Interpretations of Space-Time Spectral Energy Equations

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ABSTRACT

Interpretations are given of two different formulations of space-time spectral energy equations derived by Kao (1968) and Hayashi (1980).

Contrary to Kao's interpretation, it is argued that his formulation does not describe how spectral energy is maintained, since his equation corresponds to the *imaginary part* of the energy equation of space-time Fourier components which governs the *frequency* (time change of phase).

On the other hand, Hayashi's formulation is consistent with Saltzman's (1957) wavenumber spectral energy equation, since his formulation corresponds to the *real part* which governs the *growth rate* (time change of amplitude).

1. Introduction

Kao (1968) has formulated wavenumber–frequency (space-time) spectra and an energy equation governing the maintenance of these spectra by linear and nonlinear energy transfer spectra, extending the wavenumber spectral energy equation formulated by Saltzman (1957). Kao's energy equation has been applied to atmospheric disturbances in several papers (e.g. Kao and Lee, 1977; Chen *et al.*, 1981). This equation is different from the time-Fourier decomposition of the wavenumber energy equation as formulated by Kao (1980) and applied by Kao and Chi (1978) to analyze the evolution of wavenumber spectral energy.

Hayashi (1971) proposed a method of estimating space-time cross-spectra by the use of time-cross spectral analysis. This method gave additional information of phase and coherence. Recently, Hayashi (1980) reformulated the wavenumber (or wavenumber-frequency) spectral energy equations in such a way that both linear and nonlinear energy transfer spectra can be estimated by wavenumber (or wavenumber-frequency) cross-spectral analysis, while the Saltzman and Kao formulations involve convolutions of Fourier transforms. Although equivalence between the Hayashi and Saltzman wavenumber energy equations was discussed, inequality between Hayashi's and Kao's wavenumber-frequency energy equations was overlooked.

In Section 2, the interpretations of the two different formulations of the wavenumber-frequency energy equations are discussed. A summary and remarks are given in Section 3.

2. Space-time spectral energy equations

For the sake of elucidating the differences between Hayashi's and Kao's formulations in a simple manner, the present argument will be based on the linearized equations of motions without a basic flow and dissipation.

These equations are expanded into a space-time Fourier series and each Fourier component satisfies the following equations:

$$\partial\tilde{u}/\partial t - f\tilde{v} = -\partial\tilde{\phi}/\partial x, \quad (1)$$

$$\partial\tilde{v}/\partial t + f\tilde{u} = -\partial\tilde{\phi}/\partial y, \quad (2)$$

$$\partial\tilde{u}/\partial x + \partial\tilde{v}/\partial y + \partial\tilde{\omega}/\partial p = 0, \quad (3)$$

$$\partial\tilde{\phi}/\partial p = -\alpha, \quad (4)$$

where the tilde denotes the space-time Fourier *component* which varies with x and t as $\tilde{u}(x, t) = \hat{u}(k, \sigma) \exp(ikx + i\sigma t)$, where $\hat{u}(k, \sigma)$ is the space-time Fourier *transform*. The real part of the complex Fourier component represents a physically meaningful quantity.

Multiplying (1) and (2) by the complex conjugates $\tilde{u}^*$ and $\tilde{v}^*$ respectively, and adding them, we have

$$\tilde{u}^*\partial\tilde{u}/\partial t + \tilde{v}^*\partial\tilde{v}/\partial t - 2if\,\mathrm{Im}(\tilde{u}^*\tilde{v}) = -\tilde{u}^*\partial\tilde{\phi}/\partial x - \tilde{v}^*\partial\tilde{\phi}/\partial y. \quad (5)$$

The right-hand side of (5) can be rewritten by use of (3) and (4) as

$$-\tilde{u}^*\partial\tilde{\phi}/\partial x - \tilde{v}^*\partial\tilde{\phi}/\partial y = -\partial(\tilde{v}^*\tilde{\phi})/\partial y - \partial(\tilde{\omega}^*\tilde{\phi})/\partial p - \tilde{\omega}^*\tilde{\alpha}, \quad (6)$$

with use of the identity

$$\tilde{u}^*\partial\tilde{\phi}/\partial x = \partial(\tilde{u}^*\tilde{\phi})/\partial x - (\partial\tilde{u}^*/\partial x)\tilde{\phi}.$$

