On describing and predicting the hydraulic properties of unsaturated soils

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ABSTRACT. To accurately predict the unsaturated hydraulic conductivity (K) from measured soil water retention data using analytical (nontabular) functions, one needs a reliable descriptive model of the retention curve as well as an accurate model for K on which the prediction ultimately is based. This paper reviews several empirical expressions for describing the soil water retention curve. A five-parameter equation was used that exhibits great flexibility in matching retention data of various soils, has a simple expression for the inverse, and permits the derivation of closed-form analytical expressions for K when combined with predictive theories previously proposed by either Burdine or Mualem. The retention function graphically displays a smooth (continuously differentiable) S-shaped curve between the saturated and residual water contents. Application of the retention model to hundreds of data sets, both for disturbed (screened) and undisturbed field soils, consistently lead to an excellent description of the observed data. Of the two predictive hydraulic conductivity models, Mualem's approach was found to be applicable to a wider variety of soils than Burdine's formulation.

Key words : soil water retention curve, soil water characteristic curve, hydraulic conductivity, soil hydraulic properties, unsaturated flow.

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INTRODUCTION

Application of increasingly sophisticated mathematical techniques to the analysis of field-scale flow and transport processes points to the need for accurate yet expedient methods for quantifying the hydraulic properties of the medium to be simulated. Less time-consuming methods for estimating the unsaturated hydraulic conductivity (K) are especially important, in part because of the time and labor involved in the direct field-measurement of this parameter, and in part because of its spatial variability. Over the years, various methods have been devised to predict K from the more easily measured soil water retention curve (Childs and Collis-George, 1950 ; Burdine, 1953 ; Marshall, 1958 ; Millington and Quirk, 1961 ; Mualem, 1976*a*; among others). Although exceptions exist (e.g., Brooks and Corey, 1964 ; Campbell, 1974), direct application of these predictive methods generally leads to hydraulic conductivity data that are in tabular form, a feature that often restricts their usefulness, for example when simulating flow in one- or two-dimensional layered systems. As compared to analytical expressions, tabulated functions also do not allow for a rapid comparison (Rawls *et al.*, 1983) or the scaling (Simmons *et al.*, 1979) of hydraulic properties of different soils. In this paper we will present several closed-form predictive equations for the unsaturated hydraulic conductivity in terms of parameters that can be fitted to observed retention data.

To arrive at an accurate predictive analytical (nontabular) expression for the unsaturated hydraulic conductivity, two conditions must be satisfied. First, an analytical expression is needed that accurately describes the soil water retention curve. This first requirement is essential : any attempt to predict the unsaturated hydraulic conductivity from retention data fails if the assumed retention model cannot describe the data over the entire range of observed values. Consequently, a major part of this paper deals with the mathematical description of the soil water retention curve. The accuracy of a predictive equation for K also hinges on the accuracy of the theory on which that equation ultimately is based. Of the predictive models presently available, those advanced by Burdine (1953) and Mualem (1976*a*) are probably most suitable for deriving closed-form analytical conductivity equations. Hence, predictions based on both of these models will be discussed.

SOIL WATER RETENTION FUNCTIONS

A variety of empirical functions have been proposed to relate the volumetric water content, θ, with the soil water pressure head, h. Undoubtedly the most popular retention function used to date stems from Brooks and Corey (1964). Their function is written here in the form

$$\theta = \begin{cases} \theta_r + (\theta_s - \theta_r)(\alpha h)^{-\lambda} & \alpha h \geqslant 1 \\ \theta_s & \alpha h < 1 \end{cases} \qquad (1)$$

where θ_s is the saturated water content, θ_r is the residual water content, α is a parameter whose inverse ($h_a = \alpha^{-1}$) frequently has been referred to as the air entry value (or bubbling pressure), while λ is sometimes named the pore-size distribution index. To simplify notation, h (and hence also α) is taken positive for unsaturated soils. In some studies, θ_r in (1) was assumed to be zero (e.g., Gardner *et al.*, 1970; Campbell, 1974).

If we introduce a reduced water content, S_e (sometimes called effective saturation) of the form

$$S_e = \frac{\theta - \theta_r}{\theta_s - \theta_r} \qquad (2)$$

then equation (1) can be written as

$$S_e = \begin{cases} (\alpha h)^{-\lambda} & \alpha h \geqslant 1 \\ 1 & \alpha h < 1 \end{cases} \qquad (3)$$

Because of its simple form, equation (1) has been used frequently in unsaturated flow studies (Laliberte *et al.*, 1968; Jeppson, 1974; Brakensiek *et al.*, 1981; among others). Moreover, the equation produces acceptable results for relatively coarse-textured, often disturbed soils with relatively narrow pore-size distributions (large λ-values), especially when $\alpha h \gg 1$, i.e., outside the wet range (Brooks and Corey, 1964; 1966).

Unfortunately, equation (1) also has been shown to give relatively poor fits with observed retention data near saturation, especially for certain fine-textured, structured or other field soils exhibiting relatively broad pore-size distributions. On a logarithmic plot, equation (1) generates two straight lines, the point of connection being exactly at the bubbling pressure (h_a). The resulting discontinuity in the slope of the curve at h_a ignores the presence of a smooth transition zone at and near the bubbling pressure; such a smooth transition zone is especially characteristic for field-measured retention curves. Similar continuity limitations also hold for equations proposed by Rogowski (1971) and Farrel and Larson (1972).

To improve the description of the soil water retention curve near saturation, various continuously differentiable (smooth) S-shaped curves have been proposed. King (1964) suggested the equation

$$\theta = \theta_s \frac{\cosh[(h/h_0)^b + \varepsilon] - \gamma}{\cosh[(h/h_0)^b + \varepsilon] + \gamma} \qquad (4)$$

where h_0, b (having negative values), ε and γ are characteristic for a given soil. Equation (4) is a sigmoidal curve between the field-saturated water content, θ_s, and a residual water content, θ_r, the latter quantity being related to γ and ε by

$$\theta_r = \lim_{h \to \infty} (\theta) \equiv \theta_s \frac{\cosh(\varepsilon) - \gamma}{\cosh(\varepsilon) + \gamma}. \qquad (5)$$

King (1965) obtained excellent results with this equation for several coarse- and fine-textured soils. Gillham *et al.* (1976) found that little flexibility was lost when ε in equation (4) was assumed to be zero.

An equally flexible equation for describing retention data is given by

$$\alpha h = (1 - S_e)^a S_e^{-b}, \qquad (6)$$

particular forms of which were used by Visser (1968) and Su and Brooks (1975). The latter authors also obtained excellent agreement between theoretical curves based on (6) and measured data for several soils.

A third improved descriptive function for the retention curve was suggested by Laliberte (1969). His expression can be put into the form

$$S_e = \frac{1}{2} \operatorname{erfc}\left(a - \frac{b}{c + h/h_a}\right) \qquad (7)$$

where the parameters a, b and c were assumed to be unique functions of the pore-size distribution index λ in equation (3). Values for a, b and c were obtained by matching equations (7) and (3) at relatively small values of S_e. Unfortunately, by doing so the wet side of the curve becomes fixed for a given value of λ, and the equation loses much of its flexibity. To keep the equation flexible for different data sets, it seems more useful to keep at least one and perhaps two parameters (e.g., a and c) in addition to h_a floating in (7). Note that only two of the parameters b, c and h_a in (7) are independent. A slightly different erfc-model for the retention curve, but with similar properties as (7), was used by Varallyay and Mironenko (1979).

Finally, a fourth S-shaped model for the retention curve that possesses similar features as equations (4), (6) and (7) was suggested by van Genuchten (1980) :

$$S_e = [1 + (\alpha h)^n]^{-m} \qquad (8)$$

where α, n and m also represent empirical parameters. This equation with $m = 1$ was used earlier by Ahuja and Swartzendruber (1972), Endelman *et al.* (1974) and by Varallyay and Mironenko (1979). To obtain relatively simple predictive closed-form analytical expressions for the unsaturated hydraulic conductivity, van Genuchten (1980) assumed unique relations between the parameters m and n. This restriction is somewhat analogous to the assumption by Laliberte (1969) that the constants a, b and c in (7) are unique functions of λ. As will be shown later, restricting m and n sometimes limits the flexibility of (8) in describing retention data of several soils. Hence, for now we will assume that m and n are both independent parameters.

The discussion above shows that four alternative and equally flexible models (equations (4), (6), (7) and (8)) are available to describe observed retention data for a variety of soils. Choice of a particular model therefore should be governed by the relative simplicity of that model, as well as by its ability to permit the derivation of analytical expressions for the hydraulic conductivity when used in conjunction with the predictive models of Burdine (1953) or Mualem (1976*a*). Of the four models, equations (4) and (7) are too complicated for deriving closed-form predictive equations for K.

