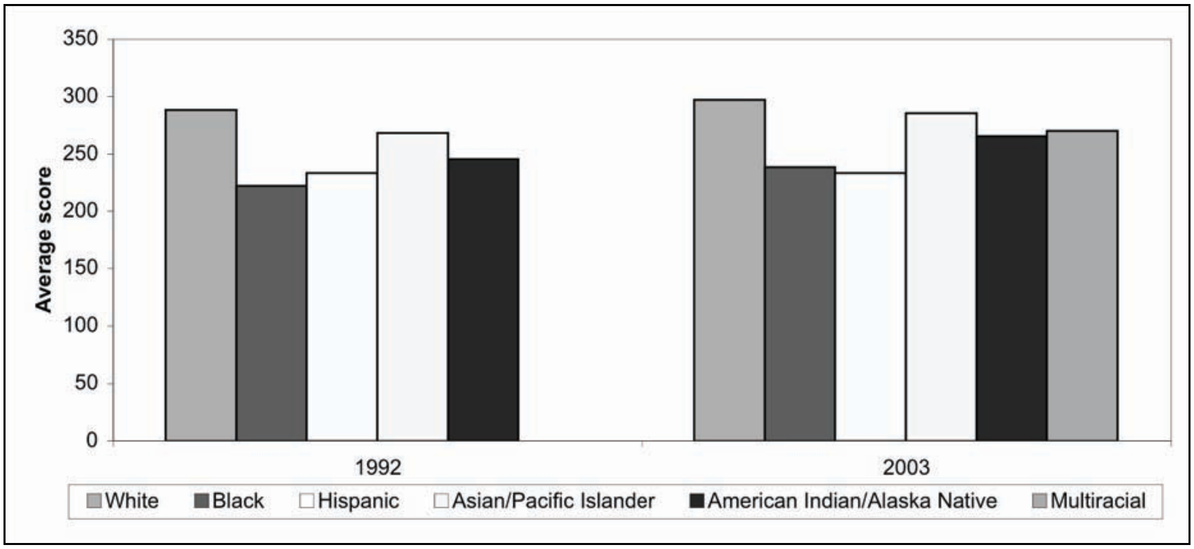
Source: Cahalan et al., 2006, p. 58.

National Adult Literacy Survey: NALS
National Assessment of Adult Literacy: NAAL

The Committee on Performance Levels for Adult Literacy set performance levels for quantitative literacy as Below Basic, Basic, Intermediate, and Proficient and defined them as follows, based on scores on NALS and NAAL:

> ***Below Basic (0–234)*** indicates no more than the most simple and concrete literacy skills.
>
> *Key abilities*—locating numbers and using them to perform simple *quantitative* operations (primarily addition) when the mathematical information is very concrete and familiar.
>
> ***Basic (235–289)*** indicates skills necessary to perform simple and everyday literacy activities.
>
> *Key abilities*—locating easily identifiable *quantitative* information and using it to solve simple, one-step problems when the arithmetic operation is specified or easily inferred.
>
> ***Intermediate (290–349)*** indicates skills necessary to perform moderately challenging literacy activities.
>
> *Key abilities*—locating less familiar *quantitative* information and using it to solve problems when the arithmetic operation is not specified or easily inferred.
>
> ***Proficient (350–500)*** indicates skills necessary to perform more complex and challenging literacy activities.
>
> *Key abilities*—locating more abstract quantitative information and using it to solve multistep problems when the arithmetic operations are not easily inferred and the problems are more complex. (Kutner et al., 2006, p. 3).

Figure D-9: Average Quantitative Literacy Scores of Adults, by Race/Ethnicity: NALS 1992 and NAAL 2003



Note: Literacy levels: Below basic 0-234, Basic 235-289, Intermediate 290-349, Proficient 350-500

Source: Kutner, Greenberg, and Baer (2006), Figure 1: Kutner et al. (2006), Figure 2-6a.

Standardized mean difference adults, race/ethnicity		
	NALS 1992	**NAAL 2003**
White-Black	1.00	0.97
White-Hispanic	0.83	1.05
White-Asian	-0.17	0.08
White-American Indian	0.65	0.52
White-Multiracial		0.44

Table D-2: Percentage of Adults in Each Quantitative Literacy Level, by Race/Ethnicity: NALS 1992 and NAAL 2003

		NALS, 1992	NAAL, 2003
Below basic			
	White	9	7
	Black	30	24
	Hispanic	35	44
	Asian/Pacific Islander	25	14
	American Indian/Alaska Native	17	19
	Multiracial		7
Basic			
	White	25	25
	Black	41	43
	Hispanic	33	30
	Asian/Pacific Islander	30	32
	American Indian/Alaska Native	43	29
	Multiracial		35
Intermediate			
	White	48	51
	Black	27	31
	Hispanic	28	23
	Asian/Pacific Islander	36	42
	American Indian/Alaska Native	35	41
	Multiracial		54
Proficient			
	White	18	17
	Black	2	2
	Hispanic	5	4
	Asian/Pacific Islander	9	12
	American Indian/Alaska Native	5	10
	Multiracial		4

Note: Below Basic (0–234)—no more than the most simple and concrete literacy skills; Basic (235–289)—skills necessary to perform simple and everyday literacy activities; Intermediate (290–349)—skills necessary to perform moderately challenging literacy activities; Proficient (350–500)—skills necessary to perform more complex and challenging literacy activities.

Source: Kutner et al., 2007, p. 16.

Program for International Student Assessment: PISA

Mathematics literacy can be classified by proficiency levels, based on scores on the PISA, as follows:

Below level 1 (*less than or equal to 357.77*)

Level 1 (*greater than 357.77 to 420.07*) At Level 1, students can answer questions involving familiar contexts where all relevant information is present and the questions are clearly defined. They are able to identify information and to carry out routine procedures according to direct instructions in explicit situations. They can perform actions that are obvious and follow immediately from the given stimuli.

Level 2 (*greater than 420.07 to 482.38*) At Level 2, students can interpret and recognize situations in contexts that require no more than direct inference. They can extract relevant information from a single source and make use of a single representational mode. Students at this level can employ basic algorithms, formula, procedures, or conventions. They are capable of direct reasoning and making literal interpretations of the results.