Hayashi's energy equation is derived by taking the *real part* of (5) along with (6) as

$$\tfrac{1}{2}\partial(|\tilde{u}|^2 + |\tilde{v}|^2)/\partial t = -\partial\,\mathrm{Re}(\tilde{v}^*\tilde{\phi})/\partial y - \partial\,\mathrm{Re}(\tilde{\omega}^*\tilde{\phi})/\partial p - \mathrm{Re}(\tilde{\omega}^*\tilde{\alpha}), \quad (7)$$

with use of the identity

$$\partial(\tilde{u}^*\tilde{u})/\partial t = 2\,\mathrm{Re}(\tilde{u}^*\partial\tilde{u}/\partial t).$$

The relation (7) can also be derived by first taking the real part of (1)–(4) and then taking only the zonal (not time) average [ ] of the resulting energy equation by use of the identity

$$[\mathrm{Re}\tilde{u}\,\mathrm{Re}\tilde{v}] = \mathrm{Re}(\tilde{u}^*\tilde{v})/2.$$

The relation (7) corresponds to the linear parts of Hayashi's energy equation. The left-hand side can be rewritten as $-\sigma_i(|\tilde{u}|^2 + |\tilde{v}|^2)$ where σ_i is the imaginary part of the complex frequency. This σ_i is actually zero for the space-time Fourier components, since their kinetic energy is constant with time. $\mathrm{Re}(\tilde{\omega}^*\tilde{\alpha})$ is proportional to the space-time cospectrum between ω and α and can be estimated by use of a conventional time-cross spectral analysis such as that proposed by Hayashi (1971). It follows that a wavenumber-frequency integration of (7) coincides with the conventional linearized perturbation kinetic energy equation averaged over x and t. If the space-time Fourier components are replaced by the space-Fourier components, (7) would correspond to the linear part of the wavenumber spectral energy equation formulated by Saltzman (1957).

On the other hand, taking the imaginary part of (5) along with (6) gives

$$(\partial\theta/\partial t)(|\tilde{u}|^2 + |\tilde{v}|^2) = 2f\,\mathrm{Im}(\tilde{u}^*\tilde{v}) - \partial\,\mathrm{Im}(\tilde{v}^*\tilde{\phi})/\partial y - \partial\,\mathrm{Im}(\tilde{\omega}^*\tilde{\phi})/\partial p - \mathrm{Im}(\tilde{\omega}^*\tilde{\phi}), \quad (8)$$

with use of the identity $\mathrm{Im}(\tilde{u}^*\partial\tilde{u}/\partial t) = (\partial\theta/\partial t)|\tilde{u}|^2$, where θ is the phase defined by $\tilde{u} = |\tilde{u}|\exp(i\theta)$ and $\partial\theta/\partial t = \sigma$.

Thus the real (7) and imaginary (8) parts govern the growth rate and frequency (time derivative of amplitude and phase), respectively.

Next, Kao's (1968) energy equation can be obtained by replacing $\partial/\partial t$ and $\partial/\partial x$ in (5) by $i\sigma$ and ik respectively and dividing by $i\sigma$ as

$$|\tilde{u}|^2 + |\tilde{v}|^2 = 2(f/\sigma)\,\mathrm{Im}(\tilde{u}^*\tilde{v}) - (k/\sigma)\tilde{u}^*\tilde{\phi} - \tilde{v}^*\partial\tilde{\phi}/\partial y/(i\sigma) \quad (9a)$$

$$= f/(i\sigma)[\tilde{u}^*(\tilde{v} - \tilde{v}_g) - \tilde{v}^*(\tilde{u} - \tilde{u}_g)], \quad (9b)$$

where u_g and v_g are the geostrophic components of the wind.

This relation (9) corresponds to the linear (ageostrophic) parts of Kao's energy equation. The terms on the right-hand side are complex valued. However, the imaginary parts of these terms cancel each other.

