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On the Discretization of Continuous-Time Filters for Non-Stationary Stock and Flow Time Series

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On the discretization of continuous-time filters for nonstationary stock and flow time series

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Abstract

This paper discusses the discretization of continuous-time filters for application to discrete time series sampled at any fixed frequency. In this approach, the filter is first set up directly in continuous-time – since the filter is expressed over a continuous range of lags, we also refer to them as continuous-lag filters. The second step is to discretize the filter itself. This approach applies to different problems in signal extraction, including trend or business cycle analysis, and the method allows for coherent design of discrete filters for observed data sampled as a stock or a flow, for nonstationary data with stochastic trend, and for different sampling frequencies. We derive explicit formulas for the Mean Squared Error optimal discretization filters. We also discuss the problem of optimal interpolation for nonstationary processes – namely, how to estimate the values of a process and its components at arbitrary times in-between the sampling times. A number of illustrations of discrete filter coefficient calculations are provided, including the Local Level Model trend filter, the Smooth Trend Model trend filter, the Band Pass filter, and the Henderson filter. The essential methodology can be applied to other kinds of signal extraction problems.

Keywords. Continuous time processes, Hodrick-Prescott filter, Interpolants, Linear filtering, Signal extraction.

1 Introduction

Economic data are typically sampled as a stock or a flow, are often nonstationary with stochastic trend, and may differ in sampling frequency. In this paper, we propose a general method for obtaining filters for such data and for addressing the interpolation problem that arises when estimates of data or components are needed in between observation times. An efficient way to initiate the general analysis is by setting up an underlying continuous-time model, as in Harvey (1989) and Harvey and Stock (1993). These authors proceed, as is usually done, by discretizing the underlying model, so that the subsequent analysis is done with discrete-time signal extraction methods. Recently, McElroy and Trimbur (2006) have proposed an alternative method based on filters set up directly in continuous-time, referred to as continuous-lag filters because the lag/lead index of the filter coefficients can take on any real value.

This paper develops methods for the discretization of continuous-lag filters. Such filters arise naturally in considering continuous-time models for economic variables[1]. One advantage of the continuous-lag approach to filtering is that one can produce filter estimates from a sampled version of the series obtained at any fixed sampling interval, and also obtain estimates at intermediate times between the samples. One simply changes the value of δ – the sampling interval – in our formulas. Our method does not depend on the continuous-lag filter being derived from a specific time series model – nonparametric filters such as the Henderson filter in Dagum and Bianconcini (2006) may also be used.

In this paper, we derive explicit formulas for the Mean Squared Error (MSE) linear optimal discretized filters. We also discuss the problem of MSE linear optimal interpolation for nonstationary processes, for values of the process and its components at arbitrary times in-between the sampling times. The aim is to allow for coherent design of discrete filters for different problems in signal extraction, such as trend or business cycle analysis, and to handle observed data sampled as either a stock or a flow. A further advantage is that we are able to work with continuous-lag filters that specifically target some property of interest, for instance the velocity and acceleration filters for turning point analysis (see McElroy and Trimbur, 2006).

Section 2 briefly reviews continuous-time processes and filters. This material sets up the background and notation; a more complete treatment is given in McElroy and Trimbur (2006). Section 3 discusses filter discretization for stocks and flows and presents the main results: Theorem 1 gives formulas for the discretization of general continuous-lag filters for stock and flow data sampled from a trend nonstationary process; Theorem 2 gives similar formulas for the trend extraction problem. These new results are formulated using frequency domain methods, and at each step we allow for a general sampling frequency δ, as well as treatment of interpolations for both stock and flow variables.

In Sections 4 and 5, several illustrations of filter coefficient calculations for sampled series are given. We show how to discretize a continuous-lag Henderson filter (Dagum and Bianconcini, 2006), and provide derivations for the well-known Local Level Model and Smooth Trend Model. This latter example amounts to a generalized Hodrick-Prescott (HP) filter (Hodrick and Prescott, 1997)

[1] Bergstrom (1988) argues that it is natural to consider continuous-time models to represent underlying economic processes. For example, for many flow variables linked to the macroeconomy, increments from production and exchange occur on a more or less constant basis.

adapted to stocks and flows and appropriately interpolated, and so the results may be compared with those in Ravn and Uhlig (2002). More generally, we treat the discretization of the continuous-time analogue of the Butterworth (BW) filters; see Gómez (2001) and Harvey and Trimbur (2003). Further, we derive discretized filters for the continuous-lag Band Pass (BP) filter. The same basic method can equally well be applied to other problems in signal extraction.

2 Processes and Filters in Continuous Time

This section presents the theoretical framework for the analysis of continuous-time processes and filters. A fuller treatment is given in the companion paper McElroy and Trimbur (2006), which derives MSE linear optimal filters for continuous time signal extraction problems. Here we just discuss some concepts and notation that will be needed in the development that follows. Let $y(t)$ for $t \in \mathbb{R}$, the set of real numbers, denote a real-valued continuous-time process that is measurable and square-integrable for each t. The process is weakly stationary by definition if it has constant mean – set to zero for simplicity – and an autocovariance function that depends only on the lag h:

$$R_y(h) = \mathbb{E}[y(t)y(t+h)] \quad h \in \mathbb{R}. \tag{1}$$

Thus if $y(t)$ is a Gaussian process, R_y completely describes the dynamics of the stochastic process. A convenient class of stationary continuous-time processes has the form

$$y(t) = (\theta * \epsilon)(t) = \int_{-\infty}^{\infty} \theta(x)\epsilon(t-x)\, dx, \tag{2}$$

where $\theta(\cdot)$ is square integrable on $\mathbb{R}$, and $\epsilon(t)$ is continuous-time white noise (WN) – see Priestley (1981, p.156). In this case, $R_y(h) = (\theta * \theta^-)(h)$, where $\theta^-(x) = \theta(-x)$ and $*$ is the convolution operator. We define the continuous-time lag operator L via the equation

$$L^x y(t) = y(t-x) \tag{3}$$

for any $x \in \mathbb{R}$ and for all times $t \in \mathbb{R}$. We denote the identity operator L^0 by 1, just as in discrete time. Then a **Continuous-Lag Filter** is an operator $\Psi(L)$ with associated **weighting kernel** ψ (an integrable function) whose effect on a process $y(t)$ is given by

$$\Psi(L)y(t) = \int_{-\infty}^{\infty} \psi(x)\, y(t-x)\, dx = (\psi * y)(t). \tag{4}$$

Heuristically, we have $\Psi(L) = \int_{-\infty}^{\infty} \psi(x)\, L^x\, dx$ in analogy with discrete-lag filters. The requirement of integrability for the function $\psi(x)$ is a mild condition that we impose throughout. Then the frequency response function (frf) is obtained by replacing L by the argument $e^{-i\lambda}$:

$$\Psi(e^{-i\lambda}) = \int_{-\infty}^{\infty} \psi(x)\, e^{-i\lambda x}\, dx, \quad \lambda \in \mathbb{R}. \tag{5}$$

Denoting the continuous-time Fourier Transform by $\mathcal{F}[\cdot]$, equation (5) can be written as $\Psi(e^{-i\lambda}) = \mathcal{F}[\psi]$. There is a similar treatment in Priestley (1981, pp. 150–183), although our notation is slightly different. The power spectrum of a weakly stationary continuous-time process $y(t)$ is the Fourier Transform of its autocovariance function R_y:

$$f_y(\lambda) = \mathcal{F}[R_y](\lambda), \quad \lambda \in \mathbb{R}. \tag{6}$$

This exists when R_y is integrable, and the process $y(t)$ is said to be stochastically continuous (see Priestley (1981, p.151)). We will also need to consider *generalized spectra*, which includes non-integrable functions f_y – see Hannan (1971, Section II.9). For example, continuous-time white noise has generalized spectrum given by a constant – which is bounded but not integrable (Priestley, p.156, 1981). In this case (6) is interpreted as being true in the sense of tempered distributions (Folland, 1995).

As in discrete time series signal processing, passing an input process through the filter $\Psi(L)$ results in an output process with spectrum multiplied by the squared modulus of $\Psi(e^{-i\lambda})$. Note that in contrast to the discrete case where the domain is restricted to the interval $[-\pi, \pi]$, the frf (6) is defined over the entire real line. The inverse Fourier Transform of an integrable function g is defined by

$$\mathcal{F}^{-1}[g](x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\lambda) e^{i\lambda x} \, d\lambda, \quad x \in \mathbb{R}. \tag{7}$$

For nonstationary models, the mean-square derivative operator D plays a key role (again, see Priestley (1981)). In the time domain, it is expressed as $D = -\log L$, and its frequency response is $i\lambda$. The operator D gives the continuous-time analogue of the difference operator used in discrete-time to define nonstationary processes such as the standard ARIMA class, which are difference equations built on discrete-time white noise. In the same way, nonstationary continuous-time models may be defined as stochastic differential equations built on continuous-time white noise. See Brockwell and Marquardt (2005) for further discussion.

Let $\mathcal{C}^d$ denote the space of all processes that are dth order stochastically differentiable (see Priestley (1981, p. 153)); when $d = 0$, $\mathcal{C}^0$ refers to the space of stochastically continuous processes – for these an orthogonal increments representation exists by Theorem 4.11.1 of Priestley (1981). If $y \in \mathcal{C}^d$, we say that y is integrated of order d, or is $I(d)$, if $w(t) = D^d y(t)$ is a weakly stationary continuous-time process (but any lesser number of derivatives does not result in a stationary process). The CARIMA class of processes (Brockwell and Marquardt, 2005) furnish convenient examples of $I(d)$ processes. The major topic of this paper is the optimal discretization of a given continuous-lag filter $\Psi(L)$ that is applied to an $I(d)$ process $y(t)$. A secondary topic is concerned with the case that $\Psi(L)$ is a signal extraction filter, when $y(t)$ has a signal plus noise decomposition

$$y(t) = s(t) + n(t) \tag{8}$$

with signal $s(t)$ and noise $n(t)$. It is assumed that s is $I(d)$ just like y, but that n is weakly stationary – possibly $n \notin \mathcal{C}^0$. Letting $D^d s(t) = u(t)$, it will be assumed that $u(t)$ and $n(t)$ are uncorrelated with each other. Even though it is possible that w is not stochastically continuous, we let $f_w(\lambda) = f_u(\lambda) + \lambda^{2d} f_n(\lambda)$, which is interpreted as a generalized spectrum when it is non-integrable.

Examples of continuous-lag filters can be found in McElroy and Trimbur (2006), which discusses BW and HP filters, as well as model-based low-pass and band-pass filters for estimating trends and business cycles. Recently, Dagum and Bianconcini (2006) have proposed a weighting kernel to define a continuous-time version of the Henderson trend filters that are commonly used in seasonal adjustment methods such as X-12-ARIMA. Stock and flow discretizations of all these filters are presented in Sections 4 and 5, for any sampling interval length δ.

3 Discretization

In this section we discuss how a given continuous-lag filter $\Psi(L)$ can be optimally discretized. The problem is formulated as follows: the filter output is $x(t) = \Psi(L)y(t) = (\psi * y)(t)$, and the available sample of data is Y. This sample consists of a bi-infinite collection of regularly sampled stock or flow observations. We obtain the MSE optimal linear estimate of $x(t)$ given the sample Y. We also examine a related signal extraction problem, namely how to estimate $s(t)$ given (8) in an MSE optimal fashion from the given sample Y. In both cases, the time point t is allowed to come in-between the sampling times of Y.

3.1 Stock and Flow

In economics, most time series are classified as either stock or flow. This distinction expresses how the data are sampled from an underlying process. Stock observations are estimates of levels at particular time points, while flow observations are accumulations over time intervals. These properties are formalized by setting up a continuous-time model with characteristic measurement equations; see, for example, Harvey (1989, Chapter 9).

As indicated, we now suppose that the observed data are measured at evenly spaced intervals of length δ. The basic timing unit is defined in the fundamental continuous-time setting, and is set so that $\delta = 1$ corresponds to these time units. For example, if we have annual units, then for quarterly data the sampling interval is $\delta = 1/4$ and the sampling frequency is 4 observations per year. In the analysis that follows, results are derived for general δ; the effects of changing δ are therefore explicit in the formulas. Observations in the discretized series are indexed by integer values τ, so that the τth observation occurs at the time $\delta\tau$. In referring to a discrete-time process, the time index τ is thus used to represent particular points in continuous-time.

Starting with a base model in continuous-time, the classification into stock and flow is reflected in the measurement of observations. Given the sampling interval $\delta > 0$, a stock observation at the τth time point is defined as

$$y_\tau = y(\delta\tau). \tag{9}$$

We never make stock observations of processes that are not stochastically continuous, since this is not well-defined mathematically (as shown in Appendix A.1, the spectral density of the stock observations of such a process would be undefined). A series of flow observations has the form

$$y_\tau = \int_{\delta(\tau-1)}^{\delta\tau} y(v)\, dv. \tag{10}$$

Note that flow variables may also be formulated as differenced stock variables, if the stock variable can be expressed as the cumulative integral of some other underlying process.

3.2 Discretization of Filters

Now assuming that $y \in \mathcal{C}^d$ and that $D^d y(t) = w(t)$, where w is stationary, then from Hannan (1971, p.81) it follows that

$$y(t) = \sum_{j=0}^{d-1} \frac{t^j}{j!} y^{(j)}(0) + [I^d w](t), \tag{11}$$

where $y(0), y^{(1)}(0), \cdots, y^{(d-1)}(0)$ represent successive derivatives of $y(t)$ evaluated at time zero. The I operator is defined by $[Iw](t) = \int_0^t w(z)\, dz$, and $I^d w$ is obtained inductively for $d > 1$. Using integration by parts, we can write $[I^{d+1}w](t) = \int_0^t w(s)(t-s)^d\, ds/d!$, for any $t \in \mathbb{R}$. Now a stock observation of (11) at time $\delta\tau$ satisfies

$$y_\tau = \sum_{j=0}^{d-1} \frac{\delta^j}{j!} y^{(j)}(0)\tau^j + [I^d w](\delta\tau). \tag{12}$$

The process $\{y_\tau\}$ is a d-integrated discrete-time process, and is reduced to stationarity by taking d differences (with $B = L^\delta$):

$$w_\tau = (1-B)^d y_\tau = (1-B)^d [I^d w](\delta\tau). \tag{13}$$

The stationarity is proved in Appendix A.2; note that $w_\tau \neq w(\delta\tau)$. Now the process $\{y_\tau\}$ can be completely described through $\{w_\tau\}$ and d initial values, say $y^* = (y_0, y_{-1}, \cdots, y_{1-d})$ as shown in Bell (1984). Note that these initial values are distinct from the $\{y^{(j)}(0)\}$ when $d > 1$. We have two assumptions on the initial values:

$$y^* \quad \text{is uncorrelated with} \quad \{w_\tau\}_{\tau=-\infty}^{\infty} \tag{14}$$

$$y^* \quad \text{is uncorrelated with} \quad \{w(t)\}_{t\in\mathbb{R}}. \tag{15}$$

Clearly (15) implies (14), since the former is concerned with the stochastic process $w(t)$ at all times t, whereas the weaker condition is only concerned with the (differenced) sampled values. While (14) is sufficient for a purely discrete-time setup (e.g., when no interpolation is considered) and is implied by Assumption A of Bell (1984) – a commonly employed assumption in the theory of signal extraction, the stronger condition (15) is needed to establish optimality in the more general case.

If on the other hand we flow-observe (11), then

$$y_\tau = \sum_{j=0}^{d-1} \frac{\delta^{j+1}}{(j+1)!} y^{(j)}(0)(\tau^{j+1} - (\tau-1)^{j+1}) + \int_{\delta\tau-\delta}^{\delta\tau} [I^d w](v)\, dv. \tag{16}$$

This too is a d-integrated discrete-time process, and in this case d differences yields:

$$w_\tau = (1-B)^d y_\tau = (1-B)^d \int_{\delta\tau-\delta}^{\delta\tau} [I^d w](v)\, dv. \tag{17}$$

Now the initial values y^* and $\{w_\tau\}$ have different formulas, but we still refer to the initial value conditions (14) and (15), with w_τ interpreted appropriately.