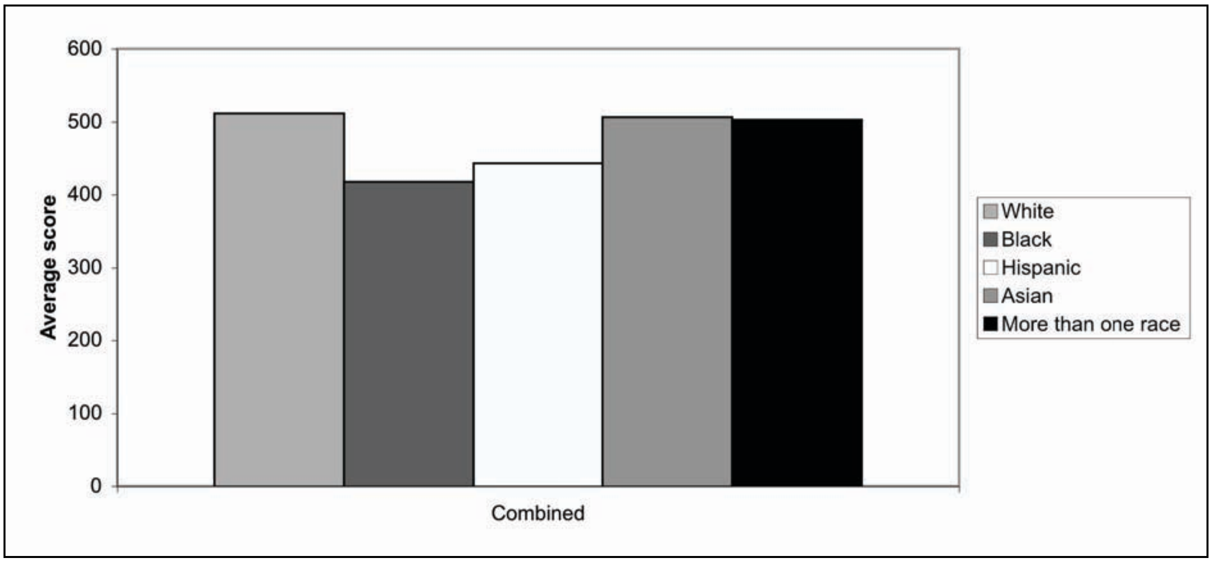
Level 3 (*greater than 482.38 to 544.68*) At Level 3, students can execute clearly described procedures, including those that require sequential decisions. They can select and apply simple problem solving strategies. Students at this level can interpret and use representations based on different information sources and reason directly from them. They can develop short communications reporting their interpretations, results, and reasoning.

Level 4 (*greater than 544.68 to 606.99*) At Level 4, students can work effectively with explicit models for complex concrete situations that may involve constraints or call for making assumptions. They can select and integrate different representations, including symbolic, linking them directly to aspects of real-world situations. Students at this level can utilize well developed skills and reason flexibly, with some insight, in these contexts. They can construct and communicate explanations and arguments based on their interpretations, arguments, and actions.

Level 5 (*greater than 606.99 to 669.3*) At Level 5, students can develop and work with models for complex situations, identifying constraints and specifying assumptions. They can select, compare, and evaluate appropriate problem solving strategies for dealing with complex problems related to these models. Students at this level can work strategically using broad, well-developed thinking and reasoning skills, appropriate linked representations, symbolic and formal characterizations, and insight pertaining to these situations. They can reflect on their actions and formulate and communicate their interpretations and reasoning.

Level 6 (*greater than 669.3*) At Level 6, students can conceptualize, generalize, and utilize information based on their investigations and modeling of complex problem situations. They can link different information sources and representations and flexibly translate among them. Students at this level are capable of advanced mathematical thinking and reasoning. These students can apply this insight and understandings along with a mastery of symbolic and formal mathematical operations and relationships to develop new approaches and strategies for attacking novel situations. Students at this level can formulate and precisely communicate their actions and reflections regarding their findings, interpretations, arguments, and the appropriateness of these to the original situations (Lemke et al., 2005, p.18).

Figure D-10: Average Mathematic Literacy Scores of U.S. 15-Year-Olds, by Race/Ethnicity: 2003 PISA



Note: Level 1 (greater than 357.77 to 420.07), Level 2 (greater than 420.07 to 482.38), Level 3 (greater than 482.38 to 544.68), Level 4 (greater than 544.68 to 606.99), Level 5 (greater than 606.99 to 669.3), Level 6 (greater than 669.3).

Source: Lemke et al. (2005) Tables B-26.

Standardized mean difference, 15-year-olds, race/ethnicity	
White-Black	0.99
White-Hispanic	0.72
White-Asian	0.06
White-Multiracial	0.10

Source: Lemke et al., 2005, Tables B-19 and B-6.

APPENDIX E: Socioeconomic Differences (SES)

The following tables and exhibits summarize the data on math performance by SES using data available on national samples. Data from the National Assessment of Educational Progress (NAEP) Long-Term Trend study illustrate performance between groups over the last 30 years. Data from the Trends in Math and Science Survey (TIMSS) illustrate the math performance of fourth- and eighth-graders. Data from the High School and Beyond (HS&B:80), National Education Longitudinal Study of 1988 (NELS:88), and Education Longitudinal Study of 2002 (ELS:2002) illustrate the math performance of 10th-grade students. Data from the National Adult Literacy Survey (NALS) and the National Assessment of Adult Literacy (NAAL) survey illustrate the quantitative literacy of adults. Data from the Program for International Student Assessment (PISA) illustrate the mathematics literacy and problem-solving proficiency of 15-year-olds. To facilitate the interpretation of the various scores, a description of the test benchmarks and performance levels associated with each test is provided.

National Assessment of Educational Progress Long-Term Trends: Mathematics Scores

This section presents the trends in long-term NAEP mathematics scores. The goal is to describe the differences in performance between groups over the last 30 years and to describe how their scores have evolved over time. For each reporting group, results are presented in the form of the average scale score for 1978–2004 and the percent of students at each achievement level in 1978, 1999, and 2004.

Methodology

All data presented in this section were obtained from the NAEP Data Explorer.[4] The Data Explorer allows users to create tables of results by custom combinations of reporting variables. The results can be reported in terms of mean score, percentage of students at or above performance levels, and score percentile.

The Data Explorer also reports standard errors and can calculate the statistical significance of changes in a variable between years or between variables in the same year. The statistical significance of changes between variables over time (e.g., the score difference between girls and boys in 1978 versus the score difference between girls and boys in 2004) is taken either directly from the *NAEP 2004 Trends in Academic Progress* or estimated using the reported standard error provided by the Data Explorer. Only differences that are statistically significant beyond the 0.05 level are described in the text of this section.

[4] http://nces.ed.gov/nationsreportcard/naepdata/

Average Scale Scores and Performance Levels

The NAEP long-term trend assessments are scored on a 0–500 point scale, but all average scale score charts presented here are ranged from 180–340 for consistency and best visibility of score differences. Charts of average scale scores are reconstructed to resemble the gap charts in *NAEP 2004 Trends in Academic Progress*.

The following text was taken verbatim from the National Center for Education Statistics website, http://nces.ed.gov/nationsreportcard/ltt/performance-levels.asp in April 2007.

More detailed information about what students know and can do in each subject area can be gained by examining their attainment of specific performance levels in each assessment year. This process of developing the performance-level descriptions is different from that used to develop *achievement-level* descriptions in the main NAEP reports.

For each of the subject area scales, performance levels were set at 50-point increments from 150 through 350. The five performance levels—150, 200, 250, 300, and 350—were then described in terms of the knowledge and skills likely to be demonstrated by students who reached each level.

A “scale anchoring” process was used to define what it means to score in each of these levels. NAEP’s scale anchoring follows an empirical procedure whereby the scaled assessment results are analyzed to delineate sets of questions that discriminate between adjacent performance levels on the scales. To develop these descriptions, assessment questions were identified that students at a particular performance level were more likely to answer successfully than students at lower levels. The descriptions of what students know and can do at each level are based on these sets of questions.

The guidelines used to select the questions were as follows: Students at a given level must have at least a specified probability of success with the questions (75% for mathematics, 80% for reading), while students at the next lower level have a much lower probability of success (that is, the difference in probabilities between adjacent levels must exceed 30%). For each curriculum area, subject-matter specialists examined these empirically selected question sets and used their professional judgment to characterize each level. The scale anchoring for mathematics trend reporting was based on the 1986 assessment.

