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A Consistent Set of Parametric Models for the Two-Phase Flow of Immiscible Fluids in the Subsurface

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Accurate estimates of the unsaturated hydraulic properties are needed for any quantitative description of multiphase flow in porous media. This paper presents a consistent set of parametric models for the isothermal, hysteretic unsaturated fluid phase content (retention) and hydraulic conductivity functions of typical two-phase systems like water and air, or water and hydrocarbons. The equations are obtained by combining expressions for the hysteretic fluid retention curves with the statistical pore size distribution model of Mualem (1976) which predicts the hydraulic conductivity from more easily measured fluid retention data. Hysteresis is described using the model of Scott et al. (1983). The existence of residual fluid saturations for both the wetting and nonwetting fluids is justified. Theoretical and experimental considerations indicate a need to match predicted and observed hydraulic conductivities at fluid phase contents less than full saturation.

INTRODUCTION

There is growing evidence that the quality of the environment is being adversely affected by agricultural, industrial, and municipal activities. In efforts to better manage and monitor the migration of chemicals in the subsurface, scientists and engineers are developing increasingly sophisticated computer models describing how water and dissolved constituents move into and through the unsaturated zone of soils. For example, models are now routinely used by agriculturalists to compare alternative management practices for the purpose of optimizing crop yield and minimizing groundwater pollution by salts and toxic elements. Numerical models have also become popular tools for evaluating groundwater contamination from existing waste disposal sites, as well as for the siting and optimal operation of newly designed disposal systems. This trend of using models as tools in research and management is likely to continue as computer costs decrease and the need for more realistic models increases.

Unfortunately, our current technology to develop sophisticated numerical models of water and solute movement in soils far exceeds our ability to accurately estimate the parameters which appear in those models. This is especially true for the unsaturated soil hydraulic properties which are by far the most important soil parameters affecting the rate with which water and dissolved chemicals move through soils. While a large number of laboratory and field methods have been developed over the last several decades to estimate the soil hydraulic functions, and especially the unsaturated hydraulic conductivity [e.g, *Klute*, 1986; *van Genuchten et al.*, 1989], most or all of these methods have remained extremely difficult to implement. Enormous investments have been made, and continue to be made, by soil scientists and engineers in measuring the hydraulic functions using laboratory or field methods that have not significantly improved over the last several decades. Thus cheaper and less time-consuming techniques for measuring these properties are needed if we are to implement improved practices for managing water and chemicals in the unsaturated zone of soils.

One major factor that complicates the accurate measurement of the unsaturated soil hydraulic properties is their hysteretic nature when two immiscible fluids like air and water, or water and hydrocarbons, simultaneously occupy the pore space. Neglecting hysteresis in fluid flow calculations may result in significant prediction errors, as has been shown by *Dane and Wierenga* [1975], *Hoa et al.*, [1977], *Kaluarachchi and Parker* [1987], among others. In this paper we develop a consistent set of parametric equations for the hysteretic, isothermal soil hydraulic properties of two immiscible fluids. Two equations are needed for the volumetric fluid contents, θ_i, as a function of the pressure difference (p_c) between the two fluids ($i = 1, 2$). The fluid contents also depend on the time derivate $\partial p_c/\partial t$ to reflect the history of the flow process, being either drying (drainage) or wetting (imbibition):

$$\theta_1 = \theta_1(p_c, \partial p_c/\partial t) \qquad (1a)$$

$$\theta_2 = \phi - \theta_1 \qquad (1b)$$

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where ϕ is the porosity, t is time, and p_c is the soil matric or capillary pressure. By definition, θ_1 in (1a) and (1b) is the wetting fluid and θ_2 is the nonwetting fluid.

Two additional equations are needed to describe the mobility of each fluid in the subsurface. The mobility of a fluid is expressed in terms of the hydraulic conductivity (K_i) as a function of the fluid content (θ_i) and selected properties of the fluid-filled pore network, notably the connectivity (l_i) and statistical distribution (m) of the pore channels; i.e.,

$$K_i = K_i(\theta_i, l_i, m) \qquad i = 1, 2 \tag{2}$$

Instead of directly measuring the hydraulic conductivity function at selected water contents, alternative theoretical approaches have been developed over the last several decades to estimate this function from more easily measured fluid retention data. Much research has been carried out to develop and experimentally evaluate statistical pore size distribution models which consider the soil as a set of randomly distributed interconnected pores of varying radii. This research includes the early works of *Childs and Collis-George* [1950], *Burdine* [1953], and *Brooks and Corey* [1964, 1966], and the more recent studies by Mualem and coauthors [*Mualem*, 1976; *Mualem and Dagan*, 1978], among others. An excellent comprehensive review of these types of statistical models is given by *Mualem* [1986]. The predictive models have significantly advanced our ability to quantitatively describe the unsaturated soil hydraulic properties, including the hysteretic nature of the soil water retention functions.

