Space-Time Cross Spectral Analysis Using the Maximum Entropy Method

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Abstract

Space-time cross spectra are experimentally estimated from given sinusoidal waves by use of the multivariate maximum entropy method. This method gives not only power spectra but also cospectra, phase difference and coherence with fine frequency resolutions from short time records. As an example of its application, a space-time spectral analysis is made of external Rossby waves simulated by a GFDL spectral general circulation model.

1. Introduction

In previous papers (Hayashi, 1977; Hayashi and Golder, 1978, 1980) space-time power spectra have been estimated from short time records by the use of the maximum entropy method (MEM) proposed by Burg (1967) and reviewed by Ulrych and Bishop (1977), Hino (1977) and Childers (1978). Recently, the MEM was generalized by Nutall (1976), Strand (1977), Morf *et al.* (1978), and Jones (1978) to estimate not only power spectra but also cross spectra, phase and coherence.

In the present paper Morf *et al*'s formulation is used to estimate space-time cross spectra. Section 2 describes the principle of the maximum entropy cross spectra, while Section 3 summarizes space-time cross spectral formulas. Section 4 tests the proposed method with given sinusoidal waves, while Section 5 applies the proposed method to external Rossby waves simulated by a GFDL spectral general circulation model. Remarks are given in Section 6.

2. Maximum entropy cross spectra

In principle, the maximum entropy spectral distribution is determined by extrapolating the known lag correlations to an infinite lag in such a way that the entropy (a measure of information) is maximized. In practice, the MEM spectrum is estimated by predicting the available data to infinite time by an autoregressive fitting.

Multivariate autoregressive process of order M is given (see Jenkins and Watts, 1968, p. 473) by

$$x_t=\sum_{m=1}^{M}\alpha_m x_{t-m}+z_t\,, \tag{2.1}$$

where x_t is a vector process, z_t is a vector process of white noise and $\alpha_1\sim\alpha_M$ are the matrices of prediction error filter coefficients.

For a bivariate process $x_t=(u_t, v_t)$, u_t depends not only on u_{t-m} but also on v_{t-m}.

The cross spectral matrix $c_x(f)$ for a continuous frequency f of x_t is given by the Fourier transform of (2.1) as

$$c_x(f)/(2\Delta t)=\left(I-\sum_{m=1}^{M}\alpha_m e^{i2\pi mf}\right)^{-1} \times R\times\left(I-\sum_{m=1}^{M}\alpha_m{}^{*}e^{-i2\pi mf}\right)^{-1}, \tag{2.2}$$

where Δt is the time increment, I is a unit matrix and the asterisk denotes the complex conjugate transpose.

In the above, α_m and R (variance matrix of z_t) are determined from available data by a covariance fitting and prediction error minimization. The formulation proposed by Morf *et al.* (1978) and coded by Jones (1978) appears to be superior to other formulations which yield different cross spectra for forward and backward predictions.

According to Jones (1978), the optimum order (M) of the bivariate autoregressive process is less than 1/4 of the length of the record and is determined by minimizing Akaike's Information Criterion. However, from the author's experience, this criterion should not be strictly followed. Rather, the order should be regarded as proper if the smoothed spectral distribution is not drastically altered with a slight change in the order.

It is also of importance to note that an integration of the above MEM cross spectrum matrix $c_x(f)$ over positive and negative frequencies results in exactly twice the cross covariance matrix of the original data.

An alternative MEM cross spectral estimation proposed by Ulrych and Jensen (1974) often gives quite erroneous results. This is due to the fact that the multivariate time series are fitted to different single-variate autoregressive processes which are independent of each other. For example the cospectrum K_ω between u and v is estimated by use of the identity

$$2K_\omega(u,v)=P_\omega(u+v)-P_\omega(u)-P_\omega(v)\,. \quad (2.3)$$

However, the three power spectra on the right-hand side estimated by the MEM are not consistent with each other and can cause a large error.