The five performance levels are applicable at all three age groups, but only three performance levels are discussed for each age: levels 150, 200, and 250 for age 9; levels 200, 250, and 300 for age 13; and levels 250, 300, and 350 for age 17. These performance levels are the ones most likely to show significant change within an age across the assessment years and do not include the levels that nearly all or almost no students attained at a particular age in each year.

The following description of each mathematics performance level was copied from http://nces.ed.gov/nationsreportcard/ltt/math-descriptions.asp in April 2007.

Level 350: Multistep Problem Solving and Algebra

Students at this level can apply a range of reasoning skills to solve multistep problems. They can solve routine problems involving fractions and percents, recognize properties of basic geometric figures, and work with exponents and square roots. They can solve a variety of two-step problems using variables, identify equivalent algebraic expressions, and solve linear equations and inequalities. They are developing an understanding of functions and coordinate systems.

Level 300: Moderately Complex Procedures and Reasoning

Students at this level are developing an understanding of number systems. They can compute with decimals, simple fractions, and commonly encountered percents. They can identify geometric figures, measure lengths and angles, and calculate areas of rectangles. These students are also able to interpret simple inequalities, evaluate formulas, and solve simple linear equations. They can find averages, make decisions based on information drawn from graphs, and use logical reasoning to solve problems. They are developing the skills to operate with signed numbers, exponents, and square roots.

Level 250: Numerical Operations and Beginning Problem Solving

Students at this level have an initial understanding of the four basic operations. They are able to apply whole number addition and subtraction skills to one-step word problems and money situations. In multiplication, they can find the product of a two-digit and a one-digit number. They can also compare information from graphs and charts, and are developing an ability to analyze simple logical relations.

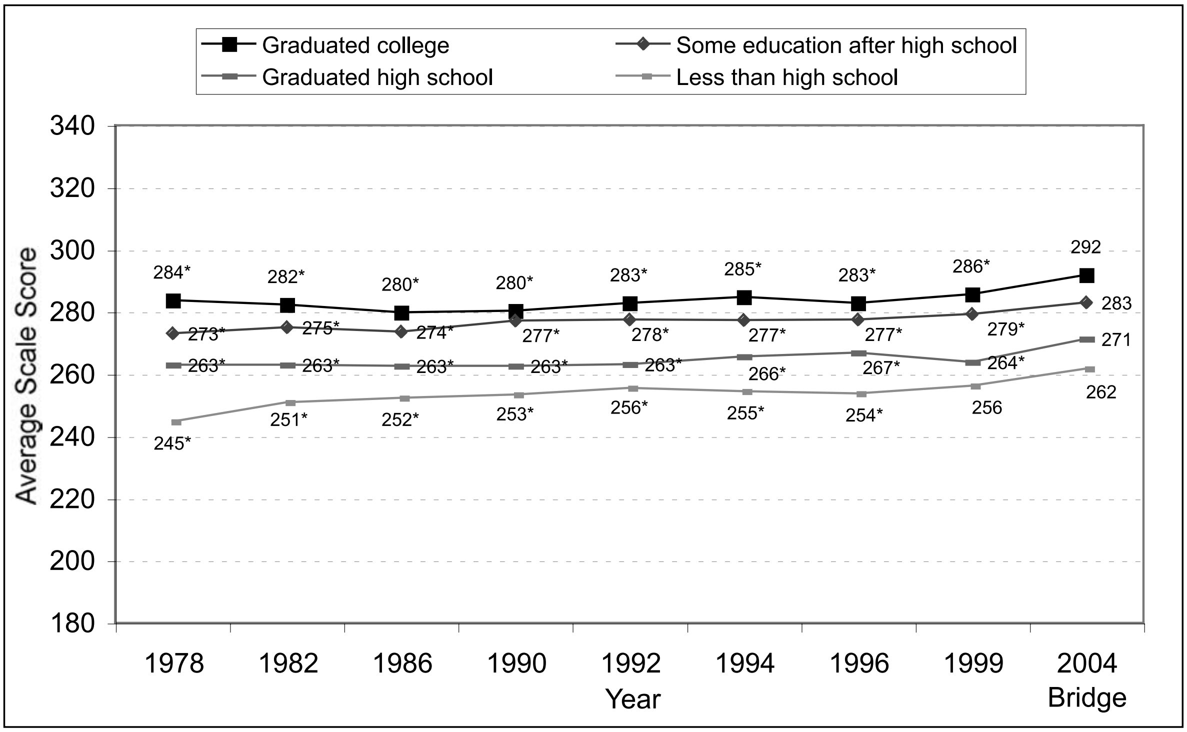
Level 200: Beginning Skills and Understandings

Students at this level have considerable understanding of two-digit numbers. They can add two-digit numbers but are still developing an ability to regroup in subtraction. They know some basic multiplication and division facts, recognize relations among coins, can read information from charts and graphs, and use simple measurement instruments. They are developing some reasoning skills.

Level 150: Simple Arithmetic Facts

Students at this level know some basic addition and subtraction facts, and most can add two-digit numbers without regrouping. They recognize simple situations in which addition and subtraction apply. They also are developing rudimentary classification skills.

Figure E-1: Average NAEP Scale Scores, by Parents' Highest Level of Education, Age 13: Intermittent Years From 1978–2004



*Indicates score is significantly different from 2004.

Note: "Bridge" refers to updates made to NAEP in 2004. The updates replaced outdated material and accommodated more students with disabilities. In order to maintain the long-term trend, test takers were randomly assigned to either the old test form, called the bridge assessment, or the modified test form. Results from the bridge assessment should be compared to results from assessments prior to 2004, while results from the modified assessment should be compared to assessments given after 2004.

Source: Created by the Institute for Defense Analysis Science and Technology Policy Institute using the NAEP Data Explorer (http://nces.ed.gov/nationsreportcard/naepdata/).

What Is This Indicator?

NAEP asks 13- and 17-year-old students to report both of their parents' highest level of education. Parental education level is the background variable on the long-term NAEP that most closely addresses SES. This indicator presents the average scale score of 13-year-old students grouped by the highest level of education attained by either parent.

Discussion

- Parents' level of education is directly related to students' average scale score.
- In 2004, 13-year-olds with at least one parent who graduated college scored 30 points higher than students whose parents had less than a high school education. This gap has not changed significantly since 1978.