The above studies all consider parts of the overall physical and mathematical problem of how to best quantify the unsaturated soil hydraulic properties. The purpose of this paper is to build upon those studies by deriving a consistent set of relatively simple parametric models for the hydraulic properties using a continuum approach to immiscible multiphase flow at the local scale. As such, we attempt to unify the main theoretical and practical concepts previously developed in some of our own publications [*Luckner*, 1977; *Luckner et al.*, 1978; *van Genuchten*, 1980; *van Genuchten and Nielsen*, 1985; *Luckner and Schestakov*, 1986]. In particular, we shall address the hysteretic nature of the soil hydraulic properties, conditions at which the fluid phases become immobile leading to the presence of residual saturations for both the wetting and nonwetting fluids, physically acceptable ranges of values for selected soil hydraulic parameters, and the problematic nature of using the measured saturated hydraulic conductivity as a matching point for the predicted unsaturated hydraulic conductivity function.

PHASE CONTENTS AND THEIR SPATIAL DISTRIBUTION

The relative contributions of two fluid phases ($i = 1, 2$) and one solid phase ($i = 3$) comprising a subsurface porous medium can be expressed in terms of their volumetric contents, i.e.,

$$\theta_i = V_i/V \qquad i = 1, 3 \tag{3a}$$

such that

$$\sum_{i=1}^{3} \theta_i = 1 \tag{3b}$$

where V_i is the volume occupied by phase i, V is the total volume occupied by the three phases combined, and θ_i is the volumetric content of phase i. Geohydraulics based upon continuum mechanics assumes that all three phases (e.g., the solid, liquid, and gaseous phases) occupy the subsurface in a spatially continuous fashion. In other words, one and the same point or part of the subsurface is assumed to be occupied homogeneously and simultaneously by two fluid phases and one solid phase in the same fashion as oxygen and nitrogen occupy the atmosphere. Of course, this picture can only be valid on a scale which allows local spatial averaging of the actual, microscopic phase distribution. It is on this scale that soil properties can be characterized locally by their volumetric phase contents (θ_i), their bulk phase densities (ρ_i), and their hydraulic phase conductivities (K_i). It is also on this scale that Darcy's law is presumed to be valid.

However, on a scale smaller than the above local scale, immiscible phases become discontinuously distributed. The solid phase at such a microscopic scale may be viewed to form a rigid skeleton of given geometry, whereas the two immiscible fluids form different networks which fill the empty (nonsolid) voids inside (or around) the solid phase. We consider the ensemble of solids in the soil as a connected phase, being fixed in time and immobile in space. The wetting fluid phase in the subsurface (e.g., water) is also considered as a connected phase. On the other hand, the nonwetting fluid phase (e.g., air or oil) becomes disconnectedly distributed if its phase content, θ_{nw}, becomes less than some limiting "residual" value, $\theta_{nw,r}$. The nonwetting fluid then loses its capability to move as a linked phase, causing the phase to become immobile, i.e, K_{nw} becomes zero. Fluid flow of this phase in response to a hydraulic gradient hence ceases when $\theta_{nw} \leq \theta_{nw,r}$. The same situation also holds for the wetting fluid if the connecting parts of its network become impervisious. This stage is reached when all or parts of the connecting films become so thin, and hence so strongly adsorbed onto the solid phase, that the wetting fluid loses its capability to respond to hydraulic gradients. Flow of the wetting fluid becomes then negligible as compared to its advective-diffusive transport as a dispersed phase in the nonwetting fluid. The wetting fluid in that case also becomes an apparently disconnected (immobile) phase, i.e., K_w becomes zero when $\theta_w \leq \theta_{w,r}$.

The above description of immiscible multifluid flow based on Darcy's law always considers the wetting and nonwetting fluid phases to be spatially continuous, with each phase being either coherently or incoherently distributed. We use the term coherent to characterize both real and apparent connectivities in the fluid phase distributions. The residual fluid contents $\theta_{w,r}$ and $\theta_{nw,r}$ characterize the transitions from coherent to incoherent fluid phase distributions. These two residual parameters are important geohydraulic variables affecting fluid flow in the subsurface. Unfortunately, they have too often been neglected or misinterpreted in previous descriptions of multiphase flow.