Much of our treatment relies on frequency domain methods. When sampling at fixed intervals, the range of frequencies considered is restricted to $[-\pi/\delta, \pi/\delta]$. For any frequency outside this interval, the discrete-time behavior is equivalent to that of an "alias" frequency within the interval; see Koopmans (1974). Below we use the concept of the "fold of a spectral density": $[f]_\delta(\lambda) = \delta^{-1} \sum_{l=-\infty}^{\infty} f(\lambda + 2\pi l/\delta)$. This is the definition; if $\mathcal{F}^{-1}[f] = R$, it follows that $[f]_\delta(\lambda) = \sum_{h=-\infty}^{\infty} R(\delta h) e^{-i\lambda\delta h}$ as well, so $[f]_\delta$ is the spectral density of the stock-sampled (stationary) series (9). A discussion of the derivation and etymology of this concept is included in Appendix A.1 (also see Koopmans, 1974).

Now we address the problem of finding the minimal mean square error *linear* estimate of $x(t)$, given the data $Y = \{y_\tau\}$, which is either stock- or flow-sampled according to (9) or (10) respectively. Here $t = \delta\tau + \delta c$, where τ is the greatest integer such that $\delta\tau \leq t$, and $c \in [0,1)$. Thus, c determines to what extent $x(t)$ is placed in-between the sampling times; allowing for $c \neq 0$ in our optimal filters provides interpolation estimates of the underlying trend. The optimal solution can then be written as $\Psi_c(B) y_\tau$, where $\Psi_c(B)$ is a discrete-lag filter with each coefficient dependent upon c. Below, we present formulas for the frf $\Psi_c(e^{-i\lambda})$ (for $\lambda \in [-\pi/\delta, \pi/\delta]$) of the optimal filter, for both the stock and flow sampling cases, given that $y(t)$ is an integrated process of order d satisfying (15). In both the stock and flow cases, $\Psi_c\left(e^{-i\lambda}\right)$ will be seen to depend on c, d, and δ, on the continuous time frf $\mathcal{F}[\psi] = g$, and on the (generalized) spectral density f_w of the continuous time process $w(t)$. We use the notation $m_j(\lambda) = \lambda^{-j}$, and $e_c(\lambda) = e^{i\delta c\lambda}$.

Theorem 1 *In the situation described above assume (15), and that f_w has d continuous derivatives and is positive and bounded. Then at time $t = \delta\tau + \delta c$, the frf of the optimal filter $\Psi_c(B)$ is given* *by*

$$\Psi_c(e^{-i\lambda\delta}) = \frac{[g e_c f_w m_{2d}]_\delta(\lambda)}{[f_w m_{2d}]_\delta(\lambda)}$$
$$\Psi_c(e^{-i\lambda\delta}) = \frac{i}{1-e^{-i\lambda\delta}} \frac{[g e_c f_w m_{2d+1}]_\delta(\lambda)}{[f_w m_{2d+2}]_\delta(\lambda)}$$

for the stock (9) and flow (10) cases, respectively.

For the case when t is a sampled time $t = \delta\tau$, so that $c = 0$, we call the filter $\Psi_0(B)$ the optimal discretization of $\Psi(L)$; also Theorem 1 is then true for stocks under the weaker assumption (14). At in-between times t with $0 < c < 1$, by setting $\Psi(L)$ equal to the identity filter $\Phi(L) = 1$, Theorem 1 provides the frfs of the filters $\Phi_c(B)$ that yield optimal interpolation. That is, for a Gaussian process (where the linear optimal estimate coincides with the conditional expectation)

$$E[y(t)|Y] = \Phi_c(B)\, y_\tau. \tag{18}$$

Explicit formulas for $\Phi_c\left(e^{-i\lambda}\right)$ are given below.

Corollary 1 *Under the assumptions of Theorem 1, the frf of the filter* $\Phi_c(B)$ *is given by*

$$\Phi_c(e^{-i\lambda\delta}) = \frac{[e_c f_w m_{2d}]_\delta(\lambda)}{[f_w m_{2d}]_\delta(\lambda)}$$
$$\Phi_c(e^{-i\lambda\delta}) = \frac{i}{1-e^{-i\lambda\delta}} \frac{[e_c f_w m_{2d+1}]_\delta(\lambda)}{[f_w m_{2d+2}]_\delta(\lambda)}$$

for the stock (9) and flow (10) cases, respectively.

3.3 Optimal Signal Extraction

The next theorem describes the frf of the filter that provides the optimal signal extraction estimate $\Psi_c(B)y_\tau$ of the signal $s(t)$ when the process $y(t)$ has a decomposition (8) with $I(d)$ signal component $s(t)$ and stationary noise component $n(t)$. It is assumed that $u(t)$ and $n(t)$ are mean zero and uncorrelated with one another – a common assumption in the signal extraction literature. We do not assume that n or u are stochastically continuous – for example, they can be white noise. Since $s \in \mathcal{C}^d$ with $s \sim I(d)$, in analogy with (11) we have

$$s(t) = \sum_{j=0}^{d-1} \frac{t^j}{j!} s^{(j)}(0) + [I^d u](t). \tag{19}$$

Note that (11) holds for $y(t)$ only if $n \in \mathcal{C}^d$, but in general we do not assume this. Were this true, we could write $D^d y(t) = u(t) + D^d n(t)$, called $w(t)$ say, and this would be a well-defined stationary process. If in addition $w \in \mathcal{C}^0$, then its spectral density would be well-defined:

$$f_w(\lambda) = f_u(\lambda) + \lambda^{2d} f_n(\lambda). \tag{20}$$

Now in the general case where $n \notin \mathcal{C}^d$, we still define the non-integrable function f_w via (20), since it still plays a role in determining the signal extraction filter. Of course f_u and f_n can be well-defined even when $u, n \notin \mathcal{C}^0$, being the Fourier Transforms of the respective autocovariance functions, with transform interpreted in the sense of tempered distributions (Folland, 1995). We only assume that f_u and f_n are bounded. Next, putting (8) and (19) together yields

$$y(t) = \sum_{j=0}^{d-1} \frac{t^j}{j!} s^{(j)}(0) + [I^d u](t) + n(t). \tag{21}$$

Now this form (21) is either stock- or flow-observed, as described by equations (9) or (10). However, it seems to make little sense mathematically to stock-observe a process that is not stochastically continuous – in this case the spectral density of the stock would be infinite, or undefined (see Appendix A.1). Therefore, we suppose that when $y(t)$ given by (21) is stock-observed, $n \in \mathcal{C}^0$; in the flow case, we only require that f_n be bounded.

Next, relations (12) and (13) hold in the stock case with s in place of y, and u in place of w; similarly in the flow case (16) and (17) hold. But these relations don't necessarily hold for y, since we do not assume $y \in \mathcal{C}^d$. Nevertheless, in either the stock or flow case y_τ is reduced to stationarity through d differences, and thus by results in Bell (1984) can be represented in terms of initial values y^* and the well-defined process w_τ. For initial value assumptions, we have

$$y^* \quad \text{is uncorrelated with} \quad \{u_\tau\}_{\tau=-\infty}^{\infty}, \{n_\tau\}_{\tau=-\infty}^{\infty} \tag{22}$$

$$y^* \quad \text{is uncorrelated with} \quad \{u(t)\}_{t\in\mathbb{R}}, \{n(t)\}_{t\in\mathbb{R}}. \tag{23}$$

Note that (22) corresponds to the (canonical) Assumption A of Bell (1984); but the stronger assumption (15) is needed to establish optimality when interpolation is being considered, or when we are estimating from flow observations (though see Section 3.5 below).

We employ similar notation to that of the previous section, so $t = \delta\tau + \delta c$, etc. The optimal signal extraction filter that accomplishes interpolation is denoted by $\Psi_c(B)$, and its frf is given in the theorem below.

Theorem 2 *In the situation described above assume (23), and that f_u/f_w has d continuous derivatives and is positive and bounded. Assume that u and n are uncorrelated with one another, mean zero, and weakly stationary, and that $f_u = \mathcal{F}[R_u]$ and $f_n = \mathcal{F}[R_n]$ (with Fourier Transform interpreted in the sense of tempered distributions) are bounded functions. In the stock case, also assume that $n \in \mathcal{C}^0$. Then at time $t = \delta\tau + \delta c$, the frf of the optimal filter $\Psi_c(B)$ is given by*

$$\Psi_c(e^{-i\lambda\delta}) = \frac{[e_c f_u m_{2d}]_\delta(\lambda)}{[f_w m_{2d}]_\delta(\lambda)}$$
$$\Psi_c(e^{-i\lambda\delta}) = \frac{i}{1-e^{-i\lambda\delta}} \frac{[e_c f_u m_{2d+1}]_\delta(\lambda)}{[f_w m_{2d+2}]_\delta(\lambda)}$$

for the stock (9) and flow (10) cases respectively.

Remark 1 The formulas in Theorem 2 are obtained from those of Theorem 1 by letting $g = f_u/f_w$, which is the frf of the optimal continuous-lag signal extraction filter described in McElroy and Trimbur (2006). Also, we note the following connection with the results of Bell (1984): in the stock case with $c = 0$, we only need to assume (22) – this is found by examining the proof of Theorem 2 – and the given frf is precisely the ratio of pseudo-spectra for signal and data process coming from the stock-discretizations of the continuous-time models. That is, we note that $[f_u m_{2d}]_\delta$ is the spectral density of $\{u_\tau\}$, the differenced stock-sample of the signal process; and $[f_w m_{2d}]_\delta$ is the spectral density of $\{w_\tau\}$, the differenced stock-sample of the data process. Thus Theorem 2 is a natural generalization of classical signal extraction results to handle interpolation, stocks and flows, and generic sampling frequency.

3.4 Computing Filter Coefficients

Now in order to obtain the filter coefficients, we must calculate integrals of the frf, i.e.,

$$\psi_k(c) = \frac{\delta}{2\pi} \int_{-\pi/\delta}^{\pi/\delta} \Psi_c(e^{-i\lambda\delta}) e^{ik\lambda\delta}\, d\lambda.$$

Noting that in general

$$\frac{\delta}{2\pi} \int_{-\pi/\delta}^{\pi/\delta} [f]_\delta(\lambda) e^{ik\lambda\delta}\, d\lambda = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\lambda) e^{ik\lambda\delta}\, d\lambda = \mathcal{F}^{-1}[f](\delta k),$$

we can compute filter coefficients given $[m_j]_\delta(\lambda)$ for various j. General formulas for these functions are provided in Section A.1 of the Appendix; they are always periodic in λ with period $2\pi/\delta$, and so it follows that for the various cases of stock and flow:

$$\psi_k(c) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\lambda) f_w(\lambda) \lambda^{-2d} [f_w m_{2d}]_\delta^{-1}(\lambda) e^{i(k+c)\lambda\delta}\, d\lambda$$

$$\psi_k(c) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{i g(\lambda) f_w(\lambda)}{\lambda^{2d+1}(1 - e^{-i\lambda\delta})} [f_w m_{2d+2}]_\delta^{-1}(\lambda) e^{i(k+c)\lambda\delta}\, d\lambda.$$

(In the case of signal extraction $g = f_u/f_w$, but the case of a generic filter is also covered by these formulas.) In order to compute the ψ_ks, one must first compute the folded quantities $[f_w m_j]_\delta$, and then use the method of residues to derive the inverse Fourier Transform. This can be quite challenging in certain cases, but in Sections 4 and 5 we give a few examples where explicit solutions are possible.

The following result gives an alternative formula for the coefficients $\psi_k(c)$, which is more useful when the frf g of the continuous-lag filter is not available, or is difficult to compute from a given kernel ψ. For this result, we suppose that the interpolation filter of Corollary 1 is available, and that $\phi_k(c)$ is known. Although such filter coefficients are defined for $c \in [0, 1)$, we will here extend them for all $c \in \mathbb{R}$ via the following rule:

$$\phi_k(c + j) = \phi_{k+j}(c) \qquad \forall c \in [0, 1),\ j \quad \text{integer}. \tag{24}$$

This permits a convenient short-hand for the filters, and is justified as follows: the estimate $\Phi_{c+1}(B)y_\tau$, if defined, should correspond to an interpolation estimate of y at time $t = \delta\tau + \delta(c+1)$, and thus should be identical to $\Phi_c(B)y_{\tau+1}$. Identifying coefficients yields $\phi_k(c+1) = \phi_{k+1}(c)$, and by extension we obtain (24).

Proposition 1 *With the interpolant coefficients extended via (24), we have*

$$\psi_k(c) = \delta \int_{-\infty}^{\infty} \psi(\delta z + \delta k)\phi_0(c - z)\, dz.$$

This result holds for either the stock or flow case, by using stock or flow interpolants as appropriate. It is particularly elegant, since it shows that filter discretization occurs as a convolution of the given kernel with the interpolant filter function. This result is useful for the discretization of the Henderson filter discussed below. As an example, with the "nearest-past neighbor" interpolant $\phi_j(c) = 1_{\{j=0\}}$, we obtain

$$\psi_k(c) = \int_{\delta c + \delta k - \delta}^{\delta c + \delta k} \psi(z)\, dz.$$

3.5 Flow Estimates

Finally, we consider the situation where we are interested in a flow estimate of the form $\int_{\delta\tau-\delta+\delta c}^{\delta\tau+\delta c} \hat{y}(v)\, dv$, rather than the stock form $\hat{y}(t)$. (The same type of estimate can be obtained for signals $\hat{s}(t)$; in the following discussion, just use $g = f_u/f_w$.) The results of Theorem 1 are easily adapted, and we obtain the filters (for stock and flow data respectively)

$$\Psi_c(e^{-i\lambda\delta}) = \frac{1 - e^{-i\lambda\delta}}{i} \frac{[g e_c f_w m_{2d+1}]_\delta(\lambda)}{[f_w m_{2d}]_\delta(\lambda)}$$
$$\Psi_c(e^{-i\lambda\delta}) = \frac{[g e_c f_w m_{2d+2}]_\delta(\lambda)}{[f_w m_{2d+2}]_\delta(\lambda)}$$

via integration. The results of Corollary 1 and Theorem 2 can be adapted in the same fashion to generating flow estimates. Of particular interest for applications is the flow-signal estimate from flow data, which has frequency response

$$\Psi_c(e^{-i\lambda\delta}) = \frac{[e_c f_u m_{2d+2}]_\delta(\lambda)}{[f_w m_{2d+2}]_\delta(\lambda)}. \tag{25}$$

If one is interested in estimates of this form and $c = 0$, we can relax condition (23) to (22) in Theorem 2, and we see at once that the stated frf exactly matches those of Bell (1984) after flow-discretizing the continuous-time processes for signal and data.

4 Illustrations of Filter Discretization

In this section we set out some explicit examples, intended as illustrations of Theorem 1; we consider illustrations where the given filter is either a Butterworth filter of order m (abbreviated BW(m)) or is a Henderson trend filter. Extended derivations of formulas are in the Appendix. When in the frequency domain, we use the notation $z = e^{-i\lambda\delta}$ and $\overline{z} = e^{i\lambda\delta}$.