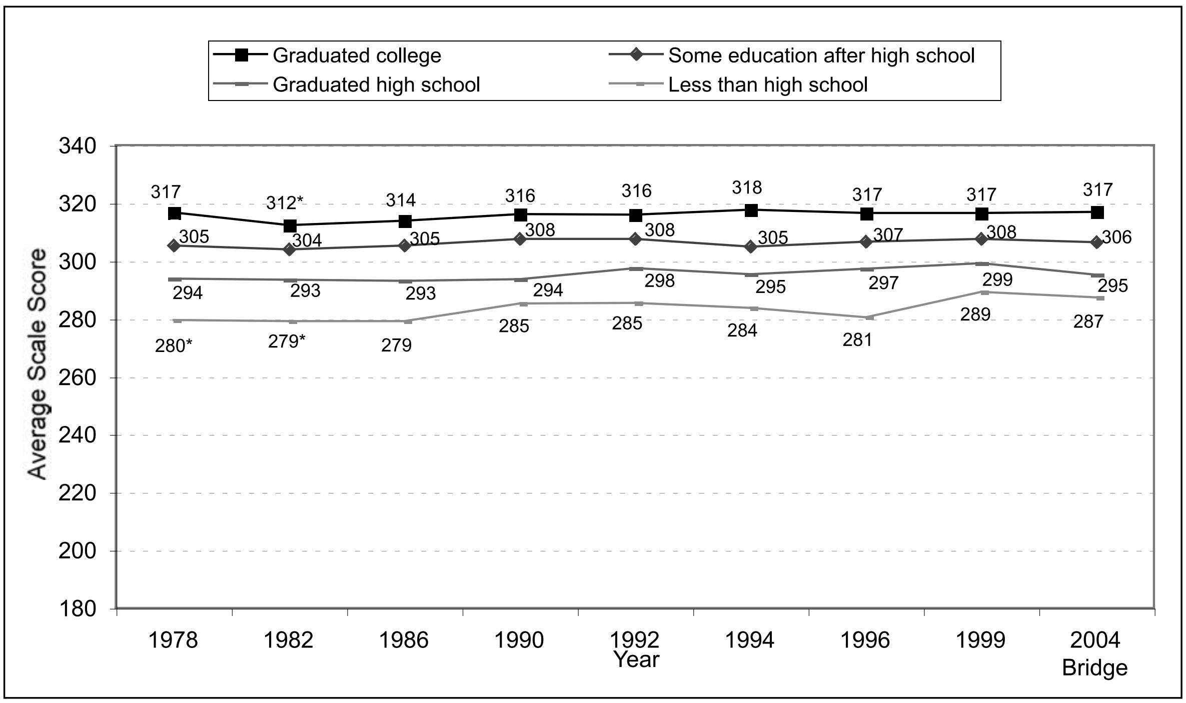
— In 1978, the gap between 13-year-olds with at least one parent who graduated from college and 13-year-olds whose parents did not complete high school was 39 points. This is a significant difference from 2004.

— The gap between 13-year-olds with at least one parent who graduated from high school and 13-year-olds whose parents did not complete high school has also improved since 1978, decreasing from 18 points in 1978 to 10 points in 2004.

- For 13-year-olds who reported that their parents completed high school, had some education after high school, or completed college, average scores were higher in 2004 than in any previous assessment year.

  — The average score for 13-year-olds whose parents did not finish high school has increased since 1978 but did not change significantly between 1999 and 2004.

Figure E-2: Average NAEP Scale Scores, by Parents' Highest Level of Education, Age 17: Intermittent Years From 1978–2004



*Indicates score is significantly different from 2004

Note: "Bridge" refers to updates made to NAEP in 2004. The updates replaced outdated material and accommodated more students with disabilities. In order to maintain the long-term trend, test takers were randomly assigned to either the old test form, called the bridge assessment, or the modified test form. Results from the bridge assessment should be compared to results from assessments prior to 2004, while results from the modified assessment should be compared to assessments given after 2004.

Source: Created by the Institute for Defense Analysis Science and Technology Policy Institute using the NAEP Data Explorer (http://nces.ed.gov/nationsreportcard/naepdata/).

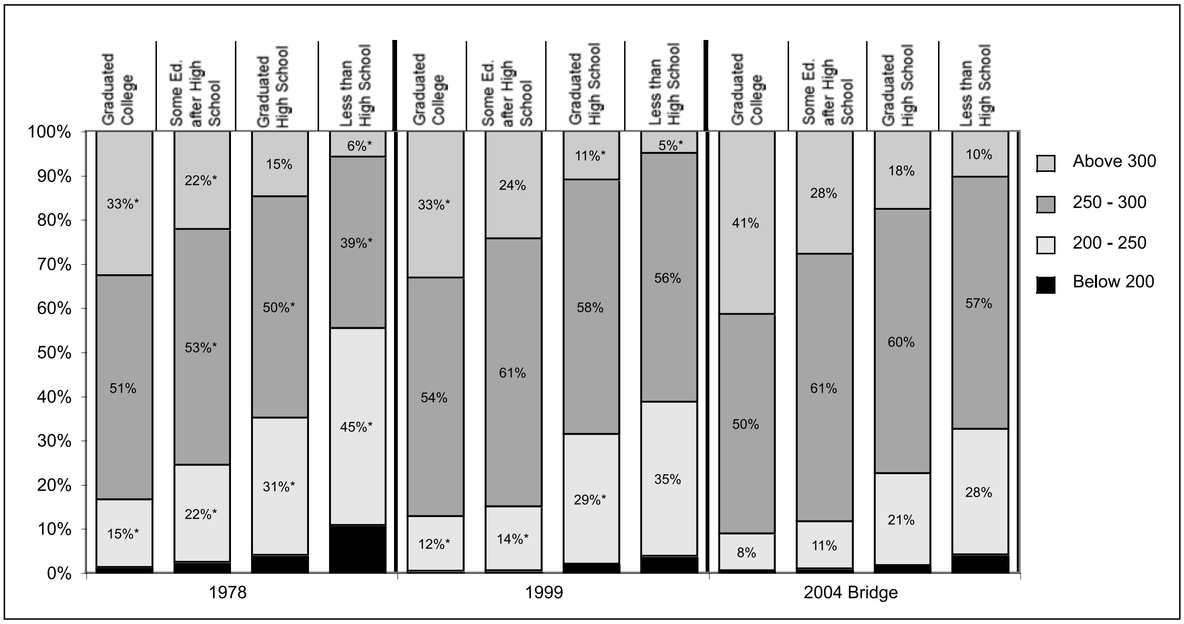
What Is This Indicator?

NAEP asks 13- and 17-year-old students to report both of their parents' highest level of education. Parental education level is the background variable on the long-term NAEP that most closely addresses SES. This indicator presents the average scale score of 17-year-old students grouped by the highest level of education attained by either parent.

Discussion

- Parents' level of education is directly related to students' average scale score.
- In 2004, 17-year-olds with at least one parent who graduated college scored 30 points higher than 17-year-olds whose parents had less than a high school education. This gap has not changed significantly since 1978.
    - — In 1978, the gap between 17-year-olds with at least one parent who graduated from college and 17-year-olds whose parents did not complete high school was 37 points. This is a significant difference from 2004.
    - — The gap between 17-year-olds with at least one parent who graduated from high school and 17-year-olds whose parents did not complete high school has also improved since 1978, decreasing from 14 points in 1978 to 8 points in 2004.
- The average scale score for 17-year-olds at all levels of parental education have generally not changed over the life of the assessment.
    - — The average scale score of 17-year-olds whose parents did not graduate from high school increased from 280 in 1978 to 187 in 2004, but the average scores of all other groups are flat when compared to 1978 and 1999.

Figure E-3: Percent at NAEP Performance Levels, by Parents' Highest Level of Education, Age 13: 1978, 1999, and 2004



*Indicates percentage is significantly different from 2004. Differences between categories may not be statistically significant.

Note: "Bridge" refers to updates made to NAEP in 2004. The updates replaced outdated material and accommodated more students with disabilities. In order to maintain the long-term trend, test takers were randomly assigned to either the old test form, called the bridge assessment, or the modified test form. Results from the bridge assessment should be compared to results from assessments prior to 2004, while results from the modified assessment should be compared to assessments given after 2004.

Source: Created by the Institute for Defense Analysis Science and Technology Policy Institute using the NAEP Data Explorer (http://nces.ed.gov/nationsreportcard/naepdata/).