Due to the division by the imaginary frequency $i\sigma$ in deriving (9a) from (5), the *real* part of (9a) virtually corresponds to the *imaginary* part of (5) and is reduced by use of (6) to

$$|\tilde{u}|^2 + |\tilde{v}|^2 = 2(f/\sigma)\,\mathrm{Im}(\tilde{u}^*\tilde{v}) - \partial\,\mathrm{Im}(\tilde{v}^*\tilde{\phi})/\partial y/\sigma - \partial\,\mathrm{Im}(\tilde{\omega}^*\tilde{\phi})/\partial p/\sigma - \mathrm{Im}(\tilde{\omega}^*\tilde{\alpha})/\sigma. \quad (10)$$

The terms on the right-hand side are proportional to the space–time quadrature spectra as pointed out by Chiu (1970).

It follows from (10) that the kinetic energy is related to the quadrature (90° out of phase) correlation between vertical velocity and temperature, since

$$\mathrm{Im}(\tilde{\omega}^*\tilde{\alpha}) = |\tilde{\alpha}||\tilde{\omega}|\cos[\theta_\alpha - \theta_\omega + (\pi/2)].$$

It also follows from (10) that the kinetic energy is related to the Coriolis force as noted by Chiu (1970). Although these conclusions are difficult to interpret physically, they are mathematically correct and not contrary to the principles of physics. Nevertheless, Kao's energy equation (10) cannot be physically interpreted as describing how spectral energy is maintained, since his equation does not describe the time change of energy.

The above argument can be clarified by rewriting Hayashi's energy equation (7) in such a way that the left-hand side represents the spectral energy rather than its time derivative as follows.

Integrating (7) with respect to time gives

$$E(t) = E(0) + \int_0^t P dt - \int_0^t D dt, \quad (11)$$

where $E(t)$ is the spectral energy at time t, while P and D represent the energy production and destruction terms, respectively.

Eq. (11) states that $E(t)$ is not altered from its initial value $E(0)$, when P and D cancel each other or when both P and D are zero. In other words (11) describes how spectral energy is maintained.

On the other hand, Kao's equation (9), which has been divided by $i\sigma$, *cannot* be interpreted as the time integration of (5), although this division corresponds to a time integration of (1) and (2). This is because (9) has been derived by multiplying (1) and (2) by $\tilde{u}^*/(i\sigma)$ and $\tilde{v}^*/(i\sigma)$, respectively. Since Kao's equation does not take the form of (11), it cannot be interpreted as describing how spectral energy is maintained.

When the right-hand side terms of (7) are balanced with the Rayleigh friction $-\kappa(|\tilde{u}|^2 + |\tilde{v}|^2)$, Eq. (7) can be rewritten as

$$|\tilde{u}|^2 + |\tilde{v}|^2 = -\partial\,\mathrm{Re}(\tilde{v}^*\tilde{\phi})/\partial y/\kappa - \partial\,\mathrm{Re}(\tilde{\omega}^*\tilde{\phi})/\partial p/\kappa - \mathrm{Re}(\tilde{\omega}^*\tilde{\alpha})/\kappa. \quad (12)$$

This equation resembles Kao's equation (10) in form except that the imaginary part and σ are replaced by the real part and κ, respectively. It should be noted that Kao's equation (10) does not explicitly involve κ even when adding the Rayleigh friction. It is incorrect to interpret the terms on the right-hand side of (12) or (10) as representing the production, destruction or transfer of energy, since neither of these equations describes the time change of energy as in (11). Neither of them can explain why spectral energy is maintained at the level given by the terms of the right-hand side, since these terms must be estimated from data.

3. Summary and remarks

Interpretations are given of two different formulations of the space-time spectral energy equations derived by Hayashi (1980) and Kao (1968).

Mathematically, the two formulations are derived from the real and imaginary parts of the energy equation of space–time Fourier components which govern the growth rate and frequency (time derivative of amplitude and phase), respectively. Physically, Hayashi's formulation (7) is consistent with the conventional perturbation energy equation as well as Saltzman's (1957) wavenumber spectral energy equation. On the other hand, Kao's energy equation (9) cannot be physically interpreted as describing *how* energy is maintained, since his equation does not describe the time change of energy. In other words, Kao's equation is a diagnostic relation such as the well-known Eliassen-Palm relation between energy and momentum fluxes. This relation does not mean that energy flux is physically maintained by momentum flux or vice versa. Kao's equation merely states that, when energy is somehow maintained, it is equal to the sum or residue of the right-hand-side terms which do *not* represent the production, destruction or transfer of energy. His equation should not be interpreted as explaining why spectral energy is maintained at the level given by the terms on the right-hand side, since these terms must be estimated from data. Moreover, when the frequency approaches zero (stationary), the individual terms on the right hand of his equation which have been divided by the frequency become infinitely large, although their residue remains finite. For quasi-stationary waves the kinetic energy is equal to the small difference of very large terms.