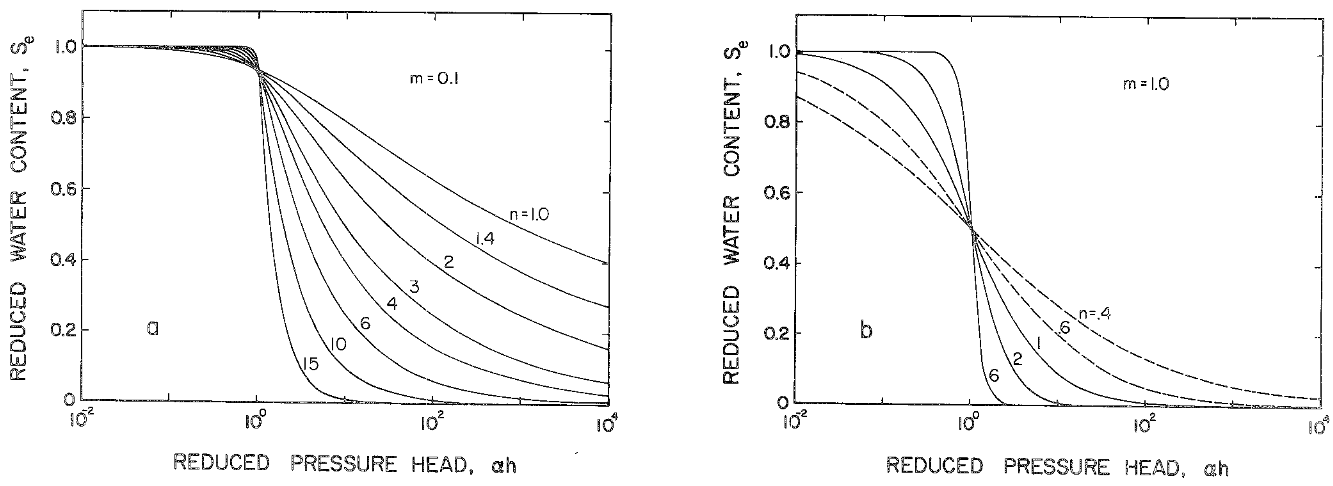

Figure 1
Retention curves based on (8) *for various values of n assuming m* = 0.1 *(a) and m* = 1.0 *(b).*

Hence, those two models will not be considered further. As shown by Su and Brooks, equation (6) leads to a closed-form analytical K_r-expression when applied to Burdine's model. Unfortunately, equation (6) is expressed in terms of $h(\theta)$, and no simple inverse function $\theta(h)$ exists. We therefore also discard (6) and focus attention only on equation (8).

SOME CALCULATED RETENTION CURVES

Calculated retention curves based on equation (8) for various values of m and n are plotted in figures 1 and 2. Plots are given using both a semi-logarithmic (figs. 1, 2*a*) and regular (fig. 2*b*) scale for the reduced pressure head (αh); actual h-values are readily obtained by simply shifting the log-scale by $\log(\alpha)$, or by multiplying the horizontal scale in figure 2*b* by $1/\alpha$. The curves in figure 1 are for two representative values of m, while in figure 2 the product mn was kept at an arbitrary but representative value of 0.4, thus resulting in a unique limiting curve at low saturation values. This limiting curve follows from (8) by removing the factor 1 from the denominator and is equivalent to the Brooks and Corey equation (3) with $\lambda = mn$. The same limiting curve also appears when we allow n in (8) to go to infinity and simultaneously decrease m such that the product mn remains 0.4. As shown in figure 2*a*, this leads to a sharp break in the curve at the « air entry value » $h_a = \alpha^{-1}$. For finite values of n ($n < \infty$) the curves remain smooth and more or less sigmoidal-shaped on a semi-logarithmic plot. However, note that the curves are markedly nonsigmoidal on the regular $\theta(h)$-plot (fig. 2*b*) when n is relatively small. Finally, also note that S_e at a reduced pressure head of 1.0 decreases with increasing values of m; this observation follows directly from (8) which for $\alpha h = 1$ simplifies to $S_e = 2^{-m}$.

An interesting feature of the retention curve is its slope when the curve approaches saturation ($h \to 0$). Upon differentiation of equation (8), one may verify that this slope (C_s) is given by

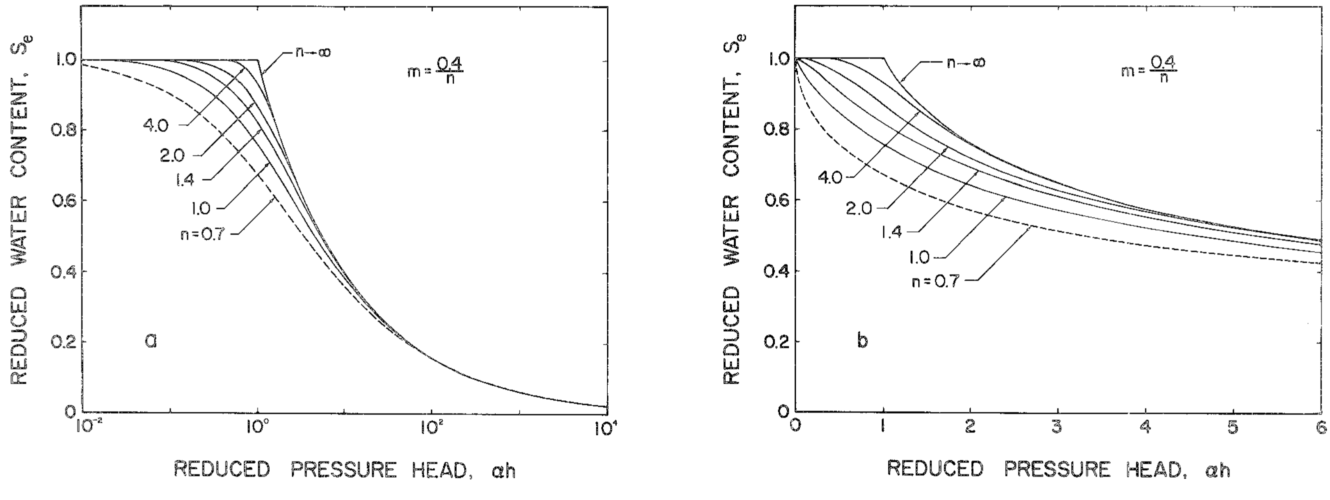

Figure 2
Semi-logarithmic (a) and regular (b) plot of various retention curves based on equation (8) *with mn* = 0.4.

Table 1

Residual sum of squares (SSQ) for nonlinear regression of equation (8) to various data sets taken from Mualem (1976b). Results are for variable m and n (SSQ_1), for m and n related by $m = 1 - 1/n$ (SSQ_2), m and n related by $m = 1 - 2/n$ (SSQ_3) and for the Brooks and Corey model ($m/n \to 0$; SSQ_4). Results for Sarpy loam (fig. 6) *are also given.*

Soil index	Mualem's Soil name	$SSQ_1 \times 10^5$	$SSQ_2 \times 10^5$	$SSQ_3 \times 10^5$	$SSQ_4 \times 10^5$
1003A	Lamberg clay, W*	5	10	20	556
1003B	Lamberg clay D*	56	58	68	294
1006	Beit Netofa clay	70	116	156	194
1101	Shluhot silty clay	42	45	44	42
2001	Silt Columbia	1	8	19	69
2004	Slate dust	45	268	249	125
3001	Weld silty clay loam	18	487	425	21
3002	Amarillo silty clay loam	159	449	346	224
3102	Yolo light clay	16	39	19	43
3301A	Caribou silt loam, D	3	5	3	29
3301B	Caribou silt loam, W	1	1	3	20
3305A	Ida silt loam, W	19	117	296	664
3305B	Ida silt loam, D	45	51	50	201
3308	Touchet silt loam	14	17	14	367
3403	Pachappa loam	263	427	313	308
4106A	Sand, D	14	14	14	140
4106B	Sand, W	4	6	8	98
4109	Botany sand	10	147	143	16
4110	Fine sand G.E. No. 13	27	27	28	363
4111	River sand (screened)	181	379	360	320
4113	Oso Flasco fine sand	34	37	41	219
4115	G.E. No. 2 sand	24	34	56	354
4121	Rehovot sand	29	117	46	154
4123	Pouder River sand	64	254	239	64
4125	Adelaide dune sand	4	5	5	54
4130	Hygiene sandstone	1	10	9	4
4132	Fragmented sandstone	32	32	34	374
4136	Fine sand G.E. No. 2	18	18	20	161
4137	Sand fraction	43	359	349	43
4142	Sand fraction	72	103	99	191
5002	Glass beads	56	149	143	155
5003A	Aggregated glass beads, D	29	165	150	223
5003B	Aggregated glass beads, W	120	337	286	330
—	Sarpy Loam	60	99	199	539

*W = main wetting branch; D = main drying branch.

$$C_s \equiv \lim_{h \to 0} \frac{dS_e}{dh} = \begin{cases} 0 & n > 1 \\ -\alpha mn & n = 1 \\ -\infty & n < 1 \end{cases} \qquad (9)$$

which shows that the retention curve approaches saturation with a zero slope only when $n > 1$. When $n < 1$ (dashed curves in figs. 1 and 2) the slope becomes $-\infty$ at saturation, which suggests that the soil water diffusivity, defined by $D = K(dh/d\theta)$, should go to zero when h approaches zero. This feature is highly unlikely for most or all soils.