What Is This Indicator?

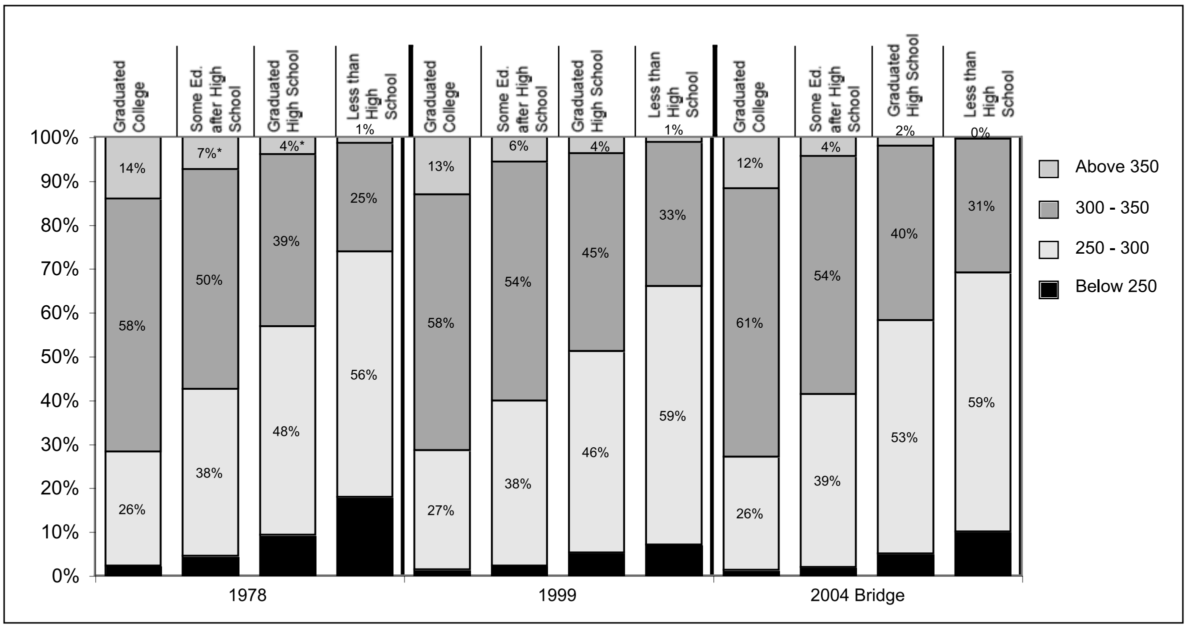
NAEP asks 13- and 17-year-old students to report both of their parents' highest level of education. Parental education level is the background variable on the long-term NAEP that most closely addresses SES. This indicator presents the percentage of 13-year-olds reaching each performance level by parents' highest level of education. The performance levels reported at age 13 are 200—Beginning Skills and Understandings, 250—Numerical Operations and Beginning Problem Solving, and 300—Moderately Complex Procedures and Reasoning.

Discussion

- Higher levels of parental education correlate with a higher percentage of 13-year-olds scoring at or above the 300 level, and a lower percentage of students at or below the 200 level. The effect of parental education on the percentage of 13-year-olds at the 250 level is not significant in most cases.

- While the percentage of students at or above 300 has increased over time for all parental education groups, the gap in achievement between the highest and lowest parental education groups has not changed significantly since 1999 or 1978.
  - — The percentage of 13-year-olds with at least one parent who graduated from college scoring at or above the 300 level increased by 9%, from 33% in 1978 and 1999 to 41% in 2004.
  - — The percentage of 13-year-olds whose parents did not finish high school scoring at or above the 300 level increased by 5%, from 5 to 6% in 1978 and 1999 to 10% in 2004.

Figure E-4: Percent at NAEP Performance Levels, by Parents' Highest Level of Education, Age 17: 1978, 1999, and 2004



*Indicates percentage is significantly different from 2004. Differences between categories may not be statistically significant.

Note: "Bridge" refers to updates made to NAEP in 2004. The updates replaced outdated material and accommodated more students with disabilities. In order to maintain the long-term trend, test takers were randomly assigned to either the old test form, called the bridge assessment, or the modified test form. Results from the bridge assessment should be compared to results from assessments prior to 2004, while results from the modified assessment should be compared to assessments given after 2004.

Source: Created by the Institute for Defense Analysis Science and Technology Policy Institute using the NAEP Data Explorer (http://nces.ed.gov/nationsreportcard/naepdata/).

What Is This Indicator?

NAEP asks 13- and 17-year-old students to report both of their parents' highest level of education. Parental education level is the background variable on the long-term NAEP that most closely addresses SES. This indicator presents the percentage of 17-year-olds reaching each performance level by parents' highest level of education. The performance levels reported at age 17 are 250—Numerical Operations and Beginning Problem Solving, 300—Moderately Complex Procedures and Reasoning, and 350—Multistep Problem Solving and Algebra.

Discussion

- Higher levels of parental education correlate with a higher percentage of 17-year-olds scoring at the 300 and 350 levels, and a lower percentage of students at or below the 250 level.
- The achievement rates of 17-year-olds in all parental education groups and performance levels have not changed since 1999 or 1978.
- Because of the small number of students whose parents did not graduate from high school reaching the 350 level, NAEP does not report statistical significance for comparisons with that subgroup.

Trends in Math and Science Survey: TIMSS

The TIMSS 2003 International Benchmarks of Mathematics Achievement are defined in Mullis et al. (2004, p. 63) as follows:

Grade 8

Advanced International Benchmark – 625

Students can organize information, make generalizations, solve non-routine problems, and draw and justify conclusions from data. They can compute percent change and apply their knowledge of numeric and algebraic concepts and relationships to solve problems. Students can solve simultaneous linear equations and model simple situations algebraically. They can apply their knowledge of measurement and geometry in complex problem situations. They can interpret data from a variety of tables and graphs, including interpolation and extrapolation.

High International Benchmark – 550

Students can apply their understanding and knowledge in a wide variety of relatively complex situations. They can order, relate, and compute fractions and decimals to solve word problems, operate with negative integers, and solve multi-step word problems involving proportions with whole numbers. Students can solve simple algebraic problems including evaluating expressions, solving simultaneous linear equations, and using a formula to determine the value of a variable. Students can find areas and volumes of simple geometric shapes and use knowledge of geometric properties to solve problems. They can solve probability problems and interpret data in a variety of graphs and tables.

Intermediate International Benchmark – 475

Students can apply basic mathematical knowledge in straightforward situations. They can add, subtract, or multiply to solve one-step word problems involving whole numbers and decimals. They can identify representations of common fractions and relative sizes of fractions. They understand simple algebraic relationships and solve linear equations with one variable. They demonstrate understanding of properties of triangles and basic geometric concepts including symmetry and rotation. They recognize basic notions of probability. They can read and interpret graphs, tables, maps, and scales.

Low International Benchmark – 400

Students have some basic mathematical knowledge (Mullis et al., 2004, p.62).

Grade 4

Advanced International Benchmark – 625

Students can apply their understanding and knowledge in a wide variety of relatively complex situations. They demonstrate a developing understanding of fractions and decimals and the relationship between them. They can select appropriate information to solve multistep word problems involving proportions. They can formulate or select a rule for a relationship. They show understanding of area and can use measurement concepts to solve a variety of problems. They show some understanding of rotation. They can organize, interpret, and represent data to solve problems.