Figure 1 schematically shows the three possible mobility domains in a soil system containing two immiscible fluids. We emphasize that incoherency of a fluid phase does not necessarily mean that the material making up the fluid phase is fixed in space or time. Incoherency only means that the fluid is immobile as a linked phase. Because of trans-phase exchange and transport as a dissolved phase in the other

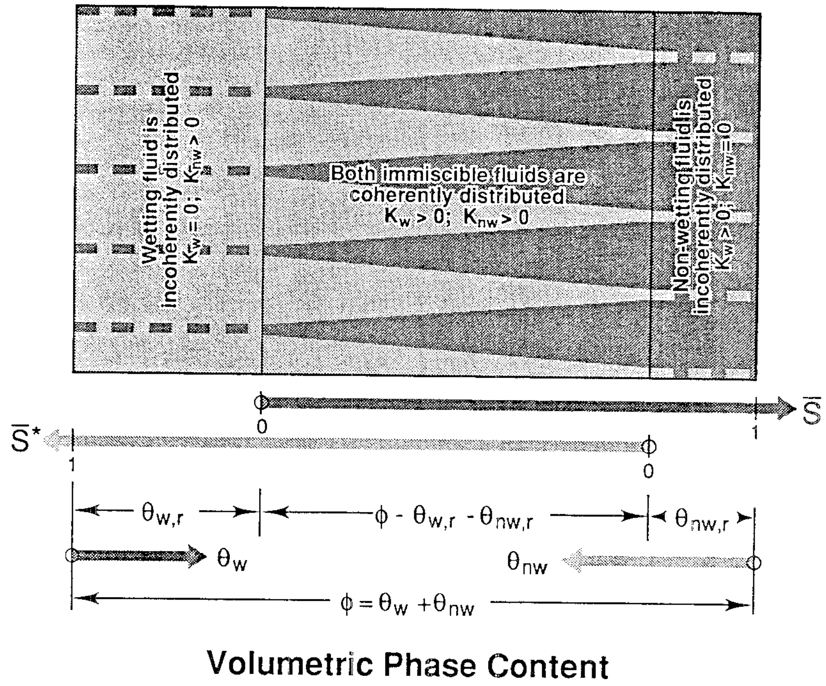

Fig. 1. Possible mobility domains for two immiscible (wetting and nonwetting) fluids in a porous medium.

fluid phase, the incoherently distributed phase in the subsurface can still undergo significant changes. Evaporation and condensation, and degassing and dissolution between the liquid and gaseous phases, are examples of such trans-phase exchange processes. Evaporation and vapor phase transport, in particular, can be the predominant cause for water movement in dry soils.

Taking water and air as the immiscible fluids in a soil, the volumetric water content in the incoherently distributed region of the wetting fluid (left domain of Figure 1) can only change through evaporation and condensation type phase exchanges, coupled simultaneously with diffusive and/or advective transport as water vapor in soil air. Liquid water in the other two domains of Figure 1 is mobile as a linked phase, and its flow is presumably governed by Darcy's law. We will use the scaled variable $\bar{S}$ to characterize the effective degree of saturation ($0 \le \bar{S} \le 1$), of that part of Figure 1 where the wetting fluid is coherently distributed. Hence from Figure 1,

$$\bar{S} = \frac{\theta_w - \theta_{w,r}}{\phi - \theta_{w,r}} \tag{4}$$

Similarly, the soil air content in the incoherently distributed nonwetting region (right domain of Figure 1) can only change through dissolution (or degassing) and its subsequent removal as dissolved air in moving soil water. Note that degassing is often a necessary trans-phase process for reaching full saturation with liquid soil water. The left and middle domains of Figure 1 represent areas in which soil air is mobile as a linked gaseous phase, and in which air transport is again governed by Darcy-type laws. As schematically shown in Figure 1, these two domains are scaled according to the reduced variable $\bar{S}^*$ ($0 \le \bar{S} \le 1$):

$$\bar{S}^* = \frac{\phi - \theta_{nw,r} - \theta_w}{\phi - \theta_{nw,r}} \tag{5}$$

PHASE DISTRIBUTION MODELS

We now define our parametric model for the hysteretic fluid phase distribution as a function of the capillary pressure $p_c = p_{nw} - p_w$. Rather than using p_c directly, we will employ the pressure head equivalent, i.e., $h = p_c/(\rho_w g)$. The water retention model is then given by

$$\bar{\theta}_w = \frac{\theta_w - A}{\phi - A - B} = \left[\frac{1}{1 + (\alpha h)^n}\right]^m \qquad m = 1 - 1/n \tag{6}$$