Figures 2*a*, *b* also demonstrate the effects on the curves when various restrictions are placed on permissible values of m and n (with a given value for the product mn, in this case 0.4). Again, when $n \to \infty$, the limiting curve of Brooks and Corey (1964) with a well-defined air entry value appears. When $m = 1 - 1/n$ as used by van Genuchten (1980) for the Mualem-based conductivity predictions, and keeping the product mn at 0.4, the wet end of the retention curve becomes fixed at $n = 1.4$ (fig. 2). Similarly, when $m = 1 - 2/n$ for the Burdine-based conductivity equation of van Genuchten (1980), the curve is fixed at $n = 2.4$. Therefore, any of these three assumptions will fix the curve for a given value of mn. Of course, the same is also true when n is taken to be unity (Endelman *et al.*, 1974).

We emphasize here that equation (8) contains five independent parameters : θ_r, θ_s, α, m and n. As will be discussed in detail later, we consider the residual and saturated water contents to be empirical parameters : they are defined only in association with the adopted retention model, and hence are to be fitted to observed data using that retention model. Of the three remaining parameters, α approximately equals the inverse of the air entry value for small m/n values, while for large m/n this parameter roughly equals the inverse of the pressure head at the inflection point (figs. 1, 2). The product mn determines the slope of the curve at large values of the pressure head ($\alpha h \gg 1$), and hence may be viewed as a parameter mostly affected by soil texture. Soil structural effects usually show up most clearly near saturation. For a given value of mn, the shape of the retention curve near saturation becomes fixed by specifying either m or n (fig. 2)

To verify the ability of equation (8) in matching experimental data, a nonlinear optimization method analogous to that described by van Genuchten (1978) was used to analyze numerous published and unpublished data sets. In one particular exercise, 102 retention curves documented by Mualem (1976*b*) were analyzed. Taking the residual sum of squares (SSQ) of the fitted *versus* observed water contents as a simple means of comparing the relative accuracy of various retention models, table 1 shows computed SSQ's for 33 typical data sets taken from Mualem's catalogue. SSQ's are given for equation (18) with variable m and n (SSQ_1 in table 1), for $m = 1 - 1/n$ (SSQ_2), for $m = 1 - 2/n$ (SSQ_3), and for $n \to \infty$ (SSQ_4), the latter case resulting in the model of Brooks and Corey (1964). For a description of the soils listed in table 1 and for appropriate references, we refer to the papers by Mualem (1976*a*, *b*).

As expected, having both m and n as variable coefficients in (8) leads to the best fit (table 1). Of the three cases with restricted m and/or n values, SSQ_2 for $m = 1 - 1/n$ had the lowest values (or was tied with the lowest value) 44 times, SSQ_3 was the lowest 35 times, while SSQ_4 had 30 times the lowest value. Clearly, neither of the three restricted cases produces the better fit for all data sets, although the relation $m = 1 - 1/n$ seems to perform better than the other two restricted cases. We note here that many or most of the data sets listed by Mualem (1976*b*) pertain to laboratory experiments involving disturbed, screened soils with rather narrow pore-size distributions. Brooks and Corey's model (see SSQ_4 in table 1) is expected to perform better for such soils than for undisturbed field soils which usually exhibit much broader pore-size distributions. We also tested equation (8) on 200 field-measured retention curves for Yolo loam, obtained at a site near Davis, California. Of the three restricted m, n cases, SSQ_2 for $m = 1 - 1/n$ was found to be the lowest (or tied with the lowest value) about 70 % of the time, SSQ_3 for $m = 1 - 2/n$ was lowest 35 % of the time, and SSQ_4 ($n \to \infty$) was lowest 13 % of the time (average SSQ-values for the 200 data sets, as a fraction of SSQ_1

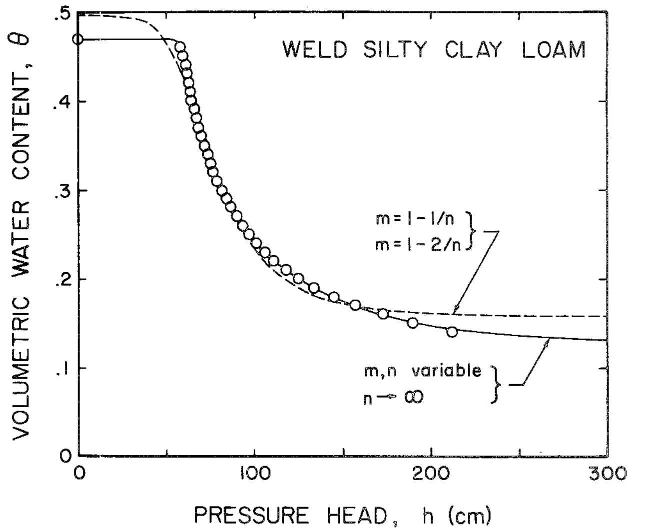

Figure 3
Observed and fitted retention curves for Weld silty clay loam.

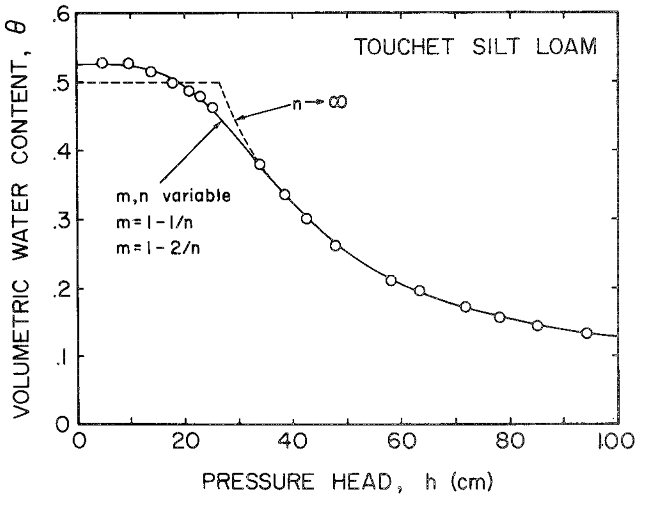

Figure 4
Observed and fitted retention curves for Touchet silt loam.

Table 2
Fitted values for the unknown independent parameters in equations (8) *and* (1) *for the retention curves plotted in figures* 3 *through* 6. *The residual sum of squares* (SSQ) *of the curve-fittings are also given.*

Soil name/Type of curve	θ_r	θ_s	α	n	m*	λ	SSQ × 10^5
Weld silty clay loam							
variable m, n	0.116	0.469	0.0173	61.54	0.0308	—	18
$m = 1 - 1/n$	0.159	0.496	0.0136	5.45	(0.816)	—	487
$m = 1 - 2/n$	0.155	0.495	0.0143	5.87	(0.659)	—	425
$n \to \infty$	0.116	0.465	0.0172	—	—	1.896	21
Touchet silt loam							
variable m, n	0.081	0.524	0.0313	3.98	0.493	—	14
$m = 1 - 1/n$	0.102	0.526	0.0278	3.59	(0.721)	—	17
$m = 1 - 2/n$	0.082	0.524	0.0312	3.98	(0.497)	—	14
$n \to \infty$	0.018	0.499	0.0377	—	—	1.146	367
G.E. No. 2 sand							
variable m, n	0.091	0.369	0.0227	4.11	4.80	—	24
$m = 1 - 1/n$	0.057	0.367	0.0364	5.05	(0.802)	—	34
$m = 1 - 2/n$	0.0	0.370	0.0382	4.51	(0.557)	—	56
$n \to \infty$	0.0	0.352	0.0462	—	—	1.757	354
Sarpy loam							
variable m, n	0.051	0.410	0.0127	1.114	0.886	—	60
$m = 1 - 1/n$	0.032	0.400	0.0279	1.60	(0.374)		99
$m = 1 - 2/n$	0.012	0.393	0.0393	2.45	(0.185)		199
$n \to \infty$	0.0	0.380	0.0444	—	—	0.387	539

* Values for m within parenthesis were calculated from the fitted n-value.