It would be controversial to argue whether or not Kao's equation is a physically meaningful diagnostic relation. At any rate, his formulation should not be confused with Hayashi's formulation which describes how spectral energy is maintained or with the time-Fourier decomposition of the wavenumber energy equation as formulated by Kao (1980) to describe the evolution of wavenumber spectral energy.

It should be remembered that the cospectral terms contribute to the maintenance of the *amplitude* of traveling or stationary waves. These terms cancel each other in such a way that the left-hand side of the energy equation (7) governing the *time change* of the kinetic energy vanishes. On the other hand the quadrature spectral terms contribute to the maintenance of the *phase* of stationary waves. These terms cancel each other so that the left-hand side of the phase equation (8) governing the *time change* of phase vanishes.

In support of the present interpretations, the following comments were offered by one of the anonymous reviewers:

"This discussion of Kao's formulation of space–time spectral energy equations makes valid points and clarifications that should be published. Kao's interpretations of his own equations are clearly in error, and it is of importance to have these errors elucidated before they are perpetuated further. Perhaps these points can be made even more forcefully in the following ways:

1) Since by "maintenance" of energy we really mean "maintenance against *frictional dissipation*", it would be more illustrative to include Navier-Stokes friction, or more simply Rayleigh friction ($\mathbf{F} = -\kappa\mathbf{V}$) in Eqs. (1) and (2). By the analysis given in Section 2 it would then be clear that the viscous dissipation which is to be balanced by all other processes (e.g., $\omega\alpha$) appears only in the *co-spectral* component of Kao's equation and is absent from the quadrature part.

2) Hayashi correctly notes that the quadrature part of the spectral equation is really an equation for the frequency or rate of change of phase. If linearized advective terms of the form $U\partial u/\partial x$ and $U\partial v/\partial x$ (U = const) are included in (1) and (2), respectively, it will be clear from the development of Section 2 that these effects will vanish in the co-spectral equations but will appear in the quadrature equation as the term representing the advective time scale U/L (L = characteristic space scale). If $U = 0$ and $\nabla\phi = 0$ we have the conditions of inertial motion in which $\sigma = f$."

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REFERENCES

Chen, T. C., H. G. Marshall and J. Shukla, 1981: Spectral analysis and diagnosis of nonlinear interactions of large-scale moving waves at 200 mb of the GLAS general circulation model. *Mon. Wea. Rev.*, **109**, 959–974.

Chiu, W.-C., 1970: On the spectral equations and the statistical energy spectrum of atmospheric motions in the frequency domain. *Tellus*, **22**, 609–619.

Hayashi, Y., 1971: A generalized method of resolving disturbances into progressive and retrogressive waves by space Fourier and time cross-spectral analyses. *J. Meteor. Soc. Japan*, **49**, 125–128.

——, 1980: Estimation of nonlinear energy transfer spectra by the cross spectral method. *J. Atmos. Sci.*, **37**, 299–307.

Kao, S. K., 1968: Governing equations and spectra for atmospheric motion and transports in frequency, wave-number space. *J. Atmos. Sci.*, **25**, 32–38.

——, 1980: Equations of kinetic and available potential energy evolution in wave-number frequency space. *Pure Appl. Geophys.*, **118**, 867–879.

——, and C. N. Chi, 1978: Mechanism for the growth and decay of long- and synoptic-scale waves in the mid-troposphere. *J. Atmos. Sci.*, **35**, 1375–1387.

——, and H. N. Lee, 1977: The nonlinear interactions and maintenance of the large-scale moving waves in the atmosphere. *J. Atmos. Sci.*, **34**, 471–485.

Saltzman, B., 1957: Equations governing the energetics of the large scales of atmospheric turbulence in the domain of wave number. *J. Meteor.*, **14**, 513–523.