4.1 Butterworth Filters Applied to a Random Walk

We now consider that the data process is a random walk, and we apply a BW(m) filter with $m = 1, 2$. Note that, although BW filters may be given a model-based interpretation – and under this interpretation, their application to a Random Walk has little motivation – they can also be viewed in their original sense, namely as simple nonparametric trend filters giving an approximation to the ideal low-pass filter (Gómez, 2001). As discussed in McElroy and Trimbur (2006) the BW(m) frf is $g(\lambda) = \left(1 + \lambda^{2m}/q\right)^{-1}$. We proceed somewhat generally at first, with general $d > 0$ and $f_w(\lambda) = \sigma^2$. Then for the stock and flow cases of Theorem 1, we have the following frequency response functions:

$$\Psi_c(e^{-i\lambda\delta}) = \frac{[g e_c m_{2d}]_\delta(\lambda)}{[m_{2d}]_\delta(\lambda)}$$
$$\Psi_c(e^{-i\lambda\delta}) = \frac{i}{1 - e^{-i\lambda\delta}} \frac{[g e_c m_{2d+1}]_\delta(\lambda)}{[m_{2d+2}]_\delta(\lambda)}.$$

If g is a rational function in λ, we can use Theorem 4.9b of Henrici (1974) to compute $[g e_c m_j]_\delta(\lambda)$ for various j. The formula is

$$[g e_c m_j]_\delta(\lambda) = -\frac{1}{\delta} \sum_\zeta Res\left(g(\lambda + 2\pi \cdot /\delta)(\lambda + 2\pi \cdot /\delta)^{-j} \frac{2\pi i e^{2\pi c i \cdot}}{e^{2\pi i \cdot} - 1}, \zeta\right) e^{i\delta c\lambda}. \quad (26)$$

As usual the sum is over the poles ζ of the functions $g(\lambda + 2\pi \cdot /\delta)$ and $(\lambda + 2\pi \cdot /\delta)^{-j}$. The latter has a pole of order j at $-\delta\lambda/2\pi$, whereas the former has a pole at $-\delta(\lambda - \omega)/2\pi$, where ω is a pole of g. For the BW(1), these poles are simple and occur at $-\delta(\lambda \pm i\sqrt{q})/2\pi$. For the BW(2), the poles are also simple and occur at $-\delta(\lambda - q^{1/4} e^{ik\pi/4})/2\pi$ with $k = 1, 3, 5, 7$. Let $\chi(x) = 2\pi i e^{2\pi c i x}/(e^{2\pi i x} - 1)$, so that for either $m = 1$ or 2 we have

$$\begin{aligned}
& Res\left(g(\lambda + 2\pi \cdot /\delta)(\lambda + 2\pi \cdot /\delta)^{-j} \chi, -\delta\lambda/2\pi\right) \\
&= \left(\frac{\delta}{2\pi}\right)^j \frac{1}{(j-1)!} \frac{\partial^{j-1}}{\partial x^{j-1}} \left[g(\lambda + 2\pi x/\delta)\chi(x)\right] |_{x = -\delta\lambda/2\pi} \\
&= \left(\frac{\delta}{2\pi}\right)^j \frac{1}{(j-1)!} \sum_{k=0}^{j-1} \binom{j-1}{k} g^{(j-1-k)}(0) (2\pi/\delta)^{j-1-k} \chi^{(k)}(-\delta\lambda/2\pi) \\
& Res\left(g(\lambda + 2\pi \cdot /\delta)(\lambda + 2\pi \cdot /\delta)^{-j} \chi, -\delta(\lambda - \omega)/2\pi\right) \\
&= Res\left(g(\lambda + 2\pi \cdot /\delta), -\delta\lambda/2\pi\right) \omega^{-j} \chi(-\delta(\lambda - \omega)/2\pi).
\end{aligned}$$

Appendix A.3 contains formulas for various higher order derivatives of χ evaluated at $-\delta\lambda/2\pi$. Applying these formulas, we can compute all the desired frfs. We present results for the stock case of Theorem 1, with $m = 1, 2$ and $d = 1$. For BW(1) we have

$$\Psi_c(e^{-i\lambda\delta}) = 1 - c + ce^{i\lambda\delta} + \frac{|1-e^{-i\lambda\delta}|^2}{2\delta\sqrt{q}} \left(\frac{e^{-\delta c\sqrt{q}}}{e^{-i\delta\lambda}e^{-\delta\sqrt{q}} - 1} - \frac{e^{\delta c\sqrt{q}}}{e^{-i\delta\lambda}e^{\delta\sqrt{q}} - 1} \right).$$

It is easily checked that $\Psi_1(e^{-i\lambda\delta}) = \Psi_0(e^{-i\lambda\delta})e^{i\lambda\delta}$. For BW(2) we have

$$\begin{aligned}\Psi_c(e^{-i\lambda\delta}) &= (c(\overline{z}-1)+1) + \frac{|1-z|^2}{2\sqrt{2}q^{1/4}\delta} \\ &\cdot \{ \frac{e^{(i-1)\delta cq^{1/4}/\sqrt{2}}}{(1-i)(ze^{(i-1)\delta cq^{1/4}/\sqrt{2}}-1)} - \frac{e^{(1-i)\delta cq^{1/4}/\sqrt{2}}}{(1-i)(ze^{(1-i)\delta cq^{1/4}/\sqrt{2}}-1)} \\ &+ \frac{e^{(-i-1)\delta cq^{1/4}/\sqrt{2}}}{(i+1)(ze^{(-i-1)\delta cq^{1/4}/\sqrt{2}}-1)} - \frac{e^{(i+1)\delta cq^{1/4}/\sqrt{2}}}{(i+1)(ze^{(i+1)\delta cq^{1/4}/\sqrt{2}}-1)} \}.\end{aligned}$$

These formulas are derived in Appendix A.3, as are the coefficients stated below. For $m = 1$ we have

$$\begin{aligned}\psi_j &= \frac{(1-e^{-\delta\sqrt{q}})^2 e^{\delta\sqrt{q}(c+j+1)}}{2\delta\sqrt{q}} \qquad j \leq -2 \\ \psi_{-1} &= c - \frac{\sinh \delta c\sqrt{q}}{\delta\sqrt{q}} + \frac{(1-e^{-\delta\sqrt{q}})^2 e^{\delta\sqrt{q}c}}{2\delta\sqrt{q}} \\ \psi_0 &= 1 - c + \frac{\sinh \delta\sqrt{q}(c-1)}{\delta\sqrt{q}} + \frac{(1-e^{-\delta\sqrt{q}})^2 e^{-\delta\sqrt{q}(c-1)}}{2\delta\sqrt{q}} \\ \psi_j &= \frac{(1-e^{-\delta\sqrt{q}})^2 e^{-\delta\sqrt{q}(c+j-1)}}{2\delta\sqrt{q}} \qquad j \geq 1.\end{aligned}$$

For $m = 2$ we have

$$\begin{aligned}\psi_j &= \frac{q^{1/4}}{\sqrt{2}}\, Re\left[(1+i)^{-1}(1-e^{(i-1)\delta q^{1/4}/\sqrt{2}})(1-e^{(1-i)\delta q^{1/4}/\sqrt{2}})e^{(1-i)\delta(c+j)q^{1/4}/\sqrt{2}}\right] \qquad j \leq -2 \\ \psi_{-1} &= c + \frac{1}{2\sqrt{2}q^{1/4}\delta} Re\left[(1+i)^{-1}\left(e^{-(1+i)c\delta q^{1/4}/\sqrt{2}} - e^{(1+i)c\delta q^{1/4}/\sqrt{2}}\right)\right] \\ &- \frac{1}{\sqrt{2}q^{1/4}\delta} Re\left[(1+i)^{-1}e^{(1+i)(c-1)\delta q^{1/4}/\sqrt{2}}(1-e^{(i+1)\delta q^{1/4}/\sqrt{2}})(1-e^{-(1+i)\delta q^{1/4}/\sqrt{2}})\right] \\ \psi_0 &= 1 - c + \frac{1}{2\sqrt{2}q^{1/4}\delta} Re\left[(1+i)^{-1}\left(e^{(1+i)(c-1)\delta q^{1/4}/\sqrt{2}} - e^{-(1+i)(c-1)\delta q^{1/4}/\sqrt{2}}\right)\right] \\ &- \frac{1}{\sqrt{2}q^{1/4}\delta} Re\left[(1+i)^{-1}e^{-(1+i)c\delta q^{1/4}/\sqrt{2}}(1-e^{(i+1)\delta q^{1/4}/\sqrt{2}})(1-e^{-(1+i)\delta q^{1/4}/\sqrt{2}})\right] \\ \psi_j &= \frac{q^{1/4}}{\sqrt{2}}\, Re\left[(1+i)^{-1}(1-e^{(i-1)\delta q^{1/4}/\sqrt{2}})(1-e^{(1-i)\delta q^{1/4}/\sqrt{2}})e^{(i-1)\delta(c+j)q^{1/4}/\sqrt{2}}\right] \qquad j \geq 1.\end{aligned}$$

4.2 Henderson Filters

The Henderson trend has a long history in actuarial science and the $X11$ seasonal adjustment procedure (Findley, Monsell, Bell, Otto, Chen (1998)); a continuous time version of the Henderson kernel has been discovered by Dagum and Bianconcini (2006) using the methods of reproducing kernel Hilbert spaces. According to that work, the third-order Henderson kernel ψ is given by

$$\psi(x) = \frac{298792800 - 16144331127x^2 + 26959307942x^4 - 17999679943x^6 + 4196775128x^8}{1786542091} 1_{[-1,1]}(x),$$

from which it is easily verified that $\Psi(L)$ passes cubic polynomials. The frequency response of the Henderson is tedious to write down, and has non-simple poles at the origin. It is thus easier to compute the discretization filter coefficients $\psi_k(c)$ using Proposition 1. We only need to know the interpolation filter coefficients, and then convolve with ψ as indicated.

For example, suppose that the data process is a simple continuous-time random walk, so that $d = 1$ and $f_w \propto 1$; then for stock observations we have

$$\Phi_c(e^{-i\lambda\delta}) = \frac{[e_c m_2]_\delta(\lambda)}{[m_2]_\delta(\lambda)} = (1-c) + ce^{i\lambda\delta}.$$

In other words, $\phi_0(c) = 1 - c$ and $\phi_{-1}(c) = c$ for $c \in [0, 1)$. A short calculation (using the fact that ψ is supported on $[-1, 1]$) yields

$$\psi_k(c) = \delta \int_0^1 \psi(\delta z + \delta c + \delta(k-1))z \, dz - \delta \int_{-1}^0 \psi(\delta z + \delta c + \delta(k+1))z \, dz.$$

From here, the exact determination of the bounds of integration will depend on δ and k. That is, z is constrained to $[(1-k-c)/\delta, (2-k-c)/\delta] \cap [0, 1]$; these types of calculations can easily be done numerically, but are not very simple to express algebraically.

5 Illustrations of Signal Extraction Discretization

In this section we consider some applications of Theorem 2 to models of great interest to econometricians. Specifically, we address the calculation of signal extraction filters for the Local Level Model (LLM), Smooth Trend Model (STM), and Band-Pass Model (BPM), and also consider a Turning Point (TP) filter for the STM.

5.1 The Local Level Model

The LLM – introduced in Harvey (1989) and further described in McElroy and Trimbur (2006) – has the following continuous time formulation:

$$Ds(t) = u(t) \sim WN(q\sigma^2)$$
$$n(t) \sim WN(\sigma^2)$$

where $WN(b)$ denotes continuous time white noise with spectral density equal to the constant b. Because of the presence of continuous time white noise $n(t)$ in the underlying process $y(t) =$

$s(t)+n(t)$, it only makes sense to consider flow observations. The flow case of Theorem 2 yields

$$\Psi_c(e^{-i\lambda\delta}) = \frac{i}{1-e^{-i\lambda\delta}} \frac{[e_c m_3]_\delta(\lambda)}{[m_4]_\delta(\lambda)+[m_2]_\delta(\lambda)/q}$$
$$= \frac{3\delta\left((e^{i\lambda\delta}+1) - c^2|1-e^{-i\lambda\delta}|^2 e^{i\lambda\delta} - 2c(1-e^{i\lambda\delta})\right)}{\delta^2(2\cos\lambda\delta+4)+6|1-e^{-i\lambda\delta}|^2/q}.$$

If we are interested in a flow-signal, then (25) yields

$$\Psi_c(e^{-i\lambda\delta}) = \frac{[e_c m_4]_\delta(\lambda)}{[m_4]_\delta(\lambda)+[m_2]_\delta(\lambda)/q}$$
$$= \frac{\delta^2\left((2\cos\lambda\delta+4) + c^3(1-e^{-i\lambda\delta})(1-e^{i\lambda\delta})^2 - 3c^2|1-e^{-i\lambda\delta}|^2 + 3c(e^{i\lambda\delta}-e^{-i\lambda\delta})\right)}{\delta^2(2\cos\lambda\delta+4)+6|1-e^{-i\lambda\delta}|^2/q}.$$

The coefficients depend upon the roots of a certain polynomial, which in turn depend upon the sign of $\delta^2 - 6/q$. We assume this is negative, since generally both δ and q are small (see McElroy and Trimbur, 2006); some of the following derivations will be changed if $\delta^2 - 6/q \geq 0$. Define η_1 and η_2 by

$$\eta_1 = \frac{-(2\delta^2+6/q)+\sqrt{3\delta^2(\delta^2+12/q)}}{\delta^2-6/q}$$
$$\eta_2 = \frac{-(2\delta^2+6/q)-\sqrt{3\delta^2(\delta^2+12/q)}}{\delta^2-6/q}.$$

Now $0 < \eta_1 < 1$ but $\eta_2 > 1$. Then the coefficients in the case of a flow-signal are:

$$\psi_j = \frac{\delta^2\left(c^3(1-\eta_1)^3 + 3c^2(1-\eta_1)^2\eta_1 + 3c(1-\eta_1^2)\eta_1 + (\eta_1^3+4\eta_1^2+\eta_1)\right)\eta_1^{-j-2}}{(\delta^2-6/q)(\eta_1-\eta_2)} \qquad j \leq -2$$

$$\psi_{-1} = \frac{\delta^2 c^3}{\delta^2-6/q} + \frac{\delta^2\left(c^3(1-\eta_1)^3 + 3c^2(1-\eta_1)^2\eta_1 + 3c(1-\eta_1^2)\eta_1 + (\eta_1^3+4\eta_1^2+\eta_1)\right)\eta_1^{-1}}{(\delta^2-6/q)(\eta_1-\eta_2)}$$

$$\psi_0 = \frac{\delta^2(1-c)^3}{\delta^2-6/q} + \frac{\delta^2\left(-c^3(1-\eta_1)^3 + 3c^2(1-\eta_1)^2 + 3c(\eta_1^2-1) + (\eta_1^2+4\eta_1+1)\right)\eta_1^{-1}}{(\delta^2-6/q)(\eta_1-\eta_2)}$$

$$\psi_j = \frac{\delta^2(1-c)^3}{\delta^2-6/q} + \frac{\delta^2\left(-c^3(1-\eta_1)^3 + 3c^2(1-\eta_1)^2 + 3c(\eta_1^2-1)\right)\eta_1^{j-1} + (\eta_1^2+4\eta_1+1)}{(\delta^2-6/q)(\eta_1-\eta_2)} \qquad j \geq 1$$

In the case of a stock-signal they are:

$$\psi_j = \frac{3\delta\left(c^2(1-\eta_1)^2 + 2c(1-\eta_1)\eta_1 + (\eta_1^2+\eta_1)\right)\eta_1^{-j-2}}{(\delta^2-6/q)(\eta_1-\eta_2)} \qquad j \leq -2$$

$$\psi_{-1} = \frac{3\delta c^2}{\delta^2-6/q} + \frac{3\delta\left(c^2(1-\eta_1)^2 + 2c(1-\eta_1)\eta_1 + (\eta_1^2+\eta_1)\right)\eta_1^{-1}}{(\delta^2-6/q)(\eta_1-\eta_2)}$$

$$\psi_j = \frac{3\delta\left(c^2(1-\eta_1)^2 - 2c(1-\eta_1) + (1+\eta_1)\right)\eta_1^{j}}{(\delta^2-6/q)(\eta_1-\eta_2)} \qquad j \geq 0$$

5.2 The Smooth Trend Model

The smooth trend model – see McElroy and Trimbur (2006) – has the following continuous time formulation:

$$D^2 s(t) = u(t) \sim WN(q\sigma^2)$$
$$n(t) \sim WN(\sigma^2).$$

As before, we must consider only flow sampling; the flow case of Theorem 2 yields

$$\begin{aligned}\Psi_c(e^{-i\lambda\delta}) &= \frac{i}{1-e^{-i\lambda\delta}} \frac{[e_c m_5]_\delta(\lambda)}{[m_6]_\delta(\lambda) + [m_2]_\delta(\lambda)/q} \\ &= 5\delta^3(c^4\overline{z}|1-z|^4 + 4c^3(1-\overline{z})|1-z|^2 + 6c^2(1-\overline{z})(z-\overline{z}) \\ &- 8c(1+z)(1-\overline{z}^2) + 4c(z+\overline{z})(1-\overline{z}) + 12\overline{z}(1+z)) \\ &\cdot \left(\delta^4(66+26z+26\overline{z}+z^2+\overline{z}^2) + 120|1-z|^4/q\right)^{-1}.\end{aligned}$$