3. Space-time cross spectra

First, the space complex Fourier transform of space-time series $w(x, t)$ is computed as

$$F_k(t)=\frac{1}{\pi}\int_0^{2\pi} w(x,t)e^{-ikx}dx \quad (3.1a)$$

$$=c_k(t)-is_k(t) \quad (3.1b)$$

where c_k and s_k are the space cosine and sine coefficients computed by a conventional method.[1]

The space-time MEM power spectrum of one set of a space-time series $w(x, t)$ is given (see Hayashi 1977, 1979) by

$$4P_{k,\pm\omega}(w)=P_{\pm\omega}(F_k)\,, \quad (3.2)$$

where $P_{\pm w}$ is the time MEM power spectrum of a complex time series. The positive and negative frequencies represent westward and eastward phase propagations, respectively.

Space-time MEM cross spectra between the progressive (or retrogressive) wave components of two sets of space-time series $w(x, t)$ and $w'(x, t)$ are given by

$$4P_{k,\pm\omega}(w)=P_{\pm\omega}(F_k)\,, \quad (3.3)$$

$$4P_{k,\pm\omega}(w')=P_{\pm\omega}(F_k')\,, \quad (3.4)$$

$$4K_{k,\pm\omega}(w,w')=K_{\pm\omega}(F_k,F_k')\,, \quad (3.5)$$

$$4Q_{k,\pm\omega}(w,w')=Q_{\pm\omega}(F_k,F_k')\,, \quad (3.6)$$

$$Ph_{k,\pm\omega}(w,w')=\tan^{-1}[Q_{k,\pm\omega}(w,w')/K_{k,\pm\omega}(w,w')]\,, \quad (3.7)$$

$$\mathrm{coh}^2{}_{k,\pm\omega}(w,w')=\frac{K^2{}_{k,\pm\omega}(w,w')+Q^2{}_{k,\pm\omega}(w,w')}{P_{k,\pm\omega}(w)P_{k,\pm\omega}(w')}\,, \quad (3.8)$$

where

$$P_{k,\pm\omega}\,,\ K_{k,\pm\omega}\,,\ Q_{k,\pm\omega}\,,\ Ph_{k,\pm\omega}\,,\ \mathrm{coh}_{k,\pm\omega}$$

are the space-time MEM power, cospectra, quadrature spectra, phase difference and coherence, respectively. $K_{\pm w}$, $Q_{\pm w}$ are the time MEM cospectra and quadrature spectra of the complex time series.[2]

The bivariate MEM power spectra (3.3) depends not only on w but also w'. This ambiguity can be avoided by using the bivariate power spectra only to compute coherence and replacing them by the univariate power spectra (3.2). In order to obtain stable estimates of phase and coherence, the cross spectra should be smoothed by a frequency band average before computing (3.7) and (3.8).

The MEM power spectra are non-negative and the coherence has a value between 0.0 and 1.0. An integration of the MEM space-time power and cospectra over positive and negative frequencies yields the exact space-time variance and covariance, respectively. If w and w' are exactly proportional to each other, pure sinusoidal waves or constant, the autoregressive process has a singularity. In this case, a small random noise must be added to the data to remove the singularity.

4. Test

In order to test the MEM space-time cross spectral method, the following two sets of space-time series are given.

$$w(x,t)=0.5\sin(kx+\omega_1 t)+[0.7\sin(kx+\omega_2 t)+\sin(kx-\omega_2 t)]+[r_1(t)\cos(kx)+r_2(t)\sin(kx)] \quad (4.1)$$

and

$$w'(x,t)=0.5\cos(kx+\omega_1 t)+[\sin(kx+\omega_2 t)+0.7\sin(kx-\omega_2 t)]+[r_3(t)\cos(kx)+r_4(t)\sin(kx)]\,, \quad (4.2)$$

[1] A modified Fourier transform (see Hayashi, 1980) should be used for correctly estimating space-time spectra from polar-orbiting satellite data.