where $\bar{\theta}_w$ is the scaled (dimensionless) wetting fluid saturation, A and B are scaling factors comparable to $\theta_{w,r}$ and $\theta_{nw,r}$, respectively, and α and n are constants affecting the shape of the retention curve. Besides the porosity ϕ, which should be estimated separately, (6) contains five unknown parameters: $\theta_{w,r}$, $\theta_{nw,r}$, α_d, α_w, where the subscripts w and d on α represent wetting and drying, respectively. Note that we momentarily assume that n is independent of the saturation (wetting or drying) history, i.e, the sign of $\partial h/\partial t$ (compare with equation (1a)). This same assumption has been invoked in many recent studies of hysteretic fluid flow [*Kool et al.*, 1986; *Kool and Parker*, 1987; *Parker and Lenhard*, 1987; *Lenhard and Parker*, 1987]. Recent literature data and results from our own experiments suggest the following reasonable ranges and first approximations of the unknown model parameters: $1.25 < n < 4.0$, $\theta_{w,r} \sim 2\theta_{nw,r}$, and $\alpha_w \sim 2\alpha_d$. Using the latter two relationships, the number of unknowns can be reduced to three. On the other hand, it may sometimes be necessary to use, similarly as for α,

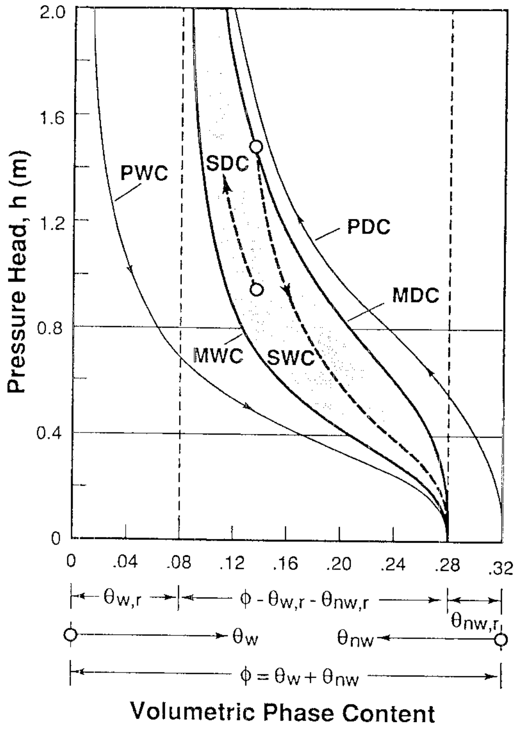

Fig. 2. Schematic of the hysteretic fluid phase retention curve for a two-phase immiscible flow system in a typical coarse-textured soil.

separate values for n during wetting (n_w) and drying (n_d), thus leading to an additional unknown in the fluid retention model.

Figure 2 shows several special cases of the hysteretic retention model based on (6). Let us denote the general vector of unknown parameters $\{b\}$ for each retention curve in Figure 2 by

$$\{b\} = \{A;\ B;\ \alpha;\ n\} \tag{7}$$

The main drying curve, MDC, and the main wetting curve, MWC, in Figure 2 are then defined by the parameter vectors

$$\{b\}_{\mathrm{MDC}} = \{A = \theta_{w,r};\ B = \theta_{nw,r};\ \alpha_d;\ n\} \tag{8a}$$

$$\{b\}_{\mathrm{MWC}} = \{A = \theta_{w,r};\ B = \theta_{nw,r};\ \alpha_w;\ n\} \tag{8b}$$

respectively. These MDC and MWC curves form geohydraulic limits of the hysteretic fluid distribution domain and approach each other asymptotically when $h \rightarrow 0$ and $h \rightarrow \infty$. It is impossible to obtain fluid contents outside this domain by normal geohydraulic fluid flow only (i.e., fluid flow governed by Darcy's law) The MDC and MWC curves therefore limit the Darcian flow domain. However, if we first generate a fully saturated soil sample by dissolving the incoherently distributed nonwetting fluid (e.g., soil air) in the wetting fluid (water) and subsequently removing dissolved air with the flowing wetting fluid, then a primary drying curve, PDC, can be produced. The PDC (sometimes also termed the "first drying curve") is often used to estimate the retention parameters from laboratory-derived data. Its parameter set is as follows:

$$\{b\}_{\mathrm{PDC}} = \{A = \theta_{w,r};\ B = 0;\ \alpha_d;\ n\} \tag{9}$$

We can similarly generate a soil sample which is saturated only with the nonwetting fluid (e.g., soil air). Full saturation in this case is reached by evaporating all residual soil water, and subsequently removing water vapor from the soil sample by diffusion and/or advection with displacing air (e.g., compare with oven drying a soil sample). By wetting such a soil sample, first perhaps with water vapor and later with liquid water, the primary wetting curve, PWC, will be generated. Of course, pressure heads along this curve in the dry region can only be measured using psychrometers rather than tensiometers. Until now, the PWC curve has been largely ignored in soil water retention studies, especially in the dry range. Notable exceptions are given by *Grismer et al.* [1986*a*, *b*]. The parameters of the PWC curve are defined by