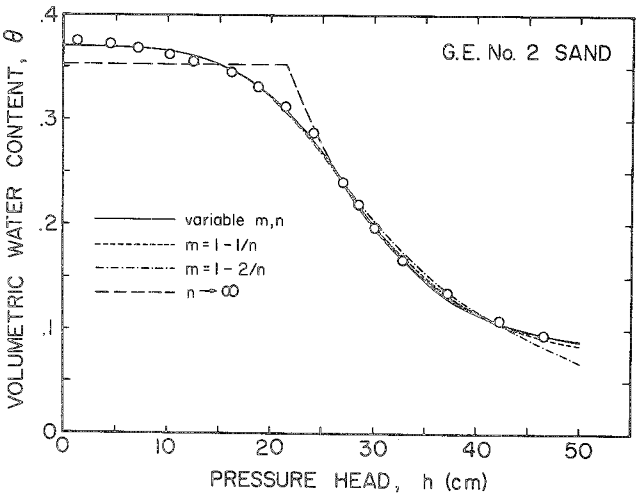

Figure 5
Observed and fitted retention curves for G.E. No. 2 sand.

for the variable m, n case, were 1.09, 1.13 and 2.01 for SSQ_2, SSQ_3 and SSQ_4, in that order).

To illustrate typical results, observed and fitted retention data for 4 soils are compared in figures 3-6. Fitted parameter values for these soils are shown in table 2. Figure 3 shows the results for Weld silty clay loam (Jensen and Hanks, 1967; Mualem's soil index 3001 in table 1). In this example, Brooks and Corey's equation matches the retention data equally well as the variable m, n case, whereas the two curves associated with SSQ_2 and SSQ_3 in table 1 produce extremely poor results. This situation is reversed with Touchet silt loam (King, 1965; Mualem's soil index 3308 in table 1), results of which are shown in figure 4. Here, Brooks and Corey's model produces an unacceptable match, while the two other cases with restricted m, n values are essentially identical to the case of variable m, n. Figure 5 shows similar results for G.E. No. 2 sand (King, 1965; Mualem's soil index 4115 of table 1); significant differences between the three smooth curves

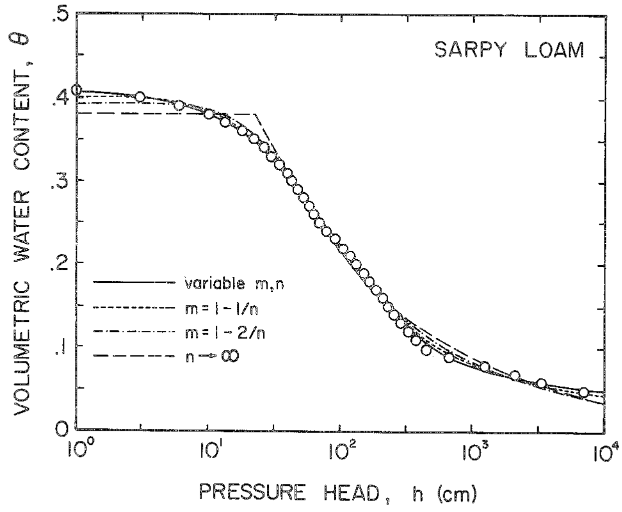

Figure 6
Observed and fitted retention curves for Sarpy loam.

are apparent only at the lower water contents. Finally, figure 6 shows a case with visible differences between all four curves based on equation (8). Data for this example (Sarpy loam) were taken from Hanks and Bowers (1962). As manifested also by the SSQ-values in table 2, there is in this case a progressively better match with the data going from SSQ_4 ($n \to \infty$) to SSQ_3 ($m = 1 - 2/n$), to SSQ_2 ($m = 1 - 1/n$), to SSQ_1 (variable m, n).

From the results in figures 3-6, from the data in table 1, and from numerous other calculations not shown here, we conclude that equation (8) with variable m, n always produces a near perfect fit of the measured retention data. Of the three cases with restricted m and n values, the relation $m = 1 - 1/n$ performs best for many but not all soils, while the Brooks and Corey equation generally leads to the worst match (except again for screened or repacked soils). Clearly, none of the three restricted cases is able to accurately match retention data for all soil types; only the variable m, n case leads to excellent results for most if not all soils.

PREDICTIONS FOR THE UNSATURATED HYDRAULIC CONDUCTIVITY

In this section we will use the predictive models of either Burdine (1953) or Mualem (1976*a*) to derive various expressions for the unsaturated hydraulic conductivity K in terms of the parameters that appear in equation (8). Of the two models, Mualem's formulation will be considered first.

Mualem's model

The predictive model of Mualem (1976*a*) can be written in the form

$$K_r = \sqrt{S_e}\left[\frac{f(S_e)}{f(1)}\right]^2 \quad (10)$$

where K_r is the relative hydraulic conductivity, K/K_s (with K_s representing the hydraulic conductivity at saturation), and where

$$f(S_e) = \int_0^{S_e} \frac{1}{h(x)}\,\mathrm{d}x\,. \quad (11)$$

Solving (8) for $h(S_e)$ and substitution into (11) gives

$$f(S_e) = \int_0^{S_e}\left[\frac{x^{1/m}}{1 - x^{1/m}}\right]^{1/n}\mathrm{d}x\,. \quad (12)$$

The substitution $x = y^m$ further reduces (12) to

$$f(S_e) = m\int_0^{S_e^{1/m}} y^{m-1+1/n}(1 - y)^{-1/n}\,\mathrm{d}y$$

$$= mI_\zeta(p, q)\,B(p, q) \quad (13)$$

where $I_\zeta(p, q)$ is the Incomplete Beta function (e.g., see Zelen and Severo, 1965), $B(p, q)$ is the Complete Beta function,

$$\zeta = S_e^{1/m} \quad (14)$$

and

$$p = m + 1/n \qquad q = 1 - 1/n\,. \tag{15}$$

Because $f(1) = mB(p, q)$, equation (10) becomes simply

$$K_r(S_e) = \sqrt{S_e}[I_\zeta(p, q)]^2\,. \tag{16}$$

which is the general solution for variable m and n.

Evaluation of the Incomplete Beta function in equation (16) is rather cumbersome. For most values of S_e, m and n, this function can be approximated accurately with *continued fractions* using equations (26.5.8) of Zelen and Severo (1965) :

$$I_\zeta(p, q) = \frac{\zeta^p(1-\zeta)^q}{pB(p, q)}\left[\frac{1}{1+}\,\frac{d_1}{1+}\,\frac{d_2}{1+}\cdots\right] \tag{17}$$

where

$$d_{2m+1} = \frac{-(p+m)(p+q+m)}{(p+2m)(p+2m+1)}\zeta$$

$$d_{2m} = \frac{m(q-m)}{(p+2m-1)(p+2m)}\zeta\,. \tag{18}$$

To increase the rate of convergence, the following transformation for $S_e > \max[2/(2+m), 0.2]$ was carried out before applying equation (17) :

$$I_\zeta(p, q) = 1 - I_{1-\zeta}(q, p)\,. \tag{19}$$

The above numerical approximation produced excellent results. Generally, only five terms of (17) were needed to obtain an accuracy of at least four significant digits; a few more terms are recommended for $m > 2$. For very small values of $\zeta = S_e^{1/m}$, we could simplify equation (16) considerably by using the following approximation for $f(S_e)$

$$f(S_e) = \frac{mn}{mn+1}S_e^{1+1/mn}\,. \tag{20}$$

This equation follows from (12) by neglecting the term $x^{1/m}$ in the denominator of equation (12) and subsequently integrating that equation. Substituting (20) into (11) and using $f(1) = mB(p, q)$ leads then to

$$K_r = \frac{n^2\,S_e^{5/2+2/mn}}{[(mn+1)\,B(p, q)]^2}\,. \tag{21}$$

As shown earlier (van Genuchten, 1980), simpler solutions for equations (10) and (11) can be derived when the permissible values for m and n are restricted such that $k = m - 1 + 1/n$ becomes an integer. The simplest case arises when $k = 0$, and hence when $m = 1 - 1/n$. The relative hydraulic conductivity in that case becomes :

$$K_r = \sqrt{S_e}[1-(1-S_e^{1/m})^m]^2 \quad (m = 1-1/n;\ 0<m<1) \tag{22}$$

The predictive equations for K_r above were obtained with equation (8). For completeness, we also give here the K_r-expression based on retention model (6) which was used earlier by Su and Brooks (1975) in connection with Burdine's model. Substitution (6) into (11), integrating, and combining the result with (10) for Mualem's model leads to

$$K_r = \sqrt{S_e}[I_{S_e}(b+1, 1-a)]^2 \quad (a < 1)\,. \tag{23}$$

Burdine's model

Results analogous to those for Mualem's model can be derived also for Burdine's theory (Burdine, 1953). Burdine's model can be written in the form

$$K_r(S_e) = S_e^2\,\frac{g(S_e)}{g(1)} \tag{24}$$

where

$$g(S_e) = \int_0^{S_e}\frac{1}{h^2(x)}\,\mathrm{d}x\,. \tag{25}$$

Substituting the inverse $h(S_e)$ of equation (8) into (25) yields

$$g(S_e) = \int_0^{S_e}\left[\frac{x^{1/m}}{1-x^{1/m}}\right]^{2/n}\mathrm{d}x\,. \tag{26}$$