For the case of a flow-signal (25) we have

$$\begin{aligned}\Psi_c(e^{-i\lambda\delta}) &= \frac{[e_c m_6]_\delta(\lambda)}{[m_6]_\delta(\lambda) + [m_2]_\delta(\lambda)/q} \\ &= \delta^4(c^5\frac{|1-z|^6}{z-1} + 5c^4|1-z|^4 + 10c^3(z-\overline{z})|1-z|^2 + 20c^2(z-\overline{z})^2 \\ &+ 10c^2(z+\overline{z})|1-z|^2 - 30c(z-\overline{z})(1+z)(1+\overline{z}) + 30c(z^2-\overline{z}^2) + 5c(z-\overline{z})|1-z|^2 \\ &+ \left(66+26z+26\overline{z}+z^2+\overline{z}^2\right)) \\ &\cdot \left(\delta^4(66+26z+26\overline{z}+z^2+\overline{z}^2) + 120|1-z|^4/q\right)^{-1}.\end{aligned}$$

The coefficient formulas are quite complicated. As in the LLM, we make the assumption that δ and q are small, such that $\delta^4 < 120/q$. Define the quantities

$$\alpha = \frac{26\delta^4 + 480/q + \sqrt{60(7\delta^8 + 960\delta^4/q)}}{2(\delta^4 - 120/q)}$$
$$\beta = \frac{26\delta^4 + 480/q - \sqrt{60(7\delta^8 + 960\delta^4/q)}}{2(\delta^4 - 120/q)}.$$

Then the following quantities are defined in terms of α and β:

$$\nu_2 = \frac{-\alpha - \sqrt{\alpha^2 - 4}}{2}, \qquad \nu_3 = \frac{-\beta + \sqrt{\beta^2 - 4}}{2}, \qquad \nu_4 = \frac{-\beta - \sqrt{\beta^2 - 4}}{2}.$$

It follows that $0 < \nu_2 < 1$ but ν_3, ν_4 are complex conjugates with unit modulus. Also, we let $\Theta_c(x)$ be the numerator polynomial in the expression for the filter frequency response function as a rational function, and similarly $\Phi_c(x)$ is the numerator polynomial of $\Psi_c(x^{-1})$ when expressed as a rational function. For the flow-signal case (25) these functions are given explicitly by

$$\begin{aligned}
\Theta_c(x) &= \delta^4 (c^5 (1-x)^5 + 5c^4 (1-x)^4 x + 10c^3 (1-x^2)(1-x)^2 + 20c^2 x (1-x^2)^2 \\
&\quad - 10c^2 (1-x)^2 (1+x^2) + 30cx(1-x^2)(1+x)^2 - 30cx(1-x^4) \\
&\quad + 5cx(1-x)^2 (1-x^2) + x\left(x^4 + 26x^3 + 66x^2 + 26x + 1\right)) \\
\Phi_c(x) &= \delta^4 (c^5 (1-x)^5 + 5c^4 (1-x)^4 - 10c^3 (1-x^2)(1-x)^2 + 20c^2 (1-x^2)^2 \\
&\quad - 10c^2 (1-x)^2 (1+x^2) - 30c(1-x^2)(1+x)^2 + 30c(1-x^4) \\
&\quad - 5c(1-x)^2 (1-x^2) + \left(x^4 + 26x^3 + 66x^2 + 26x + 1\right)).
\end{aligned}$$

Using these formulas, the coefficients in the flow-signal case are given by

$$\begin{aligned}
\psi_j &= \frac{\Theta_c(\nu_2)\nu_2^{-j-2}}{(\delta^4 - 120/q)(\nu_2 - \nu_1)(\nu_2^2 + \beta\nu_2 + 1)} + \frac{\Theta_c(\nu_3)\nu_3^{-j-2}}{2(\delta^4 - 120/q)(\nu_3 - \nu_4)(\nu_3^2 + \alpha\nu_3 + 1)} \\
&\quad + \frac{\Theta_c(\nu_4)\nu_4^{-j-2}}{2(\delta^4 - 120/q)(\nu_4 - \nu_3)(\nu_4^2 + \alpha\nu_4 + 1)} \qquad j \leq -2 \\
\psi_{-1} &= \frac{\Theta_c(0)}{\delta^4 - 120/q} + \frac{\Theta_c(\nu_2)\nu_2^{-1}}{(\delta^4 - 120/q)(\nu_2 - \nu_1)(\nu_2^2 + \beta\nu_2 + 1)} \\
&\quad + \frac{\Theta_c(\nu_3)\nu_3^{-1}}{2(\delta^4 - 120/q)(\nu_3 - \nu_4)(\nu_3^2 + \alpha\nu_3 + 1)} + \frac{\Theta_c(\nu_4)\nu_4^{-1}}{2(\delta^4 - 120/q)(\nu_4 - \nu_3)(\nu_4^2 + \alpha\nu_4 + 1)} \\
\psi_0 &= \frac{\Phi_c(0)}{\delta^4 - 120/q} + \frac{\Phi_c(\nu_2)\nu_2^{-1}}{(\delta^4 - 120/q)(\nu_2 - \nu_1)(\nu_2^2 + \beta\nu_2 + 1)} \\
&\quad + \frac{\Phi_c(\nu_3)\nu_3^{-1}}{2(\delta^4 - 120/q)(\nu_3 - \nu_4)(\nu_3^2 + \alpha\nu_3 + 1)} + \frac{\Phi_c(\nu_4)\nu_4^{-1}}{2(\delta^4 - 120/q)(\nu_4 - \nu_3)(\nu_4^2 + \alpha\nu_4 + 1)} \\
\psi_j &= \frac{\Phi_c(\nu_2)\nu_2^{j-1}}{(\delta^4 - 120/q)(\nu_2 - \nu_1)(\nu_2^2 + \beta\nu_2 + 1)} + \frac{\Phi_c(\nu_3)\nu_3^{j-1}}{2(\delta^4 - 120/q)(\nu_3 - \nu_4)(\nu_3^2 + \alpha\nu_3 + 1)} \\
&\quad + \frac{\Phi_c(\nu_4)\nu_4^{j-1}}{2(\delta^4 - 120/q)(\nu_4 - \nu_3)(\nu_4^2 + \alpha\nu_4 + 1)} \qquad j \geq 1.
\end{aligned}$$

Now for stock-signal case the roots ν_j and α, β happen to be the same, but Θ_c and Φ_c are different:

$$\begin{aligned}
\Theta_c(x) &= 5\delta^3 (c^4 (1-x)^4 + 4c^3 (1-x)^3 x + 6c^2 x(1+x)(1-x)^2 + 8cx(1+x)^2 (1-x) \\
&\quad - 4cx(1-x)(1+x^2) + 12x^2 (1+x)) \\
\Phi_c(x) &= 5\delta^3 (c^4 x(1-x)^4 - 4c^3 x(1-x)^3 + 6c^2 x(1+x)(1-x)^2 - 8cx(1-x)(1+x)^2 \\
&\quad + 4cx(1-x)(1+x^2) + 12x^2 (1+x)).
\end{aligned}$$

Then the coefficients in the stock-signal case are given by

$$\psi_j = \frac{\Theta_c(\nu_2)\nu_2^{-j-2}}{(\delta^4 - 120/q)(\nu_2 - \nu_1)(\nu_2^2 + \beta\nu_2 + 1)} + \frac{\Theta_c(\nu_3)\nu_3^{-j-2}}{2(\delta^4 - 120/q)(\nu_3 - \nu_4)(\nu_3^2 + \alpha\nu_3 + 1)}$$
$$+ \frac{\Theta_c(\nu_4)\nu_4^{-j-2}}{2(\delta^4 - 120/q)(\nu_4 - \nu_3)(\nu_4^2 + \alpha\nu_4 + 1)} \qquad j \leq -2$$
$$\psi_{-1} = \frac{5c^4\delta^3}{\delta^4 - 120/q} + \frac{\Theta_c(\nu_2)\nu_2^{-1}}{(\delta^4 - 120/q)(\nu_2 - \nu_1)(\nu_2^2 + \beta\nu_2 + 1)}$$
$$+ \frac{\Theta_c(\nu_3)\nu_3^{-1}}{2(\delta^4 - 120/q)(\nu_3 - \nu_4)(\nu_3^2 + \alpha\nu_3 + 1)} + \frac{\Theta_c(\nu_4)\nu_4^{-1}}{2(\delta^4 - 120/q)(\nu_4 - \nu_3)(\nu_4^2 + \alpha\nu_4 + 1)}$$
$$\psi_j = \frac{\Phi_c(\nu_2)\nu_2^{j-1}}{(\delta^4 - 120/q)(\nu_2 - \nu_1)(\nu_2^2 + \beta\nu_2 + 1)} + \frac{\Phi_c(\nu_3)\nu_3^{j-1}}{2(\delta^4 - 120/q)(\nu_3 - \nu_4)(\nu_3^2 + \alpha\nu_3 + 1)}$$
$$+ \frac{\Phi_c(\nu_4)\nu_4^{j-1}}{2(\delta^4 - 120/q)(\nu_4 - \nu_3)(\nu_4^2 + \alpha\nu_4 + 1)} \qquad j \geq 0.$$

All of these formulas are derived in the Appendix.

5.3 Turning Point Filters

If a given continuous time filter $\Psi(L)$ produces a smooth trend estimate, then the filter with frequency response $i\lambda\Psi(e^{-i\lambda})$ estimates the velocity of that trend (see McElroy and Trimbur (2006) for a discussion). When the velocity changes sign, this indicates a turning point in the estimated trend. Here we apply this concept to the flow case of the STM. Using Theorem 2, we see that the frequency response function of the discretized TP filter is

$$\Psi_c(e^{-i\lambda\delta}) = \frac{i[e_c m_5]_\delta(\lambda)}{[m_6]_\delta(\lambda) + [m_2]_\delta(\lambda)/q},$$

which is just $1 - e^{-i\lambda\delta}$ times the frf for the stock-signal case of the STM. So using the notation $\tilde{}$ to denote the TP functions, we have $\tilde{\Psi}_c(x) = (1-x)\Psi_c(x)$, and hence $\tilde{\Theta}_c(x) = (1-x)\Theta_c(x)$. Likewise, $\tilde{\Psi}_c(1/x) = (x-1)\Psi(1/x)/x$, which implies that $\tilde{\Phi}_c(x) = (x-1)\Phi_c(x)$, but an extra x factor shows up in the denominator. The result is the following coefficients (where the constants have the same definition as in the discussion of the STM):

$$\psi_j = \frac{\Theta_c(\nu_2)(1-\nu_2)\nu_2^{-j-2}}{(\delta^4 - 120/q)(\nu_2-\nu_1)(\nu_2^2+\beta\nu_2+1)} + \frac{\Theta_c(\nu_3)(1-\nu_3)\nu_3^{-j-2}}{2(\delta^4 - 120/q)(\nu_3-\nu_4)(\nu_3^2+\alpha\nu_3+1)}$$

$$+ \frac{\Theta_c(\nu_4)(1-\nu_4)\nu_4^{-j-2}}{2(\delta^4 - 120/q)(\nu_4-\nu_3)(\nu_4^2+\alpha\nu_4+1)} \qquad j \leq -2$$

$$\psi_{-1} = \frac{5c^4\delta^3}{\delta^4 - 120/q} + \frac{\Theta_c(\nu_2)(1-\nu_2)\nu_2^{-1}}{(\delta^4 - 120/q)(\nu_2-\nu_1)(\nu_2^2+\beta\nu_2+1)}$$

$$+ \frac{\Theta_c(\nu_3)(1-\nu_3)\nu_3^{-1}}{2(\delta^4 - 120/q)(\nu_3-\nu_4)(\nu_3^2+\alpha\nu_3+1)} + \frac{\Theta_c(\nu_4)(1-\nu_4)\nu_4^{-1}}{2(\delta^4 - 120/q)(\nu_4-\nu_3)(\nu_4^2+\alpha\nu_4+1)}$$

$$\psi_0 = \frac{5\delta^3(1-(1-c)^4)}{\delta^4 - 120/q} + \frac{\Phi_c(\nu_2)(\nu_2-1)\nu_2^{-2}}{(\delta^4 - 120/q)(\nu_2-\nu_1)(\nu_2^2+\beta\nu_2+1)}$$

$$+ \frac{\Phi_c(\nu_3)(\nu_3-1)\nu_3^{-2}}{2(\delta^4 - 120/q)(\nu_3-\nu_4)(\nu_3^2+\alpha\nu_3+1)} + \frac{\Phi_c(\nu_4)(\nu_4-1)\nu_4^{-2}}{2(\delta^4 - 120/q)(\nu_4-\nu_3)(\nu_4^2+\alpha\nu_4+1)}$$

$$\psi_j = \frac{\Theta_c(\nu_2)(\nu_2-1)\nu_2^{j-2}}{(\delta^4 - 120/q)(\nu_2-\nu_1)(\nu_2^2+\beta\nu_2+1)} + \frac{\Theta_c(\nu_3)(\nu_3-1)\nu_3^{j-2}}{2(\delta^4 - 120/q)(\nu_3-\nu_4)(\nu_3^2+\alpha\nu_3+1)}$$

$$+ \frac{\Theta_c(\nu_4)(\nu_4-1)\nu_4^{j-2}}{2(\delta^4 - 120/q)(\nu_4-\nu_3)(\nu_4^2+\alpha\nu_4+1)} \qquad j \geq 1$$

5.4 Band Pass and Low Pass Model Filters

The trend-cycle-irregular model in continuous-time is discussed in McElroy and Trimbur (2006), where continuous-lag Band Pass (BP) and Low Pass (LP) filters are derived. In this model there is a trend component m, cycle component c, and irregular component i, which have the following CARIMA models:

$$Dm(t) = u(t) \sim WN(q\sigma^2)$$
$$i(t) \sim WN(\sigma^2)$$
$$(D+\rho+i\lambda_c)(D+\rho-i\lambda_c)c(t) \sim WN(r\sigma^2).$$

We must consider only flow sampling because of the continuous white noise irregular. Here q and r are positive signal-noise ratios, and ρ indicates the strength of the cycle peak in f_c. Values of ρ closer to zero indicate a higher peak, with the location of the maximum occurring close to $\pm\lambda_c$. The component pesudo-spectra are

$$f_m(\lambda) = \frac{q\sigma^2}{\lambda^2}$$
$$f_i(\lambda) = \sigma^2$$
$$f_c(\lambda) = \frac{r\sigma^2}{(\rho^2+(\lambda-\lambda_c)^2)(\rho^2+(\lambda+\lambda_c)^2)}.$$

Using the calculus of residues, we find that

$$R_c(x) = \frac{r\sigma^2 e^{-\rho|x|}}{4\rho\lambda_c(\lambda_c^2+\rho^2)} Re\left(e^{i\lambda_c|x|}(\lambda_c - \rho i)\right).$$

The BP filter that we consider is the signal extraction filter for a process with a cyclical signal and a trend-irregular noise. So we apply Theorem 2 with the role of signal and noise swapped (so that the signal is stationary and the noise is nonstationary), which is only a change in nomenclature. The LP filter corresponds to a trend signal with cycle-irregular noise. Focusing on the case of a flow-signal (25), the BP and LP frequency response functions respectively are given by:

$$\Psi_c(e^{-i\lambda\delta}) = \frac{[e_c f_c m_4]_\delta(\lambda)}{[qm_4 + m_2 + f_c m_4]_\delta(\lambda)}$$
$$\Psi_c(e^{-i\lambda\delta}) = \frac{[e_c q m_4]_\delta(\lambda)}{[qm_4 + m_2 + f_c m_4]_\delta(\lambda)}.$$