$$\{b\}_{\mathrm{PWC}} = \{A = 0;\ B = \theta_{nw,r};\ \alpha_w;\ n\} \tag{9'}$$

Scanning curves can start at any point $P(\theta_{w0}, h_0)$ within the limits of the PDC and PWC curves when the sign of $\partial h/\partial t$ changes. Scanning wetting (SWC) and drying (SDC) curves are characterized by the parameter vectors

$$\{b\}_{\mathrm{SWC}} = \{A = A_w;\ B = \theta_{nw,r};\ \alpha_w;\ n\} \tag{10a}$$

$$\{b\}_{\mathrm{SDC}} = \{A = \theta_{w,r};\ B = B_d;\ \alpha_d;\ n\} \tag{10b}$$

respectively, where

$$A_w = \frac{\phi - \theta_{nw,r} - \theta_{w0}[1 + (\alpha_w h_0)^n]^m}{\{1 - [1 + (\alpha_w h_0)^n]^m} \tag{11a}$$

$$B_d = \phi - \theta_{w,r} - (\theta_{w0} - \theta_{w,r})[1 + (\alpha_d h_0)^n]^m \tag{11b}$$

We note that these equations for the scanning curves are identical to those proposed by *Scott et al.* [1983] and used, among others, by *Kool and Parker* [1987]. Because experiments to determine the complete MDC/MWC's at very low wetting fluid saturations can be extremely time consuming and costly in order to reach equilibrium (steady state), scanning curves are often used for identification of the unknown parameters in (6). Laboratory-type experiments generally start at complete saturation of the wetting fluid when $\theta_w = \phi$ and $h = 0$, then follow the PDC until some point $P(\theta_{w0}, h_0)$ past the inflection point on the PDC curve, and then go back along a wetting scanning curve starting at $P(\theta_{w0}, h_0)$ to end at $h = 0$ when $\theta_w = \phi - \theta_{nw,r}$ (often referred to as the satiated water content). Unlike most previous studies, our scanning curves have only one asymptote which is independent of the starting point, i.e., the SWC

TABLE 1. Values of the Scanning Parameters A and B in (6) for the Main (MDC) and Primary (PDC) Drying Curves, the Main (MWC) and Primary (PWC) Wetting Curves, and the Scanning Drying (SDC) and Scanning Wetting (SWC) Curves

	MDC	PDC	MWC	PWC	SDC	SWC
A	$\theta_{w,r}$	$\theta_{w,r}$	$\theta_{w,r}$	0	$\theta_{w,r}$	equation (11a)
B	$\theta_{nw,r}$	0	$\theta_{nw,r}$	$\theta_{nw,r}$	equation (11b)	$\theta_{nw,r}$

always attempts to go to the point $P(\phi - \theta_{nw,r}, 0)$, and the SDC to the point $P(\theta_{w,r}, \infty)$.

Table 1 summarizes the values of the scanning parameters A and B in dependence of the wetting/drying process (i.e., drainage with $\partial h/\partial t > 0$, or wetting with $\partial h/\partial t < 0$). Figure 3 gives a graphical interpretation of (6) using typical parameter values for water and air in a coarse-textured soil: $\phi = 0.32$, $\theta_{w,r} = 0.08$, $\theta_{nw,r} = 0.04$, $\alpha_d = 1.25\ \mathrm{m}^{-1}$, $\alpha_w = 2.5\ \mathrm{m}^{-1}$, and $n_w = n_d = 3.0$ ($m = 0.667$). Figure 3 also shows a series of wetting and drying scanning curves as defined by (10*a*) and (10*b*). The scanning parameters A_w and B_d in (10*a*) and (10*b*) are, for any arbitrary starting point $P(\theta_{w0}, h_0)$, calculated with (11*a*) and (11*b*). For example, for the starting point $P(\theta_{w0} = 0.20, h_0 = 0.60)$ in Figure 3 we have for the wetting ($\partial h/\partial t < 0$) curve

$$A_w = \frac{0.32 - 0.04 - 0.20[1 + (2.5 \times 0.6)^3]^{2/3}}{1 - [1 + (2.5 * 0.6)^3]^{2/3}} = 0.152$$

and for the drying ($\partial h/\partial t > 0$) curve $B_d = 0.32 - 0.08 - (0.20 - 0.08)\ [1 + (1.25 \times 0.6)^3]^{2/3} = 0.088$. The wetting and drying scanning curves going through $P(0.20, 0.60)$ are hence given by

$$\theta_w = 0.152 + \frac{0.128}{[1 + (1.25h)^3]^{2/3}}$$

$$\theta_d = 0.08 + \frac{0.152}{[1 + (1.25h)^3]^{2/3}}$$

respectively.