The substitution $x = y^m$ reduces (26) to

$$g(S_e) = m\int_0^{\zeta} y^{m-1+2/n}(1-y)^{-2/n}\,\mathrm{d}y$$

$$= mI_\zeta(r, s)\,B(r, s) \tag{27}$$

where ζ is defined as before (eq. (14)), and where

$$r = m + 2/n \qquad s = 1 - 2/n\,. \tag{28}$$

Combining (27) and (24) yields

$$K_r = S_e^2\,I_\zeta(r, s)\,. \tag{29}$$

Equation (29) is evaluated in exactly the same fashion as before for the Mualem-based expressions. In particular, $K_r(S_e)$ for small values of ζ reduces to the simple form

$$K_r(S_e) = \frac{mn}{mn+2}S_e^{3+2/mn}\,. \tag{30}$$

As before, a similar predictive equation can be obtained also when retention model (6) is used. The result in this case is (see also Su and Brooks, 1975)

$$K_r = S_e^2\,I_{S_e}(1+2b, 1-2a)\,. \tag{31}$$

SOME CALCULATED HYDRAULIC CONDUCTIVITY CURVES

Figure 7 shows calculated curves of the relative hydraulic conductivity (K_r) as a function of both the reduced pressure head (αh) and the reduced water content (S_e). The curves, obtained with equation (16) for Mualem's model, are for the same retention curves shown in figure 2, i.e., with the product mn kept at 0.4. Note that, like the retention curves, the $K_r(\alpha h)$-curves in figure 7*a* remain smooth except for the limiting case when $n \to \infty$. The two limiting curves for $n \to \infty$ in figure 7*a* and 7*b* are given by (van Genuchten, 1980) :

$$K_r(h) = (\alpha h)^{-2-5mn/2} \cong (\alpha h)^{-3} \tag{32a}$$

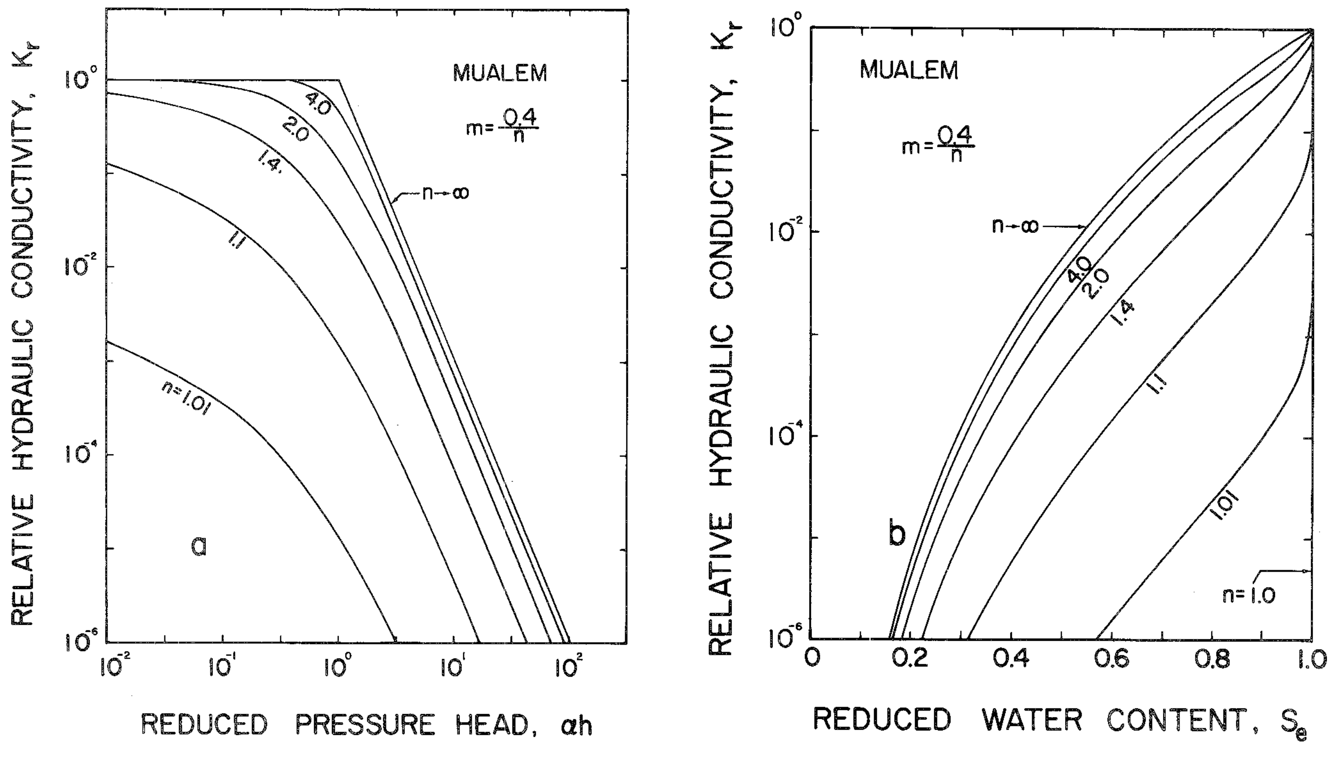

Figure 7
Calculated curves for the relative hydraulic conductivity versus *reduced pressure head* (a) *or reduced water content* (b) *as predicted from the retention curves of figure 2 using Mualem's model.*

and

$$K_r(S_e) = S_e^{5/2+2/mn} \equiv S_e^{7.5} \qquad (32b)$$

respectively; they are obtained by combining equations (3) (with $\lambda = mn$), (10) and (11).

Figures 7*a*, *b* show that the hydraulic conductivity curve decreases in value when n approaches unity. This is because the Complete Beta function $B(p, q)$ goes to infinity when $n \to 1$. Hence, Mualem's model in connection with equation (8) cannot be used when $n \leqslant 1$. Assuming variable m and n, only 8 of the 102 retention curves listed in Mualem's catalogue produced n-values that were less than 1. Similarly, of the 200 retention curves for Yolo loam, less than 10 % had n-values of less than 1.0 (most of those cases showed n-values between 0.9 and 1.0). As we noted earlier (eq. (9)), the slope of the retention function at saturation goes to $-\infty$ when $n < 1$, a physically unrealistic situation. Hence, it is likely that the inapplicability of Mualem's model for $n < 1$ can be traced back to an unreasonably shaped retention function near saturation.

As in figure 7 for Mualem's model, figure 8 shows calculated curves for $K_r(\alpha h)$ and $K_r(S_e)$ based on equation (29) for Burdine's model. The two limiting curves for $n \to \infty$ (with mn again fixed at 0.4) in this case are given by (Brooks and Corey, 1964)

$$K_r(h) = (\alpha h)^{-2-3mn} \equiv (\alpha h)^{-3.2} \qquad (33a)$$

and

$$K_r(S_e) = S_e^{3+2/mn} \equiv S_e^8 . \qquad (33b)$$

Note that for relatively large n-values, the predicted curves in figure 8 are very similar to the Mualem-based curves in figure 7. However, considerable differences between the two figures are evident when n is relatively small. In particular, note that the Burdine-based curves approach zero when $n \to 2$. This shows that Burdine's theory can be applied to far fewer soils than Mualem's model (and notably not to soils that exhibit relatively broad pore-size distributions characterized by small n-values).