The explicit expression is complicated. Formulas in Appendix A.1 can be used to compute $[e_c m_4]$ and so forth; $[e_c f_c m_4]_\delta$ is given below:

$$\begin{aligned}
[e_c f_c m_4]_\delta = & \frac{r\sigma^2}{6(\rho^2+\lambda_c^2)^2}\Big(\frac{12\delta(\lambda_c^2-\rho^2)}{(\rho^2+\lambda_c^2)^2}\left(\frac{c}{z-1}+\frac{1}{|1-z|^2}\right) \\
& - \delta^3\left(\frac{c^3}{z-1}+\frac{3c^2}{|1-z|^2}+\frac{3c(z-\overline{z})}{|1-z|^4}+\frac{2(z-\overline{z})^2}{|1-z|^6}+\frac{z+\overline{z}}{|1-z|^4}\right)\Big) \\
& - \frac{r\sigma^2}{8\lambda_c\rho}\Big((\lambda_c+i\rho)^{-5}\frac{z\big(e^{i\delta\lambda_c(c-1)}e^{-\delta\rho(c-1)} - e^{-i\delta\lambda_c(c-1)}e^{\delta\rho(c-1)}\big) - \big(e^{i\delta\lambda_c c}e^{-\delta\rho c} - e^{-i\delta\lambda_c c}e^{\delta\rho c}\big)}{\big(z-e^{\delta(\rho-i\lambda_c)}\big)\big(z-e^{-\delta(\rho-i\lambda_c)}\big)} \\
& - (\lambda_c - i\rho)^{-5}\frac{z\big(e^{i\delta\lambda_c(c-1)}e^{\delta\rho(c-1)} - e^{-i\delta\lambda_c(c-1)}e^{-\delta\rho(c-1)}\big) - \big(e^{i\delta\lambda_c c}e^{\delta\rho c} - e^{-i\delta\lambda_c c}e^{-\delta\rho c}\big)}{\big(z-e^{-\delta(\rho+i\lambda_c)}\big)\big(z-e^{\delta(\rho+i\lambda_c)}\big)}\Big).
\end{aligned}$$

This is derived in Appendix A.4. Now for a stock-signal we have

$$\Psi_c(e^{-i\lambda\delta}) = \frac{i[e_c f_c m_3]_\delta(\lambda)}{(1-z)[qm_4 + m_2 + f_c m_4]_\delta(\lambda)}$$
$$\Psi_c(e^{-i\lambda\delta}) = \frac{i[e_c q m_3]_\delta(\lambda)}{(1-z)[qm_4 + m_2 + f_c m_4]_\delta(\lambda)}.$$

The formula for $[e_c f_c m_3]_\delta$ is given below:

$$\begin{aligned}
[e_c f_c m_3]_\delta = & \frac{i\delta^2 r\sigma^2}{2(\rho^2+\lambda_c^2)^2}\left[\frac{c^2}{z-1}+\frac{2c}{|1-z|^2}+\frac{(z-\overline{z})}{|1-z|^4}\right] - \frac{2ir\sigma^2(\lambda_c^2-\rho^2)}{(\rho^2+\lambda_c^2)^4(z-1)} \\
& - \frac{r\sigma^2}{8\lambda_c\rho}\Big((\lambda_c+i\rho)^{-4}\frac{z\big(e^{i\delta\lambda_c(c-1)}e^{-\delta\rho(c-1)} + e^{-i\delta\lambda_c(c-1)}e^{\delta\rho(c-1)}\big) - \big(e^{i\delta\lambda_c c}e^{-\delta\rho c} + e^{-i\delta\lambda_c c}e^{\delta\rho c}\big)}{\big(z-e^{\delta(\rho-i\lambda_c)}\big)\big(z-e^{-\delta(\rho-i\lambda_c)}\big)} \\
& - (\lambda_c - i\rho)^{-4}\frac{z\big(e^{i\delta\lambda_c(c-1)}e^{\delta\rho(c-1)} + e^{-i\delta\lambda_c(c-1)}e^{-\delta\rho(c-1)}\big) - \big(e^{i\delta\lambda_c c}e^{\delta\rho c} + e^{-i\delta\lambda_c c}e^{-\delta\rho c}\big)}{\big(z-e^{-\delta(\rho+i\lambda_c)}\big)\big(z-e^{\delta(\rho+i\lambda_c)}\big)}\Big).
\end{aligned}$$

So the frfs for the BP and LP filters are fairly complicated, and at present the coefficients must be determined by numerical integration of these functions.

Appendix

The material in this Appendix is organized in correspondence with the order of topics in the main text. Hence we first discuss folds, then give the proofs of Theorems 1 and 2, then give details on frf and coefficient calculations for the applications in Sections 4 and 5.

A.1 Computing Folds

Much is known about folds in the engineering literature: see Solo (1983), and the literature on Laplace/Z Transform pairs in control engineering (Kuo, 1963 and Jury, 1973). The innovation here is that we incorporate the sampling frequency δ and the interpolant c into the discussion, and make distinctions between stock and flow sampling. Suppose that a mean zero stationary continuous time process $x(t)$ has orthogonal increments representation (Brockwell and Davis, 1991)

$$x(t) = \int_{-\infty}^{\infty} e^{it\lambda} d\mathbb{Z}(\lambda)$$

where $\mathbb{E}[d\mathbb{Z}(\lambda)\overline{d\mathbb{Z}}(\lambda)] = f(\lambda)d\lambda/2\pi$. For example, if x is stochastically continuous, then Theorem 4.11.1 of Priestley (1981) guarantees the existence of such a representation. Then the autocovariance function is

$$R(h) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ih\lambda} f(\lambda)\, d\lambda,$$

which is defined for all real numbers h. If we stock observe x via the equation $x_\tau = x(\delta\tau)$, where τ is integer, then

$$x_\tau = \sum_{l=-\infty}^{\infty} \int_{(2l-1)\pi/\delta}^{(2l+1)\pi/\delta} e^{i\tau\lambda\delta} d\mathbb{Z}(\lambda) = \int_{-\pi/\delta}^{\pi/\delta} e^{i\tau\lambda\delta} \sum_{l=-\infty}^{\infty} d\mathbb{Z}(\lambda + 2\pi l/\delta)$$

by change of variable. Likewise, the autocovariances are only considered at lags δh where h is now integer, and we can write

$$R_h = R(\delta h) = \frac{1}{2\pi} \int_{-\pi/\delta}^{\pi/\delta} e^{ih\lambda\delta} \sum_{l=-\infty}^{\infty} f(\lambda + 2\pi l/\delta)\, d\lambda.$$

Hence the spectral density of x_τ is given by

$$\sum_{h=-\infty}^{\infty} R_h e^{-ih\lambda\delta} = \frac{1}{\delta} \sum_{l=-\infty}^{\infty} f(\lambda + 2\pi l/\delta) \qquad \lambda \in [-\pi/\delta, \pi/\delta];$$

this latter formula is denoted by the notation $[f]_\delta(\lambda)$. It is called the "fold" of f, since it is obtained – graphically speaking – by chopping f up into contiguous domains of size $[-\pi/\delta, \pi/\delta]$ and overlaying the corresponding functions (see Koopmans, 1974 for a picture). It is obvious from the above equation that the fold is periodic if we view $[f]_\delta$ as a function on the real line. In order to be well-defined, it is necessary that the tails of f decay faster than $1/\lambda$; methods for computing folds from a given f are discussed below. Now from the expression for $[f]_\delta(\lambda)$, we see that the highest frequency that can be observed is π/δ, and at any λ the value of the spectrum $[f]_\delta(\lambda)$ is confounded by the aliases $f(\lambda + 2\pi l/\delta)$ for all integers l. The frequency $2\pi/\delta$ is referred to as the Nyquist folding frequency (Blackman and Tukey, 1958).

For a flow observation of $x(t)$, we obtain $x_\tau = \int_{-\infty}^{\infty} e^{i\delta\tau\lambda}(1-e^{-i\lambda\delta})(i\lambda)^{-1} d\mathbb{Z}(\lambda)$, with autocovariance function

$$R_h = R(\delta h) = \frac{1}{2\pi}\int_{-\pi/\delta}^{\pi/\delta} e^{ih\lambda\delta}|1-e^{-i\lambda\delta}|^2 \sum_{l=-\infty}^{\infty} \frac{f(\lambda+2\pi l/\delta)}{(\lambda+2\pi l/\delta)^2}\, d\lambda.$$

The corresponding spectral density is $[fm_2]_\delta(\lambda)|1-e^{-i\lambda\delta}|^2$.

In several places in Section 4, we need to calculate $[e_c m_j]_\delta(\lambda)$ for various $j \geq 2$. When $c = 0$ and $\lambda \neq 0$, we can use Theorem 4.9a of Henrici (1974) as follows:

$$[m_j]_\delta(\lambda) = \frac{1}{\delta}\sum_{h=-\infty}^{\infty}(\lambda+2\pi h/\delta)^{-j} = -\frac{1}{\delta}\sum_{\zeta} Res((\lambda+2\pi\cdot/\delta)^{-j}a,\zeta),$$

where the sum is over all the poles ζ of the function $(\lambda+2\pi\cdot/\delta)^{-j}$, and Res denotes the residue. The function a is defined by $a(x) = \pi\cot\pi x$. This formula does not cover the case that $\lambda = 0$, since the fold explodes to infinity at that frequency. The above application of Theorem 4.9a of Henrici (1974) can be made to compute the fold of any function g, so long as it is a rational function of λ with a zero of order at least two at infinity. In our case, we have a pole of order j at $-\delta\lambda/2\pi$, so the fold is given by

$$[m_j]_\delta(\lambda) = -\frac{1}{\delta(j-1)!}\left(\frac{\delta}{2\pi}\right)^j a^{(j-1)}\left(\frac{-\delta\lambda}{2\pi}\right).$$

We pursue a method of calculation for this below, which is easier than a brute force approach. In the case that $c \neq 0$, we instead apply Theorem 4.9b of Henrici (1974) (which also covers $c = 0$, but typically involves more computations):

$$\begin{aligned}[e_c m_j]_\delta(\lambda) &= \frac{1}{\delta}\sum_{h=-\infty}^{\infty}(\lambda+2\pi h/\delta)^{-j}e^{i\delta c(\lambda+2\pi h/\delta)} \\ &= -\frac{e^{i\delta\lambda c}}{\delta}\sum_{\zeta} Res((\lambda+2\pi\cdot/\delta)^{-j}\chi,\zeta)\end{aligned}$$

where the sum is over all the poles ζ of the function $(\lambda+2\pi\cdot/\delta)^{-j}$, and

$$\chi(x) = 2\pi i e^{2\pi icx}(e^{2\pi ix}-1)^{-1}.$$

Now letting $q_j(\lambda,c) = [e_c m_j]_\delta(\lambda)$ and using the product rule, it is easy to see that the following recursion applies:

$$q_{j+1}(\lambda,c) = -\frac{1}{j}\left(\frac{\partial}{\partial\lambda}q_j(\lambda,c) - i\delta c q_j(\lambda,c)\right).$$

Starting with $q_2(\lambda,c)$ computed through brute force, we can use this recursion to obtain these functions for $j = 2,3,4,5,6$, which are all needed in Section 4. Using the shorthand $z = e^{-i\lambda\delta}$, we have

$$[e_c m_2]_\delta(\lambda) = \frac{\delta(1 + c(\overline{z} - 1))}{|1 - z|^2}$$

$$[e_c m_3]_\delta(\lambda) = \frac{i^3\delta^2}{2}\left((\overline{z} - z)|1 - z|^{-4} - c^2(z - 1)^{-1} - 2c|1 - z|^{-2}\right)$$

$$[e_c m_4]_\delta(\lambda) = \frac{\delta^3}{6}\left((z + \overline{z} + 4)|1 - z|^{-4} - c^3(z - 1)^{-1} - 3c^2|1 - z|^{-2} + 3c(\overline{z} - z)|1 - z|^{-4}\right)$$

$$\begin{aligned}[e_c m_5]_\delta(\lambda) = \frac{i^5\delta^4}{24}&\Big(\frac{c^4}{z - 1} + \frac{4c^3}{|1 - z|^2} + \frac{6c^2(z - \overline{z})}{|1 - z|^4} + \frac{8c(z - \overline{z})^2}{|1 - z|^6} \\ &+ \frac{4c(z + \overline{z})}{|1 - z|^4} + \frac{6(z - \overline{z})^3}{|1 - z|^8} + \frac{6(z^2 - \overline{z}^2)}{|1 - z|^6} + \frac{z - \overline{z}}{|1 - z|^4}\Big)\end{aligned}$$

$$\begin{aligned}[e_c m_6]_\delta(\lambda) = \frac{\delta^5}{120}&\Big(\frac{c^5}{z - 1} + \frac{5c^4}{|1 - z|^2} + \frac{10c^3(z - \overline{z})}{|1 - z|^4} + \frac{20c^2(z - \overline{z})^2}{|1 - z|^6} \\ &+ \frac{10c^2(z + \overline{z})}{|1 - z|^4} + \frac{30c(z - \overline{z})^3}{|1 - z|^8} + \frac{30c(z^2 - \overline{z}^2)}{|1 - z|^6} + \frac{5c(z - \overline{z})}{|1 - z|^4} + \frac{24(z - \overline{z})^4}{|1 - z|^{10}} \\ &+ \frac{36(z - \overline{z})^2(z + \overline{z})}{|1 - z|^8} + \frac{z + \overline{z}}{|1 - z|^4} + \frac{12(z^2 + \overline{z}^2) + 2(z - \overline{z})^2}{|1 - z|^6}\Big).\end{aligned}$$

A.2 Proofs

Proof of the stationarity of (13) and (17). More generally, we can define the continuous-time process $\overline{w}(t) = (1 - B)^d [I^d w](t)$, with $\overline{w}(\delta\tau) = w_\tau$ in the stock case. Now using induction and the defining property of the I operator, we have

$$\overline{w}(t) = \int_0^\delta \cdots \int_0^\delta w(t - s_1 - \cdots - s_d)\, ds_1 \cdots ds_d,$$

which is stationary with autocovariance function given by

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\frac{1 - e^{-i\lambda\delta}}{i\lambda}\right|^{2d} f_w(\lambda) e^{i\lambda h}\, d\lambda. \tag{A.1}$$

So long as $d \geq 1$, this function is continuous at the origin, which shows that it has an integrable spectral density, even if f_w is non-integrable of the form $f_w(\lambda) = f_u(\lambda) + \lambda^{2d} f_n(\lambda)$. For the flow case, observe that (17) can be written as $w_\tau = (1 - B)^{d+1}[I^{d+1} w](\delta\tau)$, and apply the previous arguments with incremented d.

Proof of Theorem 1. For any $t = \delta\tau + \delta c$, the filter discretization error process is

$$\epsilon_\tau = \Psi_c(B) y_\tau - x(t);$$

it suffices to show that this error process is orthogonal to the available data Y. We first demonstrate that this error process is stationary with mean zero. Consider that y_τ is a stock (9); then the polynomial term in (12) under application of the filter $\Psi_c(B)$ is simply

$$\sum_{j=0}^{d-1} \frac{\delta^j}{j!} y^{(j)}(0) \sum_k \psi_k(c)(\tau - k)^j = \sum_{j=0}^{d-1} \frac{y^{(j)}(0)}{i^j j!} \frac{\partial^j}{\partial \lambda^j} \left(\Psi_c(e^{-i\delta\lambda}) e^{i\lambda\delta\tau} \right) |_{\lambda=0}.$$

At the same time, the polynomial term in $x(t)$ is, using (11),

$$\sum_{j=0}^{d-1} \frac{y^{(j)}(0)}{j!} \int \psi(v)(t-v)^j \, dv = \sum_{j=0}^{d-1} \frac{y^{(j)}(0)}{j!} \int \psi(v+\delta c)(\delta\tau - v)^j \, dv = \sum_{j=0}^{d-1} \frac{y^{(j)}(0)}{i^j j!} \frac{\partial^j}{\partial \lambda^j} \left(g(\lambda) e_{c+\tau}(\lambda) \right) |_{\lambda=0}.$$

So in order for the polynomial terms in the expression for ϵ_τ to cancel out, it is sufficient that $(ge_c)^{(k)}(0) = \Psi_c^{(k)}(0)$ for all $k < d$, where by $\Psi_c^{(k)}(0)$ we denote the kth derivative with respect to λ of the frf $\Psi_c(e^{-i\lambda\delta})$, evaluated at $\lambda = 0$. It turns out this condition is true for any $j < 2d$. Define the function

$$p_j(\lambda) = f_w m_j [f_w m_j]_\delta^{-1}(\lambda) = \delta \left(1 + \sum_{l \neq 0} \frac{f_w(\lambda + 2\pi l/\delta)}{f_w(\lambda)} \left(\frac{\lambda}{\lambda + 2\pi l/\delta} \right)^j \right)^{-1}.$$

Since f_w has no zeroes or poles, it can be shown that

$$p_j(0) = \delta \qquad p_j(2\pi l/\delta) = 0 \quad \text{if} \quad l \neq 0.$$

Also the first $j-1$ derivatives of p_j are all zero at $2\pi l/\delta$ for any integer l; this requires the existence of the first $j-1$ derivatives of f_w. Now using the stated formula in Theorem 1, $\Psi_c(e^{-i\lambda\delta}) = [ge_c p_{2d}]_\delta(\lambda)$. It follows that

$$\frac{\partial^k}{\partial \lambda^k} \Psi_c(e^{-i\lambda\delta})|_{\lambda=0} = (ge_c)^{(k)}(0) \quad \forall k \leq 2d-1,$$

which is obtained by expanding the fold of $\Psi_c(e^{-i\lambda\delta})$, use of the product rule, and use of the above-stated properties of p_{2d}.