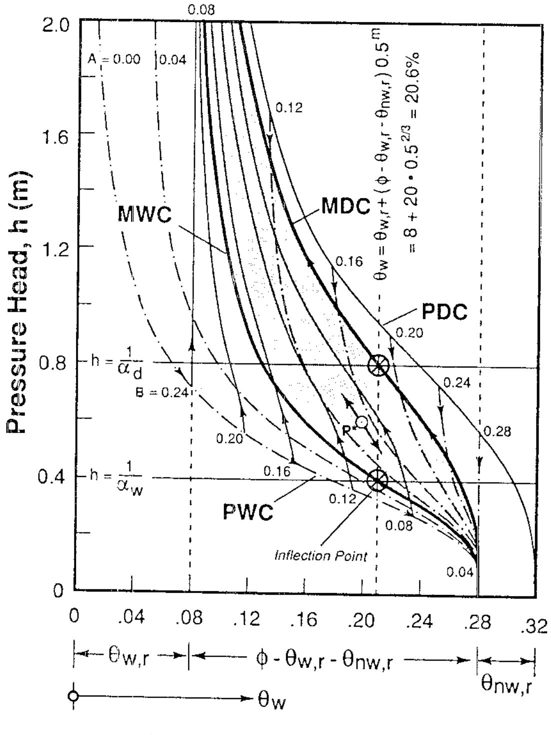

Fig. 3. Fluid saturation paths for two immiscible fluids in a typical coarse-textured soil as calculated with (6).

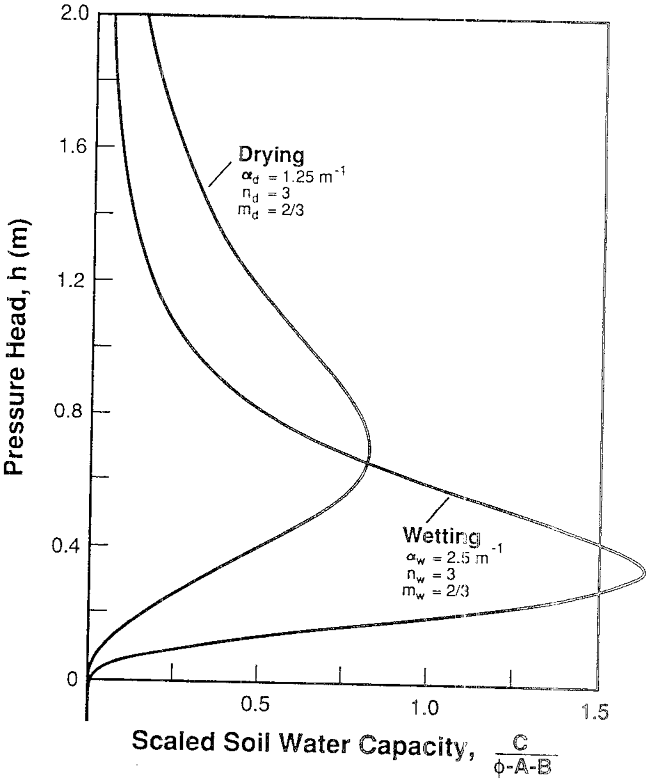

Fig. 4. Plot of the capillary storage coefficient (soil water capacity) during wetting and drying as a function of the pressure head (equation (13)).

PARAMETRIC MODELS FOR $C(h)$

Decomposition of the term $\partial\theta_w/\partial t$ in the flow equation into a parametric storage capacity term and a potential term gives

$$\frac{\partial \theta_w}{\partial t} = \frac{\partial \theta_w}{\partial h}\frac{\partial h}{\partial t} = C(h)\frac{\partial h}{\partial t} \qquad (12)$$

where C is the capillary hydraulic storage coefficient, often called the soil water capacity in water-air two-phase systems. Using (6), this function becomes

$$\begin{aligned} C &= (\phi - A - B)\frac{\alpha m n (\alpha h)^{n-1}}{[1 + (\alpha h)^n]^{m+1}} && h > 0 \\ C &= 0 && h \le 0 \end{aligned} \qquad (13)$$

One may show (e.g., *van Genuchten and Nielsen*, 1985) that C is a smooth, first-order continuous function across $h = 0$ when $n > 1$. A more severe constraint on the retention model (equation (6)) could be imposed by requiring that C is also second-order continuous at $h = 0$, thus forcing dC/dh also to be a continuous function. This last requirement leads to the constraint $n > 2$, which is contrary to experimental evidence which indicates that many soils, especially fine-textured soils, have n values between 1 and 2 [e.g., *Puckett et al.*, 1985; *Carsel and Parrish*, 1988].