The discussion above pertains only to equation (8) when both m and n are considered to be independent parameters. When fixing m and n by forcing the relation $m = 1 - 1/n$ for the Mualem-based expressions or $m = 1 - 2/n$ for the Burdine-based equations, the fitted retention parameters always lead to well-defined K_r-curves. For example, when we keep the product mn at 0.4 as before (fig. 2), the predicted curves for Mualem's model with $m = 1 - 1/n$ are marked $n = 1.4$ in figure 7, while the predicted curves for Burdine's model with $m = 1 - 2/n$ are marked $n = 2.4$ in figure 8. Like their counterparts in figure 2, these predicted curves have a limited flexibility since the shape of the curve near saturation is forced to have a unique relation with the shape (slope) of the curve in the dry range when $\alpha h \gg 1$. In other words, the position and slope of the K_r-curve near saturation is fixed for a given slope at the dry end of the curve (or vice versa).

To further illustrate the effects of restricting the relation between m and n in equation (8), figures 9-12 show predicted conductivity curves for the retention curves

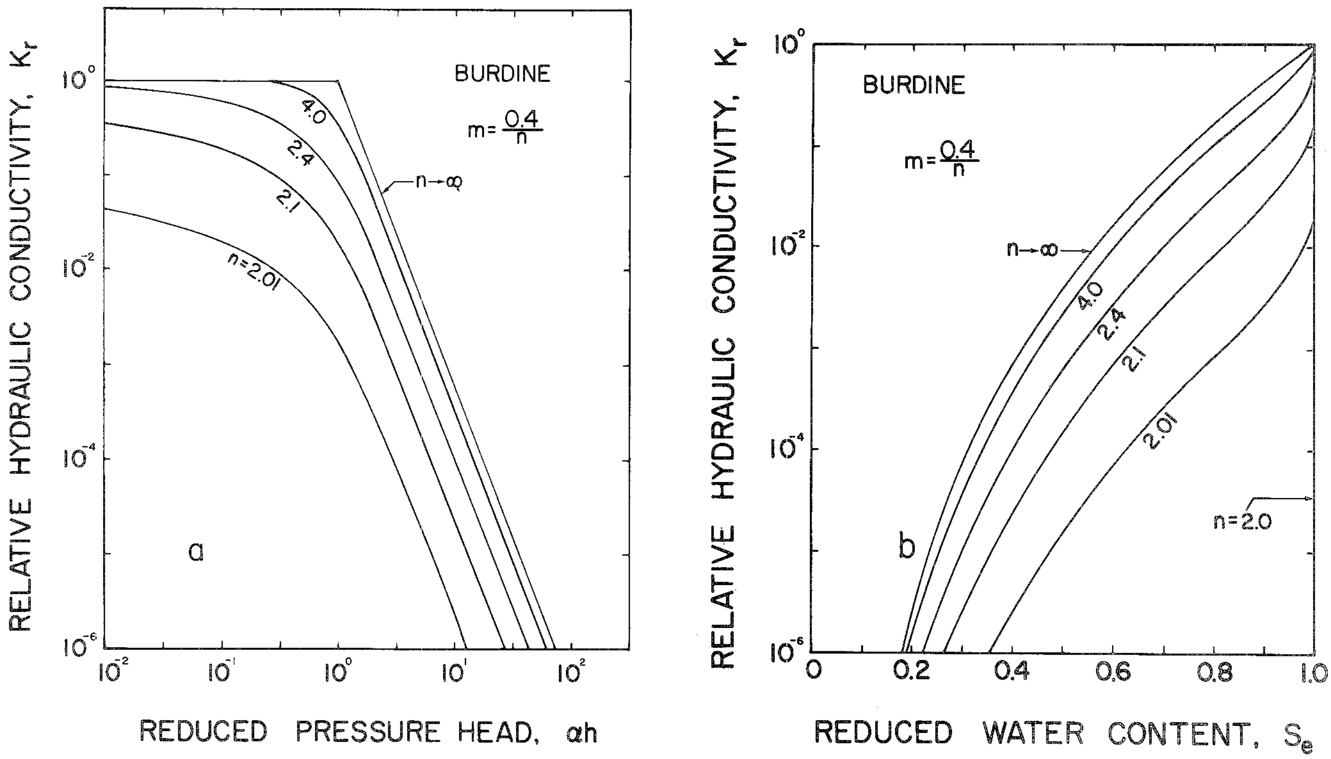

Figure 8
Calculated curves for the relative hydraulic conductivity versus *reduced pressure head (a) or reduced water content (b) as predicted from the retention curves of figure 2, using Burdine's model.*

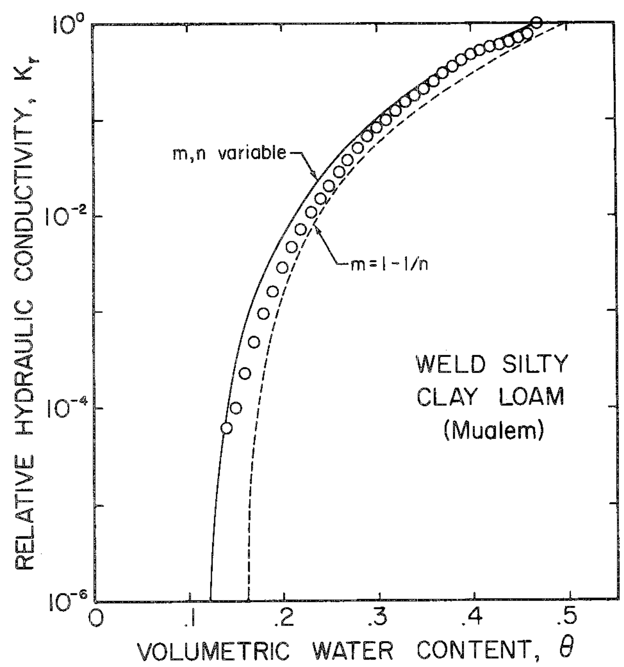

Figure 9
Observed and predicted relative hydraulic conductivity curves for Weld silty clay loam (Mualem's model).

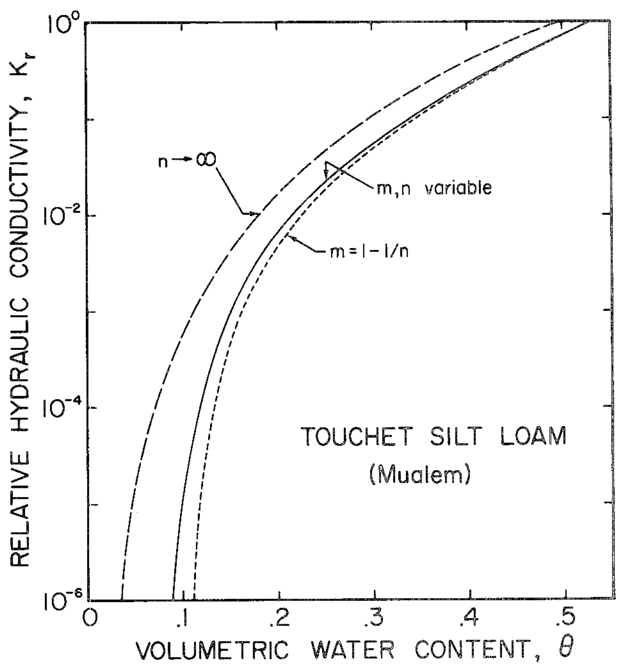

Figure 10
Predicted relative hydraulic conductivity curves for Touchet silt loam (Mualem's model).

shown in figures 3-6. Results are plotted either as a function of the volumetric water content (figs. 9, 10, 12*b*) or the pressure head (figs. 11, 12*a*). In all cases, calculations were based on Mualem's model. Except for Sarpy loam (fig. 12), application of Burdine's model lead to curves that qualitatively exhibited similar shapes as the Mualem-based curves. Because n was less than 2, no Burdine-based K_r-curve for Sarpy loam could be calculated for the variable m, n case.