In the flow case let $\Theta_c(B) = (1-B)\Psi_c(B)$; then the polynomial term in (16) under application of the filter $\Psi_c(B)$ is simply

$$\sum_{j=0}^{d-1} \frac{y^{(j)}(0)}{(j+1)!} \left(\sum_k \theta_k(c)(\delta\tau - \delta k)^{j+1} \right) = \sum_{j=0}^{d-1} \frac{y^{(j)}(0)}{i^{j+1}(j+1)!} \frac{\partial^{j+1}}{\partial \lambda^{j+1}} \left(\Theta_c(e^{-i\delta\lambda}) e^{i\lambda\delta\tau} \right) |_{\lambda=0},$$

where $\theta_k(c)$ are the coefficients of $\Theta_c(B)$. So in order for the polynomial terms in the expression for ϵ_τ to cancel out, it is sufficient that $(ge_{c+\tau})^{(j)}(0) = (\Theta_c(e^{-i\lambda\delta})e^{i\lambda\delta\tau})^{(j+1)}(0)/(i(j+1))$ for every τ. Using the stated formula for the flow frf in Theorem 1, we have $\Theta(e^{-i\lambda\delta})/i = [ge_c m_1^{-1} p_{2d+2}]_\delta(\lambda)$, whose kth derivative at $\lambda = 0$ is equal to $k(ge_c)^{(k-1)}(0)$ (or zero if $k = 0$). Then

$$\frac{1}{j+1} \sum_{k=0}^{j+1} \binom{j+1}{k} k (ge_c)^{(k-1)}(0)(i\delta\tau)^{j+1-k} = \sum_{k=0}^{j} \binom{j}{k} (ge_c)^{(k)}(0)(i\delta\tau)^{j-k} = (ge_{c+\tau})^{(j)}(0),$$

as desired. Now returning to the stock case, our error process is

$$\epsilon_\tau = \left(\Psi_c(B) - \Psi(L)L^{-\delta c}\right)[I^d w](\delta\tau). \tag{A.2}$$

In this formula, $\Psi_c(B)$ operates on $[I^d w]$ as a discrete filter on a discrete (stock-observed) process, whereas $\Psi(L)L^{-\delta c}$ operates on $[I^d w]$ as a continuous-lag filter on a continuous time process; we have shown above that $\Psi_c(B) - \Psi(L)L^{-\delta c}$ annihilates polynomials of degree $d-1$ in an integer variable τ. Now since $\overline{w}(t) = (1-B)^d [I^d w](t)$, we can apply Lemma 1 of Bell (1984) to obtain a representation

$$[I^d w](\delta\tau) = \sum_{j=1}^{d} A_{j,\tau+d}[I^d w](\delta(j-d)) + \Xi_\tau(B)\overline{w}(\delta\tau), \tag{A.3}$$

where $A_{j,\tau}$ is a deterministic coefficient sequence dependent on time τ, which is completely determined by the differencing polynomial $(1-B)^d$. Also, the time-dependent discrete filter $\Xi_\tau(B)$ is given by the formula

$$\Xi_\tau(B)(1-B)^d = 1 - \sum_{j=1}^{d} A_{j,\tau+d} B^{\tau+d-j}. \tag{A.4}$$

The coefficient sequences $A_{j,\tau}$ consist of polynomials in τ of degree at most $d-1$. Moreover, since $\overline{w}(t)$ is a stationary process with integrable spectral density, it is stochastically continuous and thus has an orthogonal increments representation $\overline{w}(t) = \int_{-\infty}^{\infty} e^{i\lambda t}\, d\zeta(\lambda)$ – see Theorem 4.11.1 of Priestley (1981). Also, from (A.1) we know that $\mathbb{E}|d\zeta(\lambda)|^2 = |1-e^{-i\lambda\delta}|^{2d}\lambda^{-2d} f_w(\lambda) d\lambda$; then by (A.4)

$$\Xi_\tau(B)\overline{w}(\delta\tau) = \int_{-\infty}^{\infty} \frac{e^{i\lambda\delta\tau} - \sum_{j=1}^{d} A_{j,\tau+d} e^{-i\lambda\delta(d-j)}}{(1-e^{-i\lambda\delta})^d}\, d\zeta(\lambda),$$

with the integrand being bounded. Putting this together with (A.2) and (A.3) yields

$$\epsilon_\tau = \int_{-\infty}^{\infty} \frac{e^{i\lambda\delta\tau}\left(\Psi_c(e^{-i\lambda\delta}) - g(\lambda)e_c(\lambda)\right)}{(1-e^{-i\lambda\delta})^d}\, d\zeta(\lambda).$$

This shows that the error process is stationary with mean zero. Moreover, (A.2) shows that the error process is orthogonal to y^* under condition (15) – but note that (14) is not sufficient in general. It only remains to show that the error process is orthogonal to $\{w_\tau\}$:

$$\begin{aligned}
\mathbb{E}[\epsilon_\tau w_{\tau+h}] &= \int_{-\infty}^{\infty} e^{-i\lambda\delta h}\left(\Psi_c(e^{-i\lambda\delta}) - g(\lambda)e_c(\lambda)\right)(1-e^{i\lambda\delta})^d \lambda^{-2d} f_w(\lambda)\, d\lambda \\
&= \delta \int_{-\pi/\delta}^{\pi/\delta} e^{-i\lambda\delta h}\left(\Psi_c(e^{-i\lambda\delta})[m_{2d}f_w]_\delta(\lambda) - [m_{2d}ge_cf_w]_\delta(\lambda)\right)(1-e^{i\lambda\delta})^d\, d\lambda,
\end{aligned}$$

using the property of folds, and the periodicity of the frf of $\Psi_c(B)$. Now the stated stock formula in the theorem makes this covariance identically zero for all τ and h.

Finally, we turn to the flow case; the argument is essentially the same, with a few computational differences. Now letting $\overline{w}(t) = (1-B)^{d+1}[I^{d+1}w](t)$, we note that by (17) $\overline{w}(\delta\tau) = w_\tau$; moreover,

$\overline{w}(t)$ is stochastically continuous with integrable spectral density, so it has an orthogonal increments representation $\int e^{i\lambda t} d\zeta(\lambda)$, with $\mathbb{E}|d\zeta(\lambda)|^2 = |1 - e^{-i\lambda\delta}|^{2d+2} \lambda^{-2d-2} f_w(\lambda) d\lambda$. The analog of (A.2) is now

$$\epsilon_\tau = \left(\Theta_c(B) - \Psi(L) L^{-\delta c} D \right) [I^{d+1} w](\delta\tau),$$

with similar interpretations of the discrete and continuous-lag filters (recall that D acts by differentiation, mapping $[I^{d+1}w]$ to $[I^d w]$). Moreover, the expression in parentheses annihilates polynomials of degree d in an integer variable τ, which is implicit in the earlier flow calculations of this proof. So we obtain that y^* is orthogonal to the error process under (15). We can also find a representation for the nonstationary process $[I^{d+1}w](\delta\tau)$ exactly analogous to the stock case; the upshot is that

$$\epsilon_\tau = \int_{-\infty}^{\infty} \frac{e^{i\lambda\delta\tau} \left(\Theta_c(e^{-i\lambda\delta}) - g(\lambda) e_c(\lambda) i\lambda \right)}{(1 - e^{-i\lambda\delta})^{d+1}} \, d\zeta(\lambda).$$

(Recall that the frf of D is $i\lambda$.) Finally,

$$\begin{aligned}
\mathbb{E}[\epsilon_\tau w_{\tau+h}] &= \int_{-\infty}^{\infty} e^{-i\lambda\delta h} \left(\Theta_c(e^{-i\lambda\delta}) - g(\lambda) e_c(\lambda) i\lambda \right) (1 - e^{i\lambda\delta})^{d+1} \lambda^{-2d-2} f_w(\lambda) \, d\lambda \\
&= \delta \int_{-\pi/\delta}^{\pi/\delta} e^{-i\lambda\delta h} \left(\Theta_c(e^{-i\lambda\delta}) [m_{2d+2} f_w]_\delta(\lambda) - i [m_{2d+1} g e_c f_w]_\delta(\lambda) \right) (1 - e^{i\lambda\delta})^{d+1} \, d\lambda,
\end{aligned}$$

which is identically zero by definition of $\Psi_c(B)$ and $\Theta_c(B)$. This completes the proof. $\square$

Proof of Theorem 2. This proof follows along the same lines as Theorem 1, though it is not a corollary. We first consider the case of a stock; then by assumption n is stochastically continuous and thus can be written as $n(t) = \int e^{it\lambda} d\zeta_n(\lambda)$ with f_n integrable. The error process is then

$$\epsilon_\tau = \Psi_c(B) y_\tau - s(t) = \Psi_c(B) n_\tau + \left(\Psi_c(B) - L^{-\delta c} \right) s_\tau.$$

Now applying the machinery of the proof of Theorem 1 to the process s_τ, we find that the operator $\Psi_c(B) - L^{-\delta c}$ annihilates polynomials of degree $d-1$ so long as $\Psi_c^{(k)}(0) = (e_c)^{(k)}(0)$ for $k < d$ – essentially we substitute $g \equiv 1$ in the calculations in the proof of Theorem 1. We likewise have $\Psi_c(e^{-i\lambda\delta}) = [e_c f_u p_{2d}/f_w]_\delta(\lambda)$, and it follows that

$$\frac{\partial^k}{\partial\lambda^k} \Psi_c(e^{-i\lambda\delta})|_{\lambda=0} = \sum_{l=0}^{k} \binom{k}{l} e_c^{(k-l)}(0) \left(\frac{f_u}{f_w} \right)^l (0) = (e_c)^{(k)}(0) \quad \forall k \leq 2d-1.$$

The last equality follows from the fact that the first $2d-1$ derivatives of f_u/f_w are zero at $\lambda = 0$, but the value of the function at zero is unity. Applying this result, we see that ϵ_τ is stationary; using techniques from the proof of Theorem 1, we easily obtain the representation

$$\epsilon_\tau = \int e^{i\lambda\delta\tau} \Psi_c(e^{-i\lambda\delta}) \, d\zeta_n(\lambda) + \int e^{i\lambda\delta\tau} \frac{\Psi_c(e^{-i\lambda\delta}) - e_c(\lambda)}{(1 - e^{-i\lambda\delta})^d} \, d\zeta_u(\lambda),$$

where $\mathbb{E}|d\zeta_u(\lambda)|^2 = |1 - e^{-i\lambda\delta}|^{2d} \lambda^{-2d} f_u(\lambda) d\lambda$. Likewise, we have

$$w_\tau = u_\tau + (1-B)^d n_\tau = \int e^{i\lambda\delta\tau}\, d\zeta_u(\lambda) + \int e^{i\lambda\delta\tau} (1 - e^{-i\lambda\delta})^d\, d\zeta_n(\lambda).$$

Now for optimality, it suffices to show that y^* and $\{w_\tau\}$ are orthogonal to the error process; in order for $\mathbb{E}[y^* \epsilon_\tau] = 0$ we need to assume (23). Then we have

$$\begin{aligned}
\mathbb{E}[\epsilon_\tau w_{\tau+h}] &= \int e^{-i\lambda\delta h} \Psi_c(e^{-i\lambda\delta}) (1 - e^{i\lambda\delta})^d f_n(\lambda)\, d\lambda \\
&\quad + \int e^{-i\lambda\delta h} \frac{\Psi_c(e^{-i\lambda\delta}) - e_c(\lambda)}{(1 - e^{-i\lambda\delta})^d} |1 - e^{-i\lambda\delta}|^{2d} \lambda^{-2d} f_u(\lambda) d\lambda \\
&= \int e^{-i\lambda\delta h} (1 - e^{i\lambda\delta})^d \left(\Psi_c(e^{-i\lambda\delta}) f_w(\lambda) - e_c(\lambda) f_u(\lambda) \right) \lambda^{-2d}\, d\lambda \\
&= \delta \int_{-\pi/\delta}^{\pi/\delta} e^{-i\lambda\delta h} (1 - e^{i\lambda\delta})^d \left(\Psi_c(e^{-i\lambda\delta}) [f_w m_{2d}]_\delta(\lambda) - [e_c f_u m_{2d}]_\delta(\lambda) \right) d\lambda,
\end{aligned}$$

which is identically zero by the stated formula for the frf of $\Psi_c(B)$. Turning to the case of a flow, we no longer have n stochastically continuous, but we can still write $n_\tau = \int e^{i\lambda\delta\tau}\, d\zeta_n(\lambda)$ with $\mathbb{E}|d\zeta_n(\lambda)|^2 = |1 - e^{-i\lambda\delta}|^2 \lambda^{-2} f_n(\lambda) d\lambda$, since the flow-sampling of n, considered as a continuous-time process, will be in $\mathcal{C}^0$. As for $\Psi_c(B) - L^{-\delta c}$ operating on s_τ, this annihilates polynomials and reduces the signal to stationarity – simply use the concepts from the proof of Theorem 1. Then we obtain

$$\epsilon_\tau = \int e^{i\lambda\delta\tau} \Psi_c(e^{-i\lambda\delta})\, d\zeta_n(\lambda) + \int e^{i\lambda\delta\tau} \frac{\Theta_c(e^{-i\lambda\delta}) - e_c(\lambda) i\lambda}{(1 - e^{-i\lambda\delta})^{d+1}}\, d\zeta_u(\lambda),$$

with $\mathbb{E}|d\zeta_u(\lambda)|^2 = |1 - e^{-i\lambda\delta}|^{2d+2} \lambda^{-2d-2} f_u(\lambda) d\lambda$. Hence y^* is orthogonal to the error process under (23), and

$$\begin{aligned}
\mathbb{E}[\epsilon_\tau w_{\tau+h}] &= \int e^{-i\lambda\delta h} \Psi_c(e^{-i\lambda\delta}) (1 - e^{i\lambda\delta})^d |1 - e^{-i\lambda\delta}|^2 \lambda^{-2} f_n(\lambda)\, d\lambda \\
&\quad + \int e^{-i\lambda\delta h} \frac{\Theta_c(e^{-i\lambda\delta}) - e_c(\lambda) i\lambda}{(1 - e^{-i\lambda\delta})^{d+1}} |1 - e^{-i\lambda\delta}|^{2d+2} \lambda^{-2d-2} f_u(\lambda) d\lambda \\
&= \int e^{-i\lambda\delta h} (1 - e^{i\lambda\delta})^{d+1} \left(\Theta_c(e^{-i\lambda\delta}) f_w(\lambda) \lambda^{-2d-2} - i e_c(\lambda) f_u(\lambda) \lambda^{-2d-1} \right) d\lambda \\
&= \delta \int_{-\pi/\delta}^{\pi/\delta} e^{-i\lambda\delta h} (1 - e^{i\lambda\delta})^{d+1} \left(\Theta_c(e^{-i\lambda\delta}) [f_w m_{2d+2}]_\delta(\lambda) - i [e_c f_u m_{2d+1}]_\delta(\lambda) \right) d\lambda.
\end{aligned}$$

This is identically zero by the stated formulas for the frf, and this concludes the proof. □

Proof of Proposition 1. We first derive the following property:

$$\begin{aligned}
\delta \int_{-\infty}^{\infty} \psi(\delta z) e_{c-z}(\lambda + 2\pi j/\delta)\, dz &= e_c(\lambda + 2\pi j/\delta) \delta \int_{-\infty}^{\infty} \psi(\delta z) e_{-z}(\lambda + 2\pi j/\delta)\, dz \\
&= e_c(\lambda + 2\pi j/\delta) g(\lambda + 2\pi j/\delta)
\end{aligned}$$

for any integer j. Thus, using this relation we can easily show, for stock or for flow, that

$$\Psi_c(e^{-i\lambda\delta}) = \delta \int_{-\infty}^{\infty} \psi(\delta z)\Phi_{c-z}(e^{-i\lambda\delta})\, dz.$$