High International Benchmark – 550

Student can apply their knowledge and understanding to solve problems. Students can solve multi-step word problems involving addition, multiplication, and division. They can use their understanding of place value and simple fractions to solve problems. They can identify a number sentence that represents situations. Students show understanding of three-dimensional objects, how shapes can make other shapes, and simple transformation in a plane. They demonstrate a variety of measurement skills and can interpret and use data in tables and graphs to solve problems.

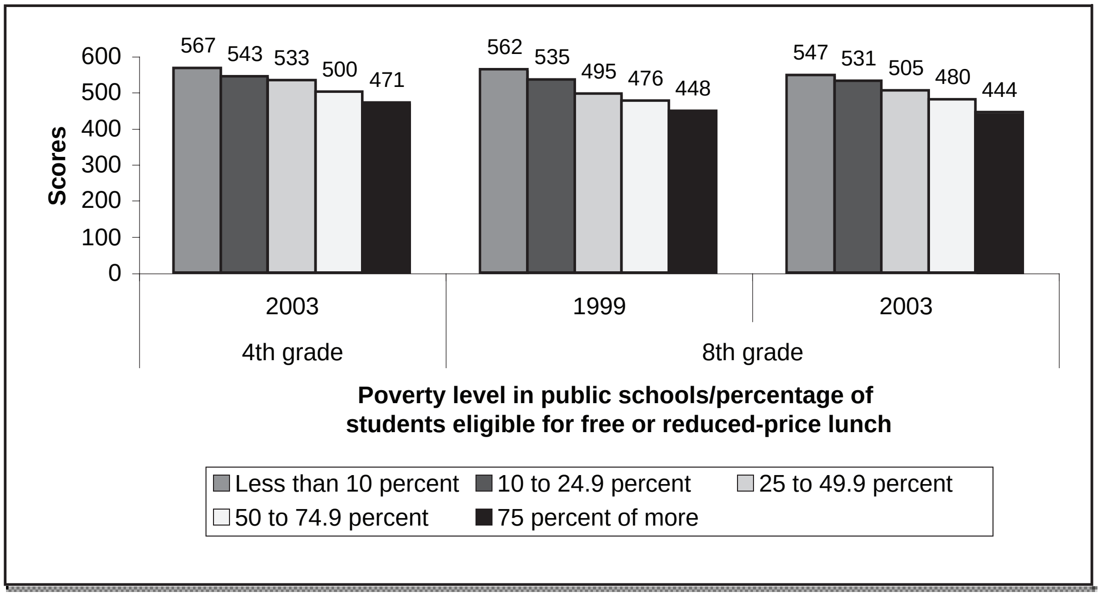
Intermediate International Benchmark – 475

Students can apply basic mathematical knowledge in straightforward situations. They can read, interpret, and use different representations of numbers. They can perform operations with three- and four-digit numbers and decimals. They can extend simple patterns. They are familiar with a range of two-dimensional shapes, and read and interpret different representations of the same data.

Low International Benchmark – 400

Students have some basic mathematical knowledge. Students demonstrate an understanding of whole numbers and can do simple computations with them. They demonstrate familiarity with the basic properties of triangles and rectangles. They can read information from simple bar graphs.

Figure E-5: Average TIMSS Mathematical Scale Scores of U.S. 4th- and 8th-Graders, by School Poverty Level: 1999 and 2003



Source: Gonzales et al., 2004, Tables C8 and C11.

High School and Beyond of 1980: HS&B:80
National Education Longitudinal Study of 1988: NELS:88
Education Longitudinal Study of 2002: ELS:2002

The scores on the HS&B:80, NELS:88, and ELS:2002 are Item Response Theory (IRT) number-right scores on the NELS:88 1990 58-item scale. IRT estimates achievement based on patterns of correct, incorrect, and unanswered questions. "The IRT-estimated number-right score reflects an estimate of the number of these 58 items that an examinee would have answered correctly if he or she had taken all of the items that appeared on the multiform 1990 NELS:88 mathematics test. The score is the probability of a correct answer on each item, summed over the total mathematics 58-item pool" (Cahalan et al., 2006, p.45). These scores are not directly translated into probability of proficiency scores. However, five probability of proficiency scores in mathematics were estimated for students using performance on clusters of four items each as follows:

Probability of Mastery, Mathematics Levels

1) Simple arithmetical operations on whole numbers, such as simple arithmetic expressions involving multiplication or division of integers;

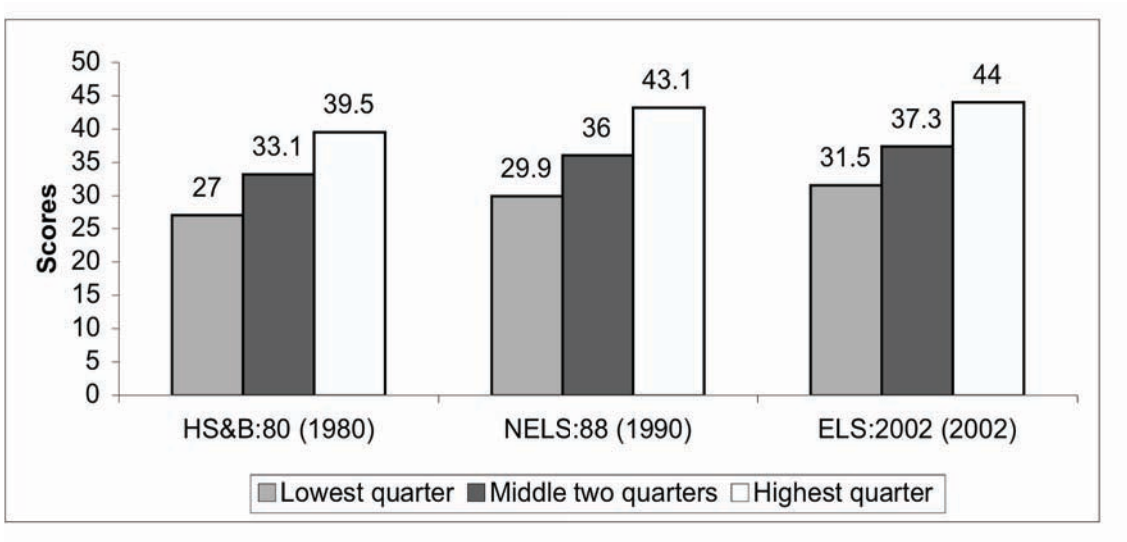
2) Simple operations with decimals, fractions, powers, and roots, such as comparing expressions, given information about exponents;

3) Simple problem solving, requiring the understanding of low-level mathematical concepts, such as simplifying an algebraic expression or comparing the length of line segments illustrated in a diagram;

4) Understanding of intermediate-level mathematical concepts and/or multistep solutions to word problems such as drawing an inference based on an algebraic expression or inequality; and

5) Complex multistep word problems and/or advanced mathematics material such as a two-step problem requiring evaluation of functions. (Cahalan et al., 2006, p. A-28)

Figure E-6: IRT—Estimated Average Math Score (10th Grade), by SES (HS&B:80, NELS:88, ELS:2002)



Source: Cahalan et al. (2006), Tables 18 and 19.