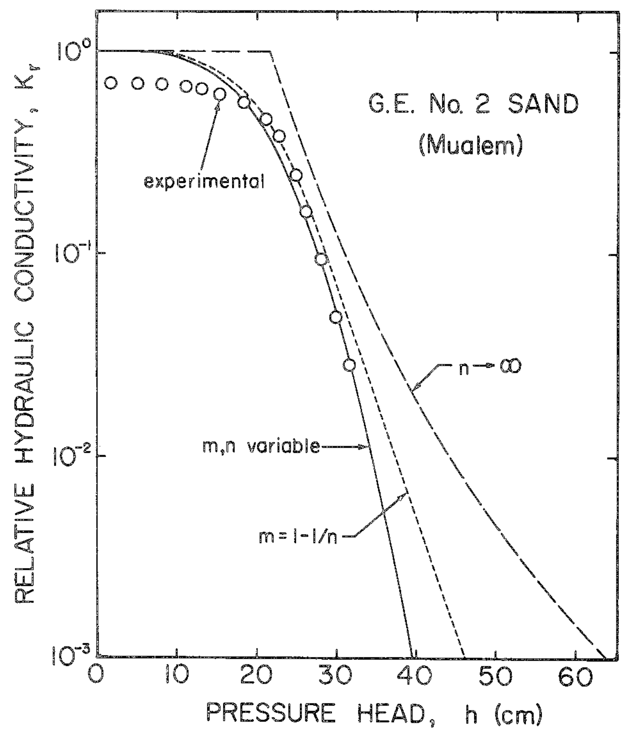

Figure 11
Observed and predicted relative hydraulic conductivity curves for G.E. *No. 2 sand (Mualem's model).*

Differences between the calculated curves in figures 9-12 parallel those found for the fitted retention curves for the same soils (figs. 3-6). For example, the predicted curve for variable m, n for Weld silty clay loam (fig. 9) and the limiting case when $n \rightarrow \infty$ were found to be nearly identical. For Touchet silt loam (figs. 4 and 10) and G.E. No. 2 sand (figs. 5 and 11), the calculated K_r-curves for $m = 1 - 1/n$ and the variable m, n case are not quite as close as the fitted retention curves. Note that for these last two soils the limiting curve $n \rightarrow \infty$ leads to much higher K_r-values than the two other cases.

For completeness, we have included in figures 9 and 11 also the experimental conductivity data as listed by Mualem (1976*b*). In both figures, the variable m, n case appears to lead to the best match with the experimental curves. However, we caution the reader from putting too much significance on these two examples only. Clearly, numerous other $K(\theta)$ or $K(h)$ data sets need to be analyzed before general conclusions about the accuracy of the different predictive equations can be made.

Figure 11 shows the calculated curves for Sarpy loam, a soil that exhibited visible differences between all four fitted retention curves, especially near saturation (fig. 6). Although those differences may appear to be relatively minor, they do lead to significant differences in the predicted curves (figs. 12*a*, *b*). Main reason for the sensitivity of the predicted curve to small changes in the slope and location of the fitted retention curve near saturation is the rather small n-value obtained for Sarpy loam (table 2). For example, n equals 1.114 for the variable m, n case which, as we already showed

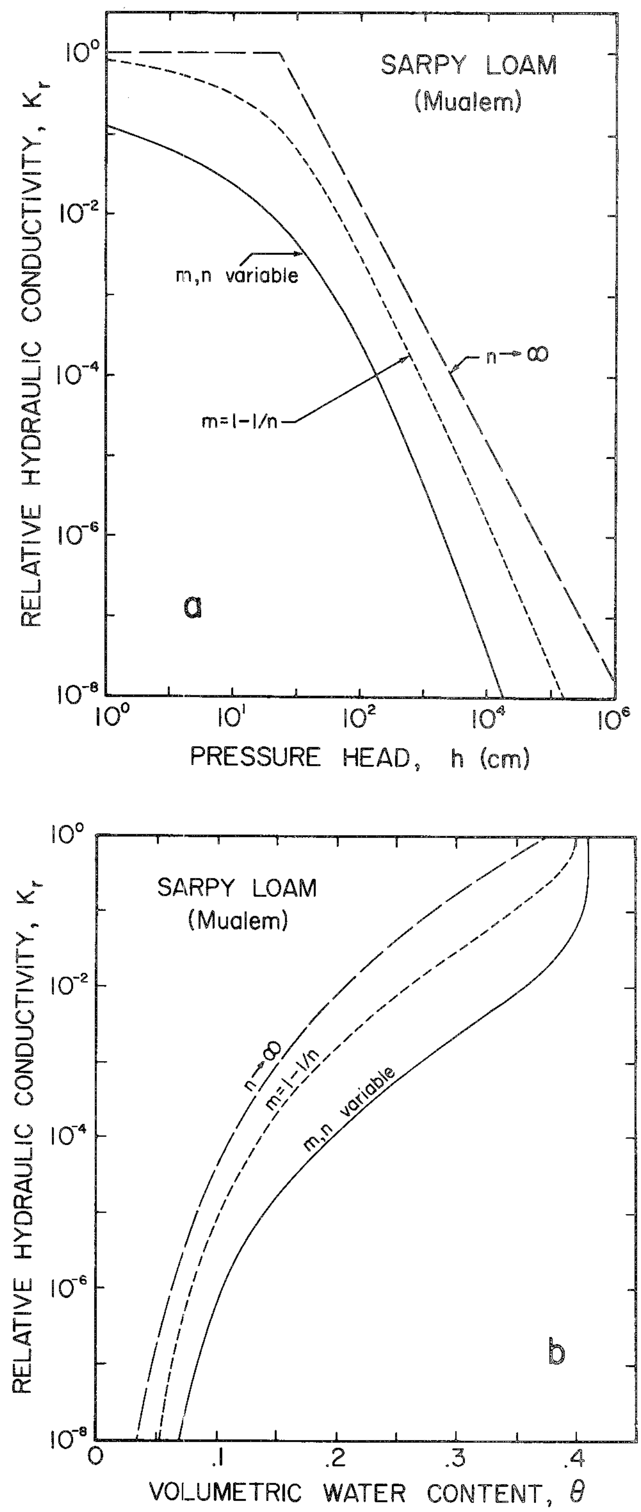

Figure 12
Predicted curves for the relative hydraulic conductivity versus *pressure head (a) and volumetric water content (b) for Sarpy loam (Mualem's model).*

in figure 7, leads to a relatively sharp drop in K_r just away from saturation. Because $n < 2$ (variable m, n), no predicted curve based on Burdine's model could be obtained for Sarpy loam. The results of figure 12 are important; they show that a small change in the slope of the fitted retention curve near saturation can greatly alter the location of the predicted K_r-curve over the entire range of conductivity values. Using a fine sandy soil, Stephens and Rehfeldt (1984) similarly demonstrated a marked sensitivity of the predicted conductivity curve to small changes in the location and slope of the retention curve at or near saturation. For Sarpy loam this sensitivity is further demonstrated in figure 13*a* where, for the same retention curves as in figure 6, the predicted curves for the soil water diffusivity (D) are compared with the experimental data of Hanks and Bowers (1962). In general terms, D is defined as

$$D = K\left|\frac{\mathrm{d}h}{\mathrm{d}\theta}\right| \equiv \frac{K_s\,K_r}{\alpha m n(\theta_s-\theta_r)}\,S_e^{-1-1/mn}(1-S_e^{1/m})^{1-1/n} \qquad (34)$$

where K_r is given by one of the predicted K_r-expressions. Specific equations for D applicable to the restricted m, n cases are given elsewhere (van Genuchten, 1980). Equation (34) requires a value for the saturated hydraulic conductivity, K_s, which for Sarpy loam was taken to be 0.0015 cm s^{-1} (Hanks and Ashcroft, 1980). Figure 13*a* shows that the variable m, n case severely underpredicts the curve, while the two restricted cases $n \to \infty$ and $m = 1 - 1/n$ describe the data equally well, the latter case ($m = 1 - 1/n$) having a somewhat more realistic shape near saturation (that is, $D \to \infty$ as the curve approaches saturation).

The curves in figure 13*a* were obtained by assuming that K_s is known, thus forcing the theoretical and experimental curves to be matched at saturation (even though the theoretical diffusivity functions go to infinity when the soil approaches saturation). Unfortunately, the value of K_s is frequently ill-defined or difficult to measure (see discussion below); in that case it is more appropriate to match the $K(\theta)$ or $D(\theta)$ curve at some other point. In figure 13*b*, the measured and theoretical curves were matched at the point ($\theta = 0.33$; $D = 0.0792$ cm^2 s^{-1}). The three calculated curves now match the measured data well, except near saturation where the limiting curve $n \to \infty$ severely underpredicts the observed values. Note that this limiting curve remains finite at saturation while the other two calculated curves become infinite.