This is obtained by using the explicit formulas in Theorem 1 and Corollary 1. Now

$$\Phi_{c-z}(e^{-i\lambda\delta}) = \sum_k \phi_k(c-z)e^{-i\lambda\delta k} = \sum_k \phi_0(c-z+k)e^{-i\lambda\delta k},$$

utilizing (24). Hence we have

$$\Psi_c(e^{-i\lambda\delta}) = \sum_k \delta \int_{-\infty}^{\infty} \psi(\delta z)\phi_0(c-z+k)e^{-i\lambda\delta k}\, dz = \sum_k \delta \int_{-\infty}^{\infty} \psi(\delta z+\delta k)\phi_0(c-z)\, dz e^{-i\lambda\delta k}.$$

Now Fourier Inversion concludes the proof. □

A.3 Derivations for Section 4

The derivatives of χ. By definition $\chi(x) = 2\pi i e^{2\pi icx}(e^{2\pi ix}-1)^{-1}$, so by the product rule

$$\chi^{(k)}(x) = 2\pi i \sum_{l=0}^{k} \binom{k}{l} e^{2\pi icx}(2\pi ic)^{k-l}\left[(e^{2\pi ix}-1)^{-1}\right]^{(l)}.$$

We list the first five derivatives of $(e^{2\pi ix}-1)^{-1}$:

$$\frac{\partial}{\partial x}(e^{2\pi ix}-1)^{-1} = \frac{2\pi i}{(1-e^{2\pi ix})(1-e^{-2\pi ix})}$$

$$\frac{\partial^2}{\partial x^2}(e^{2\pi ix}-1)^{-1} = \frac{(2\pi i)^2(e^{2\pi ix}-e^{-2\pi ix})}{(1-e^{2\pi ix})^2(1-e^{-2\pi ix})^2}$$

$$\frac{\partial^3}{\partial x^3}(e^{2\pi ix}-1)^{-1} = \frac{2(2\pi i)^3(e^{2\pi ix}-e^{-2\pi ix})^2}{(1-e^{2\pi ix})^3(1-e^{-2\pi ix})^3} + \frac{(2\pi i)^3(e^{2\pi ix}+e^{-2\pi ix})}{(1-e^{2\pi ix})^2(1-e^{-2\pi ix})^2}$$

$$\begin{aligned}\frac{\partial^4}{\partial x^4}(e^{2\pi ix}-1)^{-1} &= \frac{6(2\pi i)^4(e^{2\pi ix}-e^{-2\pi ix})^3}{(1-e^{2\pi ix})^4(1-e^{-2\pi ix})^4} \\ &+ \frac{6(2\pi i)^4(e^{4\pi ix}-e^{-4\pi ix})}{(1-e^{2\pi ix})^3(1-e^{-2\pi ix})^3} + \frac{(2\pi i)^4(e^{2\pi ix}-e^{-2\pi ix})}{(1-e^{2\pi ix})^2(1-e^{-2\pi ix})^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial^5}{\partial x^5}(e^{2\pi ix}-1)^{-1} &= \frac{24(2\pi i)^5(e^{2\pi ix}-e^{-2\pi ix})^4}{(1-e^{2\pi ix})^5(1-e^{-2\pi ix})^5} + \frac{36(2\pi i)^5(e^{2\pi ix}-e^{-2\pi ix})^2(e^{2\pi ix}+e^{-2\pi ix})}{(1-e^{2\pi ix})^4(1-e^{-2\pi ix})^4} \\ &+ \frac{(2\pi i)^5\left(12(e^{4\pi ix}+e^{-4\pi ix})+2(e^{2\pi ix}-e^{-2\pi ix})^2\right)}{(1-e^{2\pi ix})^3(1-e^{-2\pi ix})^3} + \frac{(2\pi i)^5(e^{2\pi ix}+e^{-2\pi ix})}{(1-e^{2\pi ix})^2(1-e^{-2\pi ix})^2}.\end{aligned}$$

Next, we evaluate at $x = -\delta\lambda/2\pi$.

$$\chi\left(-\frac{\delta\lambda}{2\pi}\right) = \frac{2\pi i z^c}{z-1}$$

$$\chi^{(1)}\left(-\frac{\delta\lambda}{2\pi}\right) = (2\pi i)^2 z^c \left(\frac{c}{z-1} + |1-z|^{-2}\right)$$

$$\chi^{(2)}\left(-\frac{\delta\lambda}{2\pi}\right) = (2\pi i)^3 z^c \left(\frac{c^2}{z-1} + \frac{2c}{|1-z|^2} + \frac{(z-\overline{z})}{|1-z|^4}\right)$$

$$\chi^{(3)}\left(-\frac{\delta\lambda}{2\pi}\right) = (2\pi i)^4 z^c \left(\frac{c^3}{z-1} + \frac{3c^2}{|1-z|^2} + \frac{3c(z-\overline{z})}{|1-z|^4} + \frac{2(z-\overline{z})^2}{|1-z|^6} + \frac{(z+\overline{z})}{|1-z|^4}\right)$$

$$\chi^{(4)}\left(-\frac{\delta\lambda}{2\pi}\right) = (2\pi i)^5 z^c (\frac{c^4}{z-1} + \frac{4c^3}{|1-z|^2} + \frac{6c^2(z-\overline{z})}{|1-z|^4} + \frac{8c(z-\overline{z})^2}{|1-z|^6}$$
$$+ \frac{4c(z+\overline{z})}{|1-z|^4} + \frac{6(z-\overline{z})^3}{|1-z|^8} + \frac{6(z^2-\overline{z}^2)}{|1-z|^6} + \frac{(z-\overline{z})}{|1-z|^4})$$

$$\chi^{(5)}\left(-\frac{\delta\lambda}{2\pi}\right) = (2\pi i)^6 z^c (\frac{c^5}{z-1} + \frac{5c^4}{|1-z|^2} + \frac{10c^3(z-\overline{z})}{|1-z|^4} + \frac{20c^2(z-\overline{z})^2}{|1-z|^6} + \frac{10c^2(z+\overline{z})}{|1-z|^4}$$
$$+ \frac{30c(z-\overline{z})^3}{|1-z|^8} + \frac{30c(z^2-\overline{z}^2)}{|1-z|^6} + \frac{5c(z-\overline{z})}{|1-z|^4} + \frac{24(z-\overline{z})^4}{|1-z|^{10}} + \frac{36(z-\overline{z})^2(z+\overline{z})}{|1-z|^8}$$
$$+ \frac{12(z^2+\overline{z}^2) + 2(z-\overline{z})^2}{|1-z|^6} + \frac{(z+\overline{z})}{|1-z|^4})$$

Derivations for Butterworth Filters on a Random Walk. For the $m = 1$ case, note that $\dot{g}(0) = 0$ and $g(0) = 1$. Thus using (26), we have

$$[ge_c m_2]_\delta(\lambda) = -\frac{e^{i\delta\lambda c}}{\delta}\{\left(\frac{\delta}{2\pi}\right)^2 \left(\frac{2\pi}{\delta}\dot{g}(0)\chi\left(-\frac{\delta\lambda}{2\pi}\right) + g(0)\dot{\chi}\left(-\frac{\delta\lambda}{2\pi}\right)\right)$$
$$+ \frac{\delta\sqrt{q}}{4\pi i}\chi\left(-\frac{\delta(\lambda - i\sqrt{q})}{2\pi}\right)(i\sqrt{q})^{-2} - \frac{\delta\sqrt{q}}{4\pi i}\chi\left(-\frac{\delta(\lambda + i\sqrt{q})}{2\pi}\right)(-i\sqrt{q})^{-2}\}$$
$$= -\frac{e^{i\delta\lambda c}}{\delta}\left\{\left(\frac{\delta}{2\pi}\right)^2 \dot{\chi}\left(-\frac{\delta\lambda}{2\pi}\right) + \frac{\delta}{2\sqrt{q}}\left[\frac{e^{-i\delta\lambda c}e^{\delta\sqrt{q}c}}{e^{-i\delta\lambda}e^{\delta\sqrt{q}} - 1} - \frac{e^{-i\delta\lambda c}e^{-\delta\sqrt{q}c}}{e^{-i\delta\lambda}e^{-\delta\sqrt{q}} - 1}\right]\right\}.$$

Now using $[m_2]_\delta(\lambda) = \delta|1 - e^{-i\lambda\delta}|^{-2}$, we see that $\Psi_c(e^{-i\lambda\delta})$ simplifies to the stated formula. For $m = 2$, we have $\dot{g}(0) = 0$ and $g(0) = 1$ as well. The residue calculations at the poles of g are given below:

$$Res\left(g(\lambda + 2\pi\cdot/\delta)(\lambda + 2\pi\cdot/\delta)^{-2}\chi, \frac{-\delta(\lambda - q^{1/4}e^{ik\pi/4})}{2\pi}\right) = -\frac{\delta}{2\pi}\frac{\chi\left(\frac{-\delta(\lambda - q^{1/4}e^{ik\pi/4})}{2\pi}\right)}{4q^{1/4}e^{ik\pi/4}},$$

with $k = 1, 3, 5, 7$. So by (26) we have

$$[ge_c m_2]_\delta(\lambda) = -\frac{e^{i\delta\lambda c}}{\delta}\{\left(\frac{\delta}{2\pi}\right)^2 \left(\frac{2\pi}{\delta}\dot{g}(0)\chi\left(-\frac{\delta\lambda}{2\pi}\right) + g(0)\dot{\chi}\left(-\frac{\delta\lambda}{2\pi}\right)\right)$$
$$- \frac{\delta}{8\pi q^{1/4}}\sum_{k=1,3,5,7}\chi\left(\frac{-\delta(\lambda - q^{1/4}e^{ik\pi/4})}{2\pi}\right)e^{-ik\pi/4}\}$$

Again $\dot{g}(0) = 0$ and $g(0) = 1$, and we observe that

$$\chi\left(\frac{-\delta(\lambda - q^{1/4}e^{i\pi/4})}{2\pi}\right) - \chi\left(\frac{-\delta(\lambda - q^{1/4}e^{i5\pi/4})}{2\pi}\right)$$
$$= 2\pi i e^{-i\lambda\delta c}\left[\frac{e^{(i-1)\delta c q^{1/4}/\sqrt{2}}}{e^{-i\delta\lambda}e^{(i-1)\delta c q^{1/4}/\sqrt{2}} - 1} - \frac{e^{(1-i)\delta c q^{1/4}/\sqrt{2}}}{e^{-i\delta\lambda}e^{(1-i)\delta c q^{1/4}/\sqrt{2}} - 1}\right]$$
$$\chi\left(\frac{-\delta(\lambda - q^{1/4}e^{i3\pi/4})}{2\pi}\right) - \chi\left(\frac{-\delta(\lambda - q^{1/4}e^{i7\pi/4})}{2\pi}\right)$$
$$= 2\pi i e^{-i\lambda\delta c}\left[\frac{e^{(-i-1)\delta c q^{1/4}/\sqrt{2}}}{e^{-i\delta\lambda}e^{(-i-1)\delta c q^{1/4}/\sqrt{2}} - 1} - \frac{e^{(i+1)\delta c q^{1/4}/\sqrt{2}}}{e^{-i\delta\lambda}e^{(i+1)\delta c q^{1/4}/\sqrt{2}} - 1}\right].$$

Putting these residues together we obtain

$$[g e_c m_2]_\delta(\lambda) = \delta\left(c(\overline{z} - 1) + 1\right)|1 - z|^{-2} + \frac{1}{2\sqrt{2}q^{1/4}}$$
$$\{\frac{e^{(i-1)\delta c q^{1/4}/\sqrt{2}}}{(1-i)(ze^{(i-1)\delta c q^{1/4}/\sqrt{2}} - 1)} - \frac{e^{(1-i)\delta c q^{1/4}/\sqrt{2}}}{(1-i)(ze^{(1-i)\delta c q^{1/4}/\sqrt{2}} - 1)}$$
$$+ \frac{e^{(-i-1)\delta c q^{1/4}/\sqrt{2}}}{(i+1)(ze^{(-i-1)\delta c q^{1/4}/\sqrt{2}} - 1)} - \frac{e^{(i+1)\delta c q^{1/4}/\sqrt{2}}}{(i+1)(ze^{(i+1)\delta c q^{1/4}/\sqrt{2}} - 1)}\}.$$

Finally, dividing by $[m_2]_\delta(\lambda)$ yields the stated frequency response function. Next, we derive the filter coefficients. Since $\psi_j = \frac{\delta}{2\pi}\int_{-\pi/\delta}^{\pi/\delta}\Psi_c(e^{-i\lambda\delta})e^{i\lambda\delta j}\,d\lambda$, we have by change of variable the following formula:

$$\psi_j = \frac{1}{2\pi i}\begin{cases}\int_\Omega \Psi_c(x^{-1})x^{j-1}\,dx & j \geq 0\\ \int_\Omega \Psi_c(x)x^{-j-1}\,dx & j \leq 0.\end{cases} \tag{A.5}$$

Here Ω denotes the unit circle. Note that when $j = 0$, we can apply either case as we see fit. These formulas are easily derived by just using the change of variable formula. The formula $\Psi_c(x)$ is obtained by substituting the variable x everywhere for z (and x^{-1} for $\overline{z}$), whereas for $\Psi_c(x^{-1})$ we do the opposite. Now by considering the analytic extension of Ψ_c to the unit disk, the above integrals are computed by calculating the sum of the residues at the poles of the integrand occurring within the unit disk (see pp. 249–250 of Henrici (1974)). Focusing on the $m = 1$ case first, we have

$$\Psi_c(x) = x^{-1}\left((1-c)x + c - \frac{(1-x)^2}{2\delta\sqrt{q}}\left[\frac{e^{-\delta c\sqrt{q}}}{xe^{-\delta\sqrt{q}} - 1} - \frac{e^{\delta c\sqrt{q}}}{xe^{\delta\sqrt{q}} - 1}\right]\right)$$
$$\Psi_c(1/x) = \left((1-c) + cx - \frac{(1-x)^2}{2\delta\sqrt{q}}\left[\frac{e^{-\delta c\sqrt{q}}}{e^{-\delta\sqrt{q}} - x} - \frac{e^{\delta c\sqrt{q}}}{e^{\delta\sqrt{q}} - x}\right]\right).$$

Suppose that $j \leq -2$. Then the only poles of $\Psi_c(x)x^{-j-1}$ that are inside the unit circle occur at $e^{-\delta\sqrt{q}}$. Hence

$$\psi_j = Res\left(\Psi_\delta(x)x^{-j-1}, e^{-\delta\sqrt{q}}\right) = \frac{(1 - e^{-\delta\sqrt{q}})^2 e^{\delta\sqrt{q}(c+j+1)}}{2\delta\sqrt{q}}.$$

The calculations are simple because the poles are simple. Now let $j = -1$; the integrand is just $\Psi_c(x)$, which has a simple pole at the origin (in addition to the other pole). So

$$\psi_{-1} = Res\left(\Psi_c(x), e^{-\delta\sqrt{q}}\right) + Res\left(\Psi_c(x), 0\right) = c - \frac{\sinh(\delta c\sqrt{q})}{\delta\sqrt{q}} + \frac{\left(1 - e^{-\delta\sqrt{q}}\right)^2 e^{\delta\sqrt{q}c}}{2\delta\sqrt{q}}.$$

Next suppose that $j \geq 1$, and we use the other formula. There is only one simple pole at $e^{-\delta\sqrt{q}}$ in the unit disk, so

$$\psi_j = Res\left(\Psi_c(1/x)x^{j-1}, e^{-\delta\sqrt{q}}\right) = \frac{\left(1 - e^{-\delta\sqrt{q}}\right)^2 e^{-\delta\sqrt{q}(c+j-1)}}{2\delta\sqrt{q}}.$$

Finally, when $j = 0$ we use the first formula in (A.5) (since then the pole at zero will be simple, rather than of order two), and we obtain

$$\begin{aligned}\psi_0 &= Res\left(\Psi_c(1/x)x^{-1}, e^{-\delta\sqrt{q}}\right) + Res\left(\Psi_c(1/x)x^{-1}, 0\right) \\ &= 1 - c + \frac{\sinh(\delta(c-1)\sqrt{q})}{\delta\sqrt{q}} + \frac{\left(1 - e^{-\delta\sqrt{q}}\right)^2 e^{-\delta\sqrt{q}(c-1)}}{2\delta\sqrt{q}}.\end{aligned}$$