Figure 4 shows a plot of $C(h)$ for both wetting and drying using the same parameters as employed in Figure 3. Notice that for a particular set of model parameters the area ($\int C\, dh$) under each $C(h)$ curve is the same and equal to $\phi - A - B$. This property follows immediately from (13).

PARAMETRIC MODELS FOR K_w AND K_{nw}

Numerous studies have shown that the geometrical structure of the fluid channel network in a porous medium formed by two immiscible fluids is primarily determined by the fluid phase contents, and only minimally by the saturation history, i.e., by the sign of $\partial\theta/\partial t$. This observation implies that the hydraulic conductivity functions of the wetting and nonwetting fluids are single-valued and nonhysteretic. Consequently, the same model should be applicable to the hydraulic conductivity function during both drying and wetting.

Of several predictive hydraulic conductivity models available, Mualem's model appears to be the most accurate and phenomenologically the most realistic [*Mualem*, 1976; *van Genuchten and Nielsen*, 1985; *Journel and Alabert*, 1988]. *Mualem*'s [1976] original model may be expressed in the form

$$K_w(\theta_w) = K_s\bar{S}^{\,l}\left[\frac{f(\bar{S})}{f(1)}\right]^2 \tag{14}$$

where

$$f(\bar{S}) = \int_0^{\bar{S}} \frac{1}{h(\tau)}\,d\tau \tag{15}$$

in which $\bar{S}$ is given by (4), K_s is the hydraulic conductivity at full saturation when $\theta_w = \phi$, and l is a pore connectivity parameter estimated by *Mualem* [1976] to be about 0.5 as an average for many soils.

Equation (14) assumes that an independent estimate for K_s at full saturation can be easily obtained experimentally. Unfortunately, this is generally not true, especially for undisturbed field soils. Recent studies by *van Genuchten and Nielsen* [1985] and others suggest that the accuracy of K_w near saturation is generally impaired because of the steep slope of (14) when $\bar{S} \to 1$. Very small measurement errors in the water content here can lead to unacceptably large errors in the hydraulic conductivity. Also, the hydraulic conductivity near saturation is determined primarily by soil-structural soil properties which are known to be subject to spatial and temporal variability in the field. This is in contrast to soil textural properties which generally are less variable, and which have a more dominant effect on K in the dry range. Thus both theoretical and experimental considerations suggest that K_s at full saturation should not be used as a matching point in (14), or in other predictive models for the unsaturated hydraulic conductivity. Instead, a measured unsaturated hydraulic conductivity value at a water content somewhat less than saturation, but still in the relatively wet water content range, is far more suitable as a matching point. Having this point in the relatively wet range still enables one to rather quickly execute field or laboratory experiments without being subject to some of the experimental or theoretical uncertainties discussed above.

If we take the matching point at some arbitrary fluid phase content θ_{w0}, then Mualem's model may be redefined as

$$K_w(\theta_w) = K(\theta_{w0})(\bar{S}/\bar{S}_0)^l[f(\bar{S})/f(\bar{S}_0)]^2 \tag{16}$$

where $\bar{S}_0 = \bar{S}(\theta_{w0})$. Substitution of (6) into (15) gives

$$K_w(\theta_w) = K_w(\theta_{w0})(\bar{S}/\bar{S}_0)^l\left[\frac{1-(1-\bar{S}^{1/m})^m}{1-(1-\bar{S}_0^{1/m})^m}\right]^2 \tag{17}$$

We can apply Mualem's models also to the nonwetting fluid, similar to what was done by *Brooks and Corey* [1966] when they derived their own equations using *Burdine*'s [1953] model. Analogous to (16), the hydraulic conductivity of the nonwetting fluid becomes now

$$K_{nw}(\theta_w) = K_{nw}(\theta_{w0})(\bar{S}^*/\bar{S}_0^*)^\gamma[g(\bar{S})/g(\bar{S}_0)]^2 \tag{18}$$