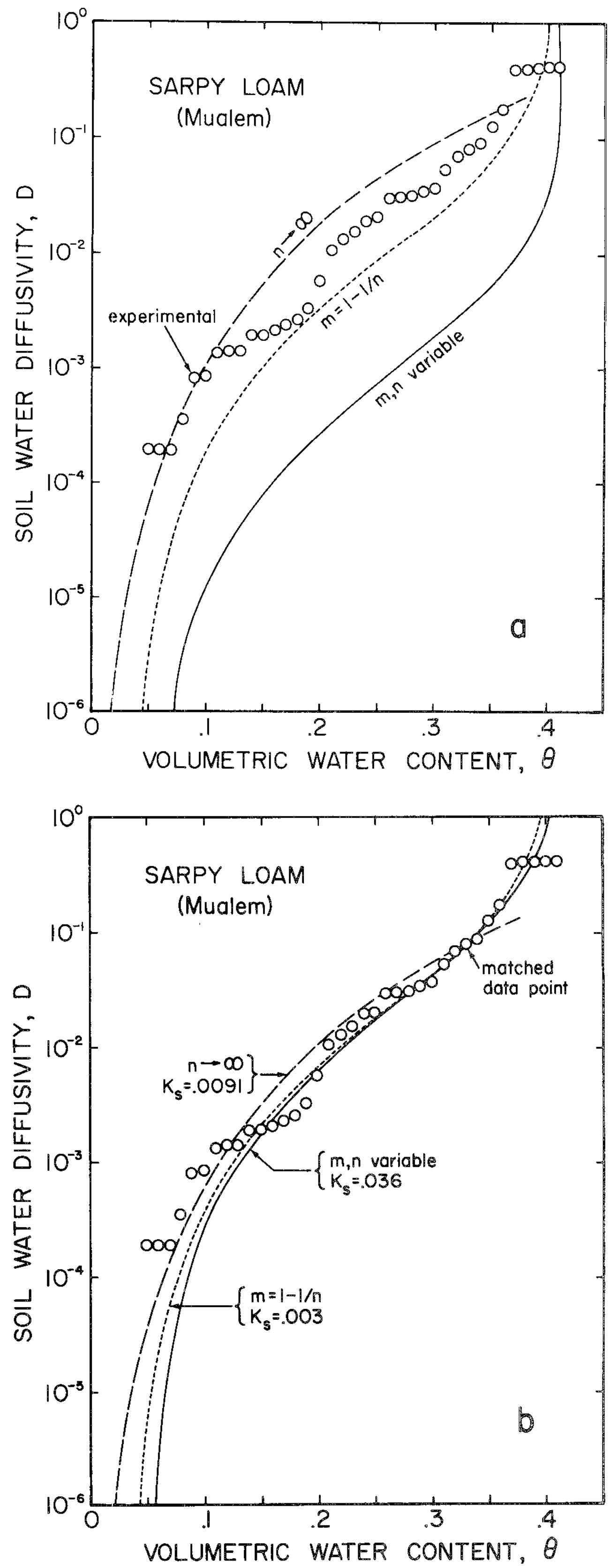

Figure 13
Observed and predicted curves for the soil water diffusivity of Sarpy loam. The predicted curves were obtained (a) by using the measured K_s-value in equation (34), *and (b) by directly matching the curves to an experimental point as shown.*

Of the soil hydraulic functions discussed in this paper, those for Sarpy loam probably are the most characteristic for a soil exhibiting a relatively broad pore-size distribution. Assuming that several large pores are present, some of those may desaturate quickly with the application of a small (negative) pressure. The water content near saturation thus drops quickly, which in turn could result in an *apparent* nonzero slope of the curve at saturation. If we ignore any matrix or fluid compressibilities and thus limit the discussion to non-swelling soils, one may reason, however, that always a finite and nonzero pressure is needed before a pore of finite dimensions can lose its water : desaturation can only start when the « air entry value » of that pore is reached. Consequently, the slope of the retention curve at saturation must be zero, and hence $n > 1$. Irrespective of what happens microscopically in such large pores upon desaturation, it is clear that from a macroscopic point of view the situation remains ill-defined, especially in view of our measurement techniques. Application of a certain retention model, such as equation (8) or any other model, further obscures the description of water retention near saturation. Using that retention model, at best we can extrapolate the observed data towards saturation. This procedure leaves the conceptual definition of a « saturated water content » in the context of a retention model as a mere academical exercise, not only because of entrapped or dissolved air, but also because of the presence of a few large pores or cracks. This is exactly the reason why in this study we consider θ_s, like θ_r and the other constants in equation (8), to be an empirical parameter that should be fitted to the data.

The above definition problem of « saturation » is even more acute when we attempt to quantify the parameter K_s in a predictive hydraulic conductivity model. For a disturbed, repacked soil with a narrow pore-size distribution and having large n-values, K_s should be well-defined, even when extrapolated from slightly unsaturated conditions (e.g., see the example of Weld silty clay loam). However, for undisturbed and especially aggregated soils, direct field measurement of K_s may prove to be extremely difficult : K could easily change several orders of magnitude with the application of only a few cm pressure. Ignoring for now the theoretical basis of the plots in figures 7 and 8, the rapid decrease of the hydraulic conductivity near saturation when n approaches 1 (fig. 7*a*) or 2 (fig. 8*a*) is intuitively quite realistic : K at saturation is determined by only a few large pores or cracks with little direct relation to the overall pore-size distribution that determines the shape of the K_r-curve at intermediate saturation values. Hence, inasmuch as θ_s in practice is a poorly defined physical parameter that should be considered in the context of a chosen retention model, one can argue even more convincingly that K_s should be an extrapolated parameter that must be fitted to a particular hydraulic conductivity model, even if that model were to be a purely empirical equation. Moreover, a fitted K_s using unsaturated conductivity data should better reflect the unsaturated flow properties of the medium, a consideration that may prove to be especially important when the hydraulic functions subsequently are used to predict unsaturated flow.

The above comments about the definition of θ_s and K_s are most opportune for soils with relatively small n-values, Sarpy loam being only a typical example. Clearly, the « saturated hydraulic conductivity » of such soils is not the best point for matching predicted and measured curves (e.g., see Jackson *et al.*, 1965; Green and Corey, 1971; among others). Instead, it seems more accurate to match the theoretical and experimental curves at some other point of the curve, perhaps still somewhere in the wet range but definitely not at saturation (Roulier *et al.*, 1972; Carvallo *et al.*, 1976). The latter procedure was followed for Sarpy loam, and perhaps should have been applied also to G.E. No. 2 sand (fig. 10).

SUMMARY AND CONCLUSIONS

The successful prediction of the unsaturated hydraulic conductivity from soil water retention data using analytical (nontabular) functions requires a reliable descriptive model of the retention curve as well as an accurate model for K_r on which the prediction ultimately is based. This paper shows that equation (8) has great flexibility in describing retention data from various soils, has a simple inverse function, and permits the derivation of closed-form analytical equations for $K(\theta)$ when combined with the predictive theories of either Burdine (1953) or Mualem (1976*a*). The 5-parameter retention model graphically displays a smooth, continuously differentiable S-shaped curve between the saturated and residual water contents. Application of the model to numerous data sets, both for disturbed, screened soils as well as for undisturbed field soils, consistently lead to an excellent match with the observed data.

Of the two hydraulic conductivity models, Mualem's model was found to be applicable to a wider variety of soils than Burdine's model. For example, Mualem's model can be combined with equation (8), provided the parameter n in that equation exceeds 1, whereas Burdine's model requires this coefficient to be larger than 2. When $n < 1$, the slope of the retention curve near saturation becomes $-\infty$, a situation that is physically unrealistic; for $n > 1$, the slope at saturation is always zero. Analysis of hundreds of data sets revealed n-values of less than 1 for only about 5 % of the cases. We tend to conclude that such low n-values are due to poorly defined or incomplete data sets, rather than a result of theoretical limitations inherent in Mualem's model.

Application of equation (8) with variable m and n to the models of Burdine or Mualem leads to predictive equations for K_r that contain the Incomplete Beta function. Although accurate approximations are available (eq. (17)), routine application of this function to unsaturated flow problems may prove to be too cumbersome. Except for soils characterized by very small (e.g., $n < 1.25$) or very large n-values (e.g., $n > 6$), the unsaturated hydraulic properties are approximated reasonably well with the more restrictive case where $m = 1 - 1/n$; the predictive K_r-expression based on Mualem's model can then be simplified considerably (eq. (22)). Data often are available over only a small part of the retention curve (usually the wet range), especially when the curve is based on direct field measurements. Estimation of all 5 independent parameters in equation (8) from such data sets may lead to inversion problems manifested by coefficient values with extremely large standard errors. For such data sets, we also recommend that the restricted case $m = 1 - 1/n$ be used.

Finally, this study concentrated on the analysis of soil water retention data. Although additional studies of the retention curve are needed, notably studies that deal with the mathematical description of the curve near saturation, we feel that equally or more research should focus on improved predictive models for the relative hydraulic conductivity. For example, while we concluded that Mualem's model has more potential than Burdine's model in predicting the unsaturated hydraulic conductivity of widely different soils, a more detailed evaluation of Mualem's formulation remains necessary.

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