Table E-1: Probability of 10th-Grade Proficiency in Mathematics, by SES

	NELS:88 (1990)	ELS:2002 (2002)
Level 1		
Lowest quarter	83.1	84.5
Middle quarters	91.1	92.5
Highest quarter	97.1	97.1
Level 2		
Lowest quarter	41.3	46.4
Middle quarters	62.6	67.8
Highest quarter	83.3	86.2
Level 3		
Lowest quarter	20.4	25.1
Middle quarters	41.4	44.7
Highest quarter	67.4	70.9
Level 4		
Lowest quarter	5.7	7.6
Middle quarters	15.9	17.7
Highest quarter	36.2	38.7
Level 5		
Lowest quarter	0.1	0.2
Middle quarters	0.2	0.5
Highest quarter	1.0	2.6

Note: Proficiency levels —1) Simple arithmetical operations with whole numbers; 2) Simple operations with decimals, fractions, powers, and roots; 3) Simple problem solving, requiring the understanding of low-level mathematical concepts; 4) Understanding of intermediate-level mathematical concepts and/or multistep solutions to word problems; and 5) Complex multistep word problems and/or advanced mathematics material.

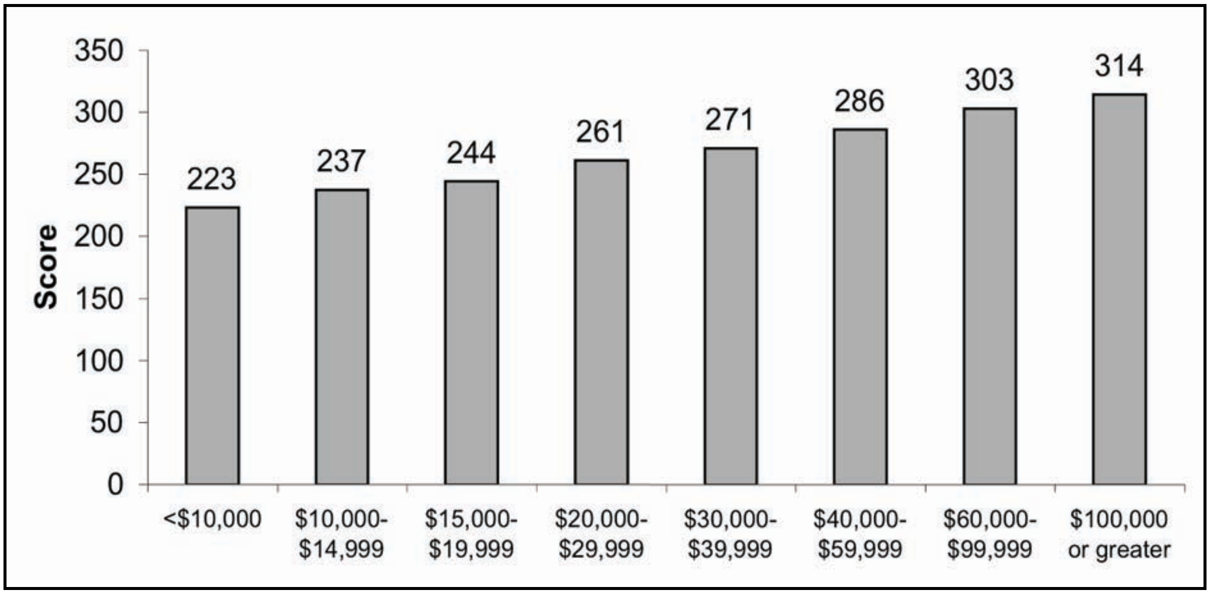
Source: Cahalan et al., 2006, p. 57-58.

National Adult Literacy Survey: NALS
National Assessment of Adult Literacy: NAAL

The Committee on Performance Levels for Adult Literacy set performance levels for quantitative literacy as Below Basic, Basic, Intermediate, and Proficient and defined them as follows, based on scores on NALS and NAAL:

> ***Below Basic (0–234)*** indicates no more than the most simple and concrete literacy skills.
>
> *Key abilities*—locating numbers and using them to perform simple *quantitative* operations (primarily addition) when the mathematical information is very concrete and familiar.
>
> ***Basic (235–289)*** indicates skills necessary to perform simple and everyday literacy activities.
>
> *Key abilities*—locating easily identifiable *quantitative* information and using it to solve simple, one-step problems when the arithmetic operation is specified or easily inferred.
>
> ***Intermediate (290–349)*** indicates skills necessary to perform moderately challenging literacy activities.
>
> *Key abilities*—locating less familiar *quantitative* information and using it to solve problems when the arithmetic operation is not specified or easily inferred.
>
> ***Proficient (350–500)*** indicates skills necessary to perform more complex and challenging literacy activities.
>
> *Key abilities*—locating more abstract quantitative information and using it to solve multistep problems when the arithmetic operations are not easily inferred and the problems are more complex (Kutner et al., 2006, p. 3).

Figure E-7: Average Quantitative Literacy Scores of Adults, by Household Income: NAAL 2003



Note: Literacy levels: Below basic 0–234, Basic 235–289, Intermediate 290–349, Proficient 350–500

Source: Kutner et al. (2007), Figure 2-17.

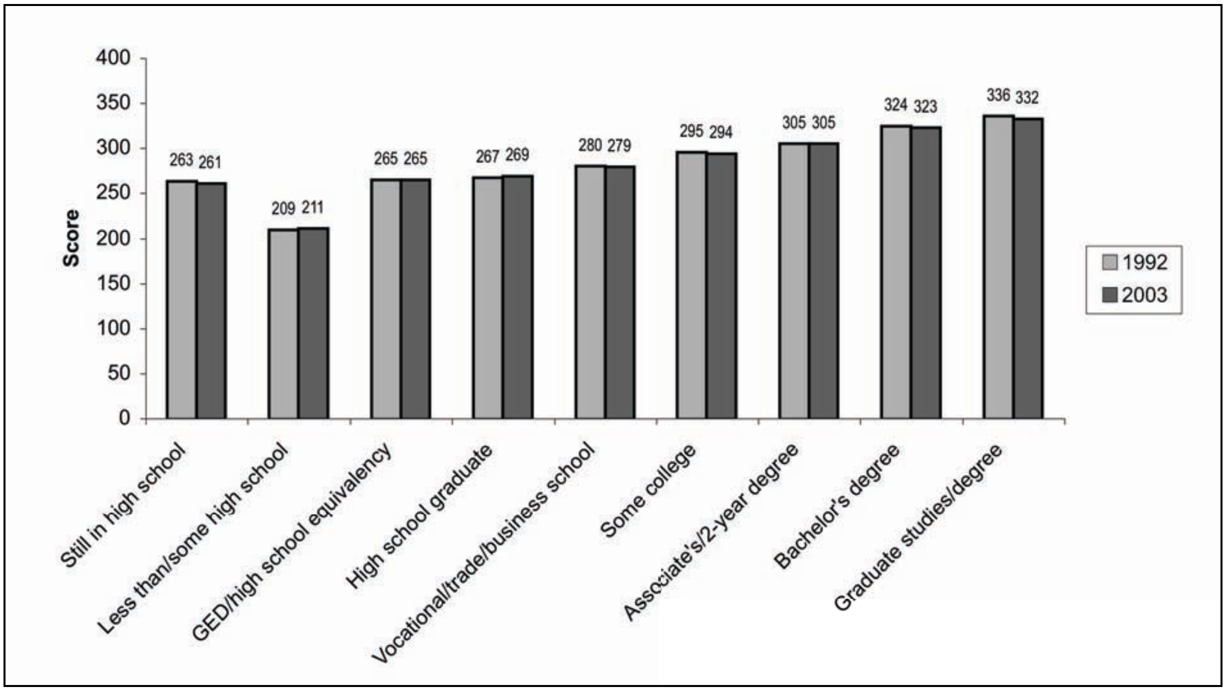
Table E-2: Percentage of Adults in Each Quantitative Literacy Level, by Household Income: NAAL 2003