Next, we turn to the $m = 2$ case, noting that of course (A.5) still applies. Letting $j \leq -2$, the relevant poles are $e^{(\pm i-1)\delta q^{1/4}/\sqrt{2}}$. They are still simple poles, though the residues are a bit more difficult to compute:

$$\begin{aligned}&Res\left(\Psi_c(x)x^{-j-1}, e^{(i-1)\delta q^{1/4}/\sqrt{2}}\right) \\ &= -\frac{e^{(1-i)(c+j)\delta q^{1/4}/\sqrt{2}}(1 - e^{(i-1)\delta q^{1/4}/\sqrt{2}})(1 - e^{(1-i)\delta q^{1/4}/\sqrt{2}})}{2\sqrt{2}q^{1/4}\delta(1-i)} \\ &Res\left(\Psi_c(x)x^{-j-1}, e^{(-i-1)\delta q^{1/4}/\sqrt{2}}\right) \\ &= -\frac{e^{(1+i)(c+j)\delta q^{1/4}/\sqrt{2}}(1 - e^{(i+1)\delta q^{1/4}/\sqrt{2}})(1 - e^{-(1+i)\delta q^{1/4}/\sqrt{2}})}{2\sqrt{2}q^{1/4}\delta(1+i)}.\end{aligned}$$

Adding these we obtain

$$\psi_j = -\frac{1}{\sqrt{2}q^{1/4}\delta}Re\left[(1+i)^{-1}e^{(1+i)(c+j)\delta q^{1/4}/\sqrt{2}}(1 - e^{(i+1)\delta q^{1/4}/\sqrt{2}})(1 - e^{-(1+i)\delta q^{1/4}/\sqrt{2}})\right].$$

For $j = -1$, we must compute the residue of $\Psi_c(x)$ at the origin:

$$Res\left(\Psi_c(x), 0\right) = c + \frac{1}{2\sqrt{2}q^{1/4}\delta}Re\left[(1+i)^{-1}\left(e^{-(1+i)c\delta q^{1/4}/\sqrt{2}} - e^{(1+i)c\delta q^{1/4}/\sqrt{2}}\right)\right].$$

Combining with the other residues yields the stated result for ψ_{-1}. When $j \geq 1$ the residues are

$$\begin{aligned}Res\left(\Psi_c(1/x)x^{j-1}, e^{(i-1)\delta q^{1/4}/\sqrt{2}}\right) &= -\frac{e^{(i-1)(c+j)\delta q^{1/4}/\sqrt{2}}(1 - e^{(i-1)\delta q^{1/4}/\sqrt{2}})(1 - e^{(1-i)\delta q^{1/4}/\sqrt{2}})}{2\sqrt{2}q^{1/4}\delta(1-i)} \\ Res\left(\Psi_c(1/x)x^{j-1}, e^{(-i-1)\delta q^{1/4}/\sqrt{2}}\right) &= -\frac{e^{-(1+i)(c+j)\delta q^{1/4}/\sqrt{2}}(1 - e^{(i+1)\delta q^{1/4}/\sqrt{2}})(1 - e^{-(1+i)\delta q^{1/4}/\sqrt{2}})}{2\sqrt{2}q^{1/4}\delta(1+i)}.\end{aligned}$$

Adding these we obtain

$$\psi_j = -\frac{1}{\sqrt{2}q^{1/4}\delta} Re\left[(1+i)^{-1} e^{-(1+i)(c+j)\delta q^{1/4}/\sqrt{2}}(1-e^{(i+1)\delta q^{1/4}/\sqrt{2}})(1-e^{-(1+i)\delta q^{1/4}/\sqrt{2}})\right].$$

Finally, for $j=0$ we use the first formula in (A.5), and compute the residue at zero:

$$Res\left(\Psi_c(1/x)x^{-1}, 0\right) = 1 - c + \frac{1}{2\sqrt{2}q^{1/4}\delta} Re\left[(1+i)^{-1}\left(e^{(1+i)(c-1)\delta q^{1/4}/\sqrt{2}} - e^{-(1+i)(c-1)\delta q^{1/4}/\sqrt{2}}\right)\right].$$

This yields the stated formula for ψ_0.

A.4 Derivations for Section 5

Derivations for the LLM. Using the formulas derived at the end of $A.1$, the formulas for the frequency response functions are immediate. To get the coefficients, we express $\Psi_c(x)$ as a rational function:

$$\Psi_c(x) = \frac{\delta^2\left((x^3+4x^2+x) + c^3(1-x)^3 + 3c^2x(1-x)^2 + 3cx(1-x^2)\right)}{x\left(\delta^2(x^2+4x+1) - 6(1-x)^2/q\right)}$$

$$\Psi_c(x) = \frac{3\delta\left((x^2+x) + c^2(1-x)^2 + 2cx(1-x)\right)}{x\left(\delta^2(x^2+4x+1) - 6(1-x)^2/q\right)}$$

for the flow-signal and stock-signal cases, respectively. So there is a simple pole at the origin; for the other poles we must analyze the quadratic $\delta^2(x^2+4x+1) - 6(1-x)^2/q$. It follows from the quadratic formula that the roots are η_1, η_2 as expressed in 5.1, and it is easy to see that $0 < \eta_1 < 1$ and $\eta_2 > 1$, using our assumptions on δ and q. Now making use of (A.5), we must also compute $\Psi_c(1/x)$, which is expressed as a rational function below:

$$\Psi_c(1/x) = \frac{\delta^2\left((x^2+4x+1) - c^3(1-x)^3 + 3c^2(1-x)^2 - 3c(1-x^2)\right)}{\left(\delta^2(x^2+4x+1) - 6(1-x)^2/q\right)}$$

$$\Psi_c(1/x) = \frac{3\delta\left((x^2+x) + c^2x(1-x)^2 - 2cx(1-x)\right)}{x\left(\delta^2(x^2+4x+1) - 6(1-x)^2/q\right)}$$

for the flow-signal and stock-signal cases, respectively. So there is no pole at the origin, but there are two simple poles at η_1, η_2 just as for $\Psi_c(x)$. So for the flow-signal case we must compute the following residues to get the coefficients:

$$\begin{aligned}
\psi_j &= Res(\Psi_c(x)x^{-j-1}, \eta_1) \qquad j \le -2 \\
\psi_{-1} &= Res(\Psi_c(x), \eta_1) + Res(\Psi_c(x), 0) \\
\psi_0 &= Res(\Psi_c(1/x)x^{-1}, \eta_1) + Res(\Psi_c(1/x)x^{-1}, 0) \\
\psi_j &= Res(\Psi_c(1/x)x^{j-1}, \eta_1) \qquad j \ge 1
\end{aligned}$$

These residues are easy to calculate, and we obtain the stated formulas for the coefficients. For the stock-signal case, the same formulas apply (although the residues themselves are slightly different), noting that in the $j = 0$ case there is no pole at the origin, so that

$$\psi_0 = Res(\Psi_c(1/x)x^{-1}, \eta_1).$$

Derivations for the STM and Turning Point. The frequency response formulas follow immediately from the formulas derived in $A.1$. To get the coefficients, we express $\Psi_c(x)$ as a rational function:

$$\Psi_c(x) = \frac{\Theta_c(x)}{x\left(\delta^4(x^4 + 26x^3 + 66x^2 + 26x + 1) - 120(1-x)^4/q\right)},$$

noting that $\Theta_c(x)$ has a different definition for the flow-signal and stock-signal cases. This rationalization is obtained by multiplying numerator and denominator by x^3. Likewise for $\Psi_c(1/x)$ we have

$$\Psi_c(1/x) = \frac{\Phi_c(x)}{\delta^4(x^4 + 26x^3 + 66x^2 + 26x + 1) - 120(1-x)^4/q}$$

for either of the cases, obtained by multiplying numerator and denominator by x^2. So $\Psi_c(x)$ has a simple pole at the origin, whereas $\Psi_c(1/x)$ does not. For their other poles, we must consider the quartic function $\delta^4(x^4 + 26x^3 + 66x^2 + 26x + 1) - 120(1-x)^4/q$. We employ the following factorization:

$$\begin{aligned}
&\delta^4(x^4 + 26x^3 + 66x^2 + 26x + 1) - 120(1-x)^4/q \\
&= (\delta^4 - 120/q)\left(x^4 + bx^3 + dx^2 + bx + 1\right) \\
&= (\delta^4 - 120/q)(x^2 + \alpha x + 1)(x^2 + \beta x + 1),
\end{aligned}$$

where $b = (26\delta^4 + 480/q)/(\delta^4 - 120/q)$ and $d = (66\delta^4 - 720/q)/(\delta^4 - 120/q)$. Note that the symmetry in the quartic allows us to proceed without recourse to Cardano's formula. Now in the second equality, α and β are generally given by

$$\alpha = \frac{b - \sqrt{b^2 - 4d + 8}}{2} \qquad \beta = \frac{b + \sqrt{b^2 - 4d + 8}}{2},$$

which simplify to the stated formulas for α and β in 5.2. Each of the two quadratics is then further factored using the quadratic formula, which yields the stated form for the roots ν_2, ν_3, ν_4 (and $\nu_1 = (-\alpha + \sqrt{\alpha^2 - 4})/2$). Now from the formula for α, we can show that $\alpha < -2$, which implies that $\nu_1 > 1$. Playing with the inequalities shows that $0 < \nu_2 < 1$. As for the other roots, observe that $|\beta| < 2$ (after some more inequality calculations), so that

$$\nu_3 = \frac{-\beta + i\sqrt{4 - \beta^2}}{2} \qquad \nu_4 = \frac{-\beta - i\sqrt{4 - \beta^2}}{2}.$$

Hence they are complex conjugate, and their product is unity. So the coefficients are just given by sums of residues, where the poles under consideration are ν_2, ν_3, ν_4 and zero. These are all simple poles, so the calculations are easy. Note that since the poles ν_3, ν_4 occur on the unit circle, we should multiply their contribution to the residue sum by 1/2 (Henrici, 1974). From here the determination of the coefficients is straightforward.

For the turning point calculations, the coefficient formulas are straightforward using the new numerator polynomials $\tilde{\Theta}_c(x)$ and $\tilde{\Phi}_c(x)$. When $j = 0$, there is now a simple pole at the origin, so we must also add in the residue

$$Res(\Psi_c(1/x)(x-1)/x^2, 0) = -\frac{\Psi_c(1/x)}{x}|_{x=0} = -\frac{5\delta^3(c^4 - 4c^3 + 6c^2 - 4c)}{\delta^4 - 120/q}.$$

Derivations for the BP and LP. For the bandpass filter, we first derive the autocovariance function for $c(t)$. Write

$$f_c(\omega) = \frac{r\sigma^2}{[\rho + i(\omega - \lambda_c)][\rho - i(\omega - \lambda_c)][\rho + i(\omega + \lambda_c)][\rho - i(\omega + \lambda_c)]},$$

where $\omega \in \mathbb{C}$ by analytic extension. The poles in $\mathbb{C}^+$ are therefore $\pm\lambda_c + i\rho$. So for $x \geq 0$ we have

$$\begin{aligned} R_c(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f_c(\lambda)e^{i\lambda x}\, d\lambda = i \sum_{\zeta \in \mathbb{C}^+} Res(f_c e^{ix\cdot}, \zeta) \\ &= \frac{r\sigma^2}{8\rho\lambda_c} e^{-\rho x} \left(\frac{e^{ix\lambda_c}}{\lambda_c + i\rho} + \frac{e^{-ix\lambda_c}}{\lambda_c - i\rho} \right) \\ &= \frac{r\sigma^2 e^{-\rho x}}{4\rho\lambda_c(\lambda_c^2 + \rho^2)} Re\left(\frac{e^{ix\lambda_c}}{\lambda_c + i\rho} \right). \end{aligned}$$

Noting that R_c is an even function, we obtain the stated result. In order to obtain the spectrum of the flow-sampled cycle, we must compute $|1-z|^2 [f_c m_2]_\delta$. The fold is

$$\begin{aligned} [f_c m_2]_\delta(\lambda) &= \frac{\delta r\sigma^2}{(\rho^2 + \lambda_c^2)^2}|1-z|^{-2} \\ &- \frac{r\sigma^2}{8\lambda_c\rho}\left((\lambda_c + i\rho)^{-3} \frac{1 - e^{2\delta(\rho - i\lambda_c)}}{|z - e^{\delta(\rho - i\lambda_c)}|^2} + (\lambda_c - i\rho)^{-3} \frac{1 - e^{2\delta(\rho + i\lambda_c)}}{|z - e^{\delta(\rho + i\lambda_c)}|^2} \right), \end{aligned}$$

which is determined by using Theorem 4.9a of Henrici (1974). Alternatively, one can differentiate with respect to c the expressions given below, evaluating at $c = 0$. Now multiplying the above formula by $|1-z|^2$, we see that the spectrum corresponds to an ARMA(2,2) process.

We next compute $[e^{ic\delta\cdot} f_c m_4]_\delta(\lambda)$. By Theorem 4.9*b* of Henrici (1974) we have the fold equal to

$$-\frac{1}{\delta}\sum_{\zeta} Res\left(f_c(\lambda + 2\pi\cdot/\delta)(\lambda + 2\pi\cdot/\delta)^{-4}\chi, \zeta \right) e^{i\delta c\lambda},$$

where the sum is over all poles ζ. There is a pole of order 4 at $-\frac{\delta\lambda}{2\pi}$, and the residue is

$$\frac{1}{6}\left(\frac{\delta}{2\pi}\right)^4 \left(f_c(0)\dddot{\chi}(-\frac{\delta\lambda}{2\pi}) + 3(\frac{2\pi}{\delta})^2 \ddot{f}_c(0)\dot{\chi}(-\frac{\delta\lambda}{2\pi})\right)$$
$$= \frac{1}{6}\left(\frac{\delta}{2\pi}\right)^4 (\frac{r\sigma^2}{(\rho^2+\lambda_c^2)^2}(2\pi i)^4 e^{-i\delta c\lambda} \left(\frac{c^3}{z-1} + \frac{3c^2}{|1-z|^2} + \frac{3c(z-\overline{z})}{|1-z|^4} + \frac{2(z-\overline{z})^2}{|1-z|^6} + \frac{z+\overline{z}}{|1-z|^4} \right)$$
$$+ (\frac{2\pi}{\delta})^2 \frac{12r\sigma^2(\lambda_c^2-\rho^2)}{(\rho^2+\lambda_c^2)^4}(2\pi i)^2 e^{-i\delta c\lambda} \left(\frac{c}{z-1} + \frac{1}{|1-z|^2}\right)).$$

There are also simple poles at $\frac{\delta}{2\pi}(-\lambda \pm \lambda_c \pm i\rho)$, whose residues are given below:

$$-\frac{\delta}{2\pi i}\frac{(\lambda_c - i\rho)^{-5} r\sigma^2}{8\rho\lambda_c}\chi(\frac{\delta}{2\pi}(-\lambda+\lambda_c-i\rho))$$
$$\frac{\delta}{2\pi i}\frac{(\lambda_c + i\rho)^{-5} r\sigma^2}{8\rho\lambda_c}\chi(\frac{\delta}{2\pi}(-\lambda+\lambda_c+i\rho))$$
$$-\frac{\delta}{2\pi i}\frac{(\lambda_c + i\rho)^{-5} r\sigma^2}{8\rho\lambda_c}\chi(\frac{\delta}{2\pi}(-\lambda-\lambda_c-i\rho))$$
$$\frac{\delta}{2\pi i}\frac{(\lambda_c - i\rho)^{-5} r\sigma^2}{8\rho\lambda_c}\chi(\frac{\delta}{2\pi}(-\lambda-\lambda_c+i\rho)).$$

Summing these and simplifying gives the stated result. Now we also note that

$$\frac{\partial}{\partial c}[e^{ic\delta\cdot} f_c m_4]_\delta(\lambda) = i\delta[e^{ic\delta\cdot} f_c m_3]_\delta(\lambda) \qquad \frac{\partial^2}{\partial c^2}[e^{ic\delta\cdot} f_c m_4]_\delta(\lambda) = -\delta^2[e^{ic\delta\cdot} f_c m_2]_\delta(\lambda),$$

which allows us to obtain the other folds fairly easily.

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