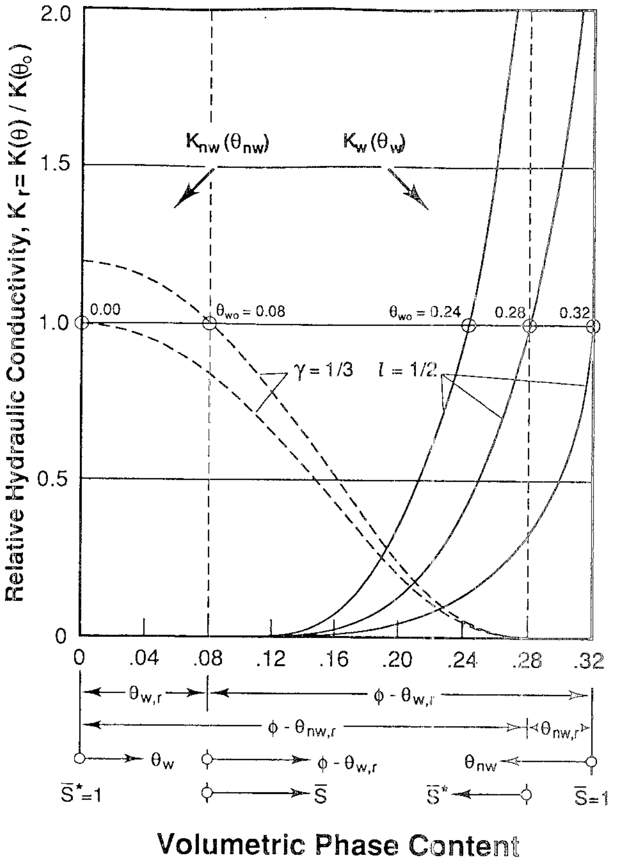

Fig. 5. Relative permeabilities of two immiscible fluids in a porous medium as calculated with (17) for the wetting fluid and (20) for the nonwetting fluid.

where the scaled nonwetting fluid saturation variable $\bar{S}^*$ is defined by (5), γ is a pore connectivity parameter for the nonwetting fluid (similar to l for the wetting fluid), and where

$$g(\bar{S}) = \int_{\bar{S}}^1 \frac{1}{h(\tau)}\,d\tau = (1-\bar{S}^{1/m})^m \tag{19}$$

Substituting (19) into (18), and approximating $\bar{S}$ by $1-\bar{S}^*$, leads finally to

$$K_{nw}(\theta_w) = K_{nw}(\theta_{w0})[\bar{S}^*/\bar{S}_0^*]^\gamma\left\{\frac{[1-(1-\bar{S}^*)^{1/m}]^m}{[1-(1-\bar{S}_0^*)^{1/m}]^m}\right\}^2 \tag{20}$$

Figure 5 shows the relative permeability functions obtained with (17) and (20) assuming $n_w = n_d = 3$ as in Figures 3 and 4 and using 1/2 for l and 1/3 for γ. The relative conductivity functions for different θ_{w0} are given by

$$K_{w,r} = K_w(\theta_w)/K_w(\theta_{w0}) \qquad K_{nw,r} = K_{nw}(\theta_w)/K_{nw}(\theta_{w0}) \tag{21}$$

The plots in Figure 5 show that very small errors in the measurement of θ_{w0} can lead to extremely large errors in the $K_{w,r}$ functions. This again suggests that measurements of $K_w(\theta_{w0})$ should not be made at or close to full fluid saturation. Actually, the use of two or more measured points at different

θ_{w0} appears very effective in improving the accuracy of the predictive unsaturated hydraulic conductivity function.

Additional experiments, especially carefully controlled laboratory studies, are indispensable for validating the concepts elaborated upon in this paper. The partly conceptual and partly phenomenological basis of the above parametric models cannot be justified or validated by purely theoretical considerations only, or exclusively by empirical work. Particularly useful would be laboratory experiments in which one and the same soil sample is used for estimating the drying and wetting fluid phase retention curves, as well as for measurement of the unsaturated hydraulic conductivity. Theoretical and experimental studies are also needed to determine reasonable ranges for the parameter m, especially if allowed to be independent in (6), and the connectivity parameters l and γ. Finally, statistical dependencies and sensitivities of the model parameters by identification using optimization procedures must be evaluated.

SUMMARY AND CONCLUSIONS

This paper presents a consistent set of parametric models for the unsaturated fluid phase content and hydraulic conductivity functions of typical immiscible two-phase systems like water and air, or water and hydrocarbons. The equations are obtained by combining physically realistic expressions for the hysteretic wetting and nonwetting fluid retention curves with a statistical pore size distribution model based on *Mualem*'s [1976] theory. Mualem's theory predicts the hydraulic conductivity from more easily measured fluid retention data. The fluid saturation (soil water retention) curves were described with modifications of the higher-order continuous S-shaped retention function of *van Genuchten* [1980]. The functions were modified to allow inclusion of both wetting and nonwetting residual fluid saturations. The use of residual saturations is shown to be appropriate when a fluid phase becomes incoherently distributed, in which case the fluid does not move anymore as a connected (or linked) phase as described by Darcy's law in response to imposed pressure gradients. To avoid several theoretical and experimental complications, we again emphasize the need to match the predicted and experimental hydraulic conductivities at fluid phase contents less than full saturation.

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