	<$10,000	$10,000–$14,999	$15,000–$19,999	$20,000–$29,999	$30,000–$39,999	$40,000–$59,999	$60,000–$99,999	$100,000 or greater
Below Basic	26	16	11	16	11	12	7	2
Basic	9	8	6	14	14	21	19	9
Intermediate	4	4	3	10	11	22	28	18
Proficient	2	2	2	5	6	18	37	29

Note: Below Basic (0–234) no more than the most simple and concrete literacy skills; Basic (235–289) skills necessary to perform simple and everyday literacy activities; Intermediate (290–349) skills necessary to perform moderately challenging literacy activities; Proficient (350–500) skills necessary to perform more complex and challenging literacy activities.

Source: Kutner et al. (2007), Table 2-3.

Figure E-8: Average Quantitative Literacy Scores of Adults, by Highest Educational Attainment: NALS 1992 and NAAL 2003



Source: Kutner et al., 2007, Table 3-2.

Table E-3: Percentage of Adults in Each Quantitative Literacy Level, by Highest Education Attainment: NALS 1992 and NAAL 2003

	Still in high school	Less than/ some high school	GED/ high school equivalency	High school graduate	Vocational/ trade/ business school	Some college	Associate's/ 2-year degree	Bachelor's degree	Graduate studies/ degree
1992	(NALS)								
Below Basic	31	65	25	26	18	11	8	5	2
Basic	37	25	46	41	39	34	29	21	15
Intermediate	27	9	26	29	35	42	45	44	43
Proficient	6	1	3	5	8	13	18	31	39
2003	(NAAL)								
Below Basic	31	64	26	24	18	10	7	4	3
Basic	38	25	43	42	41	36	30	22	18
Intermediate	25	10	28	29	35	43	45	43	43
Proficient	5	1	3	5	6	11	18	31	36

Source: Kutner et al., 2007, Figure 3-1c.

Program for International Student Assessment: PISA

Mathematics literacy can be classified by proficiency levels, based on scores on the PISA, as follows:

Level 1 (*greater than 357.77 to 420.07*) At Level 1, students can answer questions involving familiar contexts where all relevant information is present and the questions are clearly defined. They are able to identify information and to carry out routine procedures according to direct instructions in explicit situations. They can perform actions that are obvious and follow immediately from the given stimuli.

Level 2 (*greater than 420.07 to 482.38*) At Level 2, students can interpret and recognize situations in contexts that require no more than direct inference. They can extract relevant information from a single source and make use of a single representational mode. Students at this level can employ basic algorithms, formula, procedures, or conventions. They are capable of direct reasoning and making literal interpretations of the results.

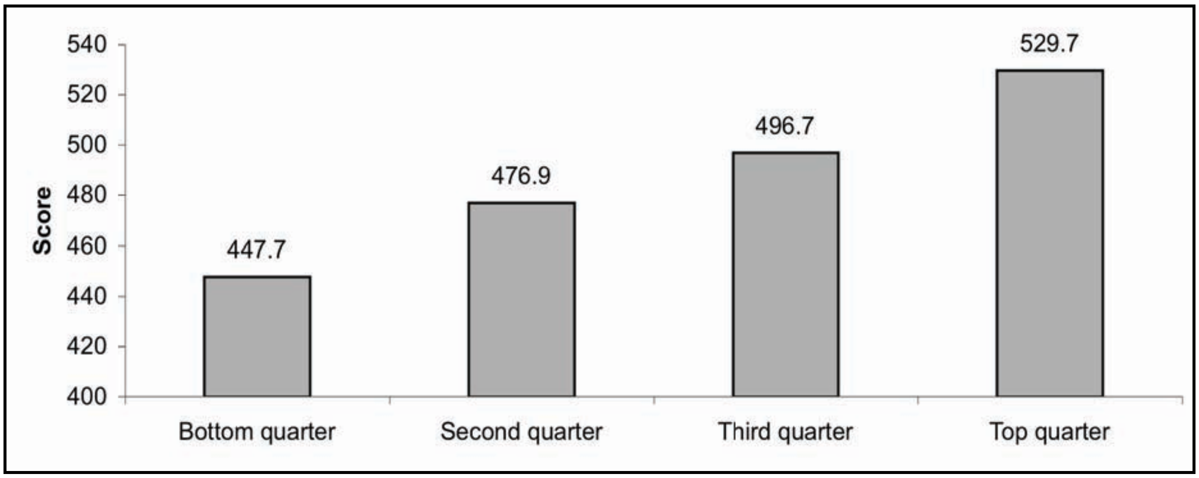
Level 3 (*greater than 482.38 to 544.68*) At Level 3, students can execute clearly described procedures, including those that require sequential decisions. They can select and apply simple problem solving strategies. Students at this level can interpret and use representations based on different information sources and reason directly from them. They can develop short communications reporting their interpretations, results, and reasoning.

Level 4 (*greater than 544.68 to 606.99*) At Level 4, students can work effectively with explicit models for complex concrete situations that may involve constraints or call for making assumptions. They can select and integrate different representations, including symbolic, linking them directly to aspects of real-world situations. Students at this level can utilize well developed skills and reason flexibly, with some insight, in these contexts. They can construct and communicate explanations and arguments based on their interpretations, arguments, and actions.

Level 5 (*greater than 606.99 to 669.3*) At Level 5, students can develop and work with models for complex situations, identifying constraints and specifying assumptions. They can select, compare, and evaluate appropriate problem solving strategies for dealing with complex problems related to these models. Students at this level can work strategically using broad, well-developed thinking and reasoning skills, appropriate linked representations, symbolic and formal characterizations, and insight pertaining to these situations. They can reflect on their actions and formulate and communicate their interpretations and reasoning.

Level 6 (*greater than 669.3*) At Level 6, students can conceptualize, generalize, and utilize information based on their investigations and modeling of complex problem situations. They can link different information sources and representations and flexibly translate among them. Students at this level are capable of advanced mathematical thinking and reasoning. These students can apply this insight and understandings along with a mastery of symbolic and formal mathematical operations and relationships to develop new approaches and strategies for attacking novel situations. Students at this level can formulate and precisely communicate their actions and reflections regarding their findings, interpretations, arguments, and the appropriateness of these to the original situations (Lemke et al., 2005, p.18).

Figure E-9: Average Mathematics Literacy Scores of U.S. 15-Year-Olds, by Quarters on the International Socioeconomic Index: 2003 PISA



Note: Level 1 (greater than 357.77 to 420.07), Level 2 (greater than 420.07 to 482.38), Level 3 (greater than 482.38 to 544.68), Level 4 (greater than 544.68 to 606.99), Level 5 (greater than 606.99 to 669.3), Level 6 (greater than 669.3)

Source: Lemke et al., 2005, Tables B-24.