[2] The computer code of the MEM cross spectra of complex time series written by the author is available upon request. It is also available from Dr. T. Maruyama at the Meteorological Research Institute, Tateno/Nagamine 1-1, Yatabe-cho, Tsukuba-gun, Ibaraki-ken 305, Japan.

where k corresponds to zonal wavenumber 1 and ω_1, ω_2 correspond to periods 5 and 20 days, respectively. r_1, r_2, r_3, r_4 are the time series of white noise which are added to remove the singularity and have a value between -0.1 and 0.1.

Fig. 1 shows the space-time cross spectra for wavenumber 1 estimated by the MEM and the conventional lag correlation method. The data length is taken to be 30 days with a 1 day increment. The order of the autoregressive process (MAXM) is shosen to be 7. The MEM cross spectra are smoothed by a frequency band average at intervals of 1/60 day^{-1} to obtain stable estimates. The maximum lag for the lag method is chosen to be 10 days. It is seen that the MEM gives much sharper spectral peaks than the lag method and, in fact, shows spectral peaks which are not detected by the lag method. This is also true with nonsinusoidal oscillations (see Hayashi, 1977, Fig. 2). For the present example, both methods give the correct phase difference (90° and 0° for 5 and 20 day periods, respectively) and show a high coherence for the given frequencies.

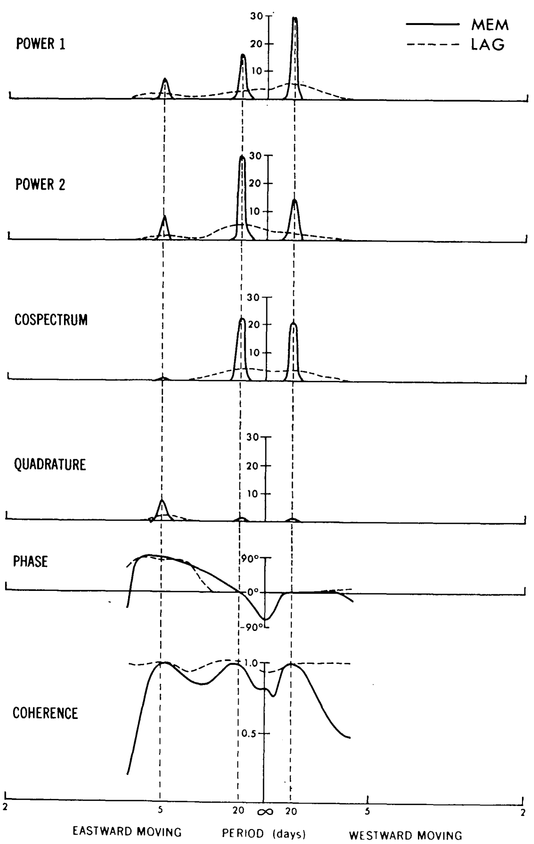

Fig. 1. Space-time cross spectra of sinusoidal waves with white noise estimated by the MEM (solid line) and by a lag correlation method (dashed line). See the text for detail.

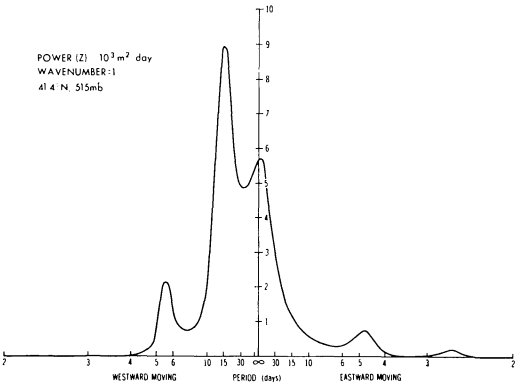

Fig. 2. Space-time power spectra density (wavenumber 1) of simulated geopotential height at 515 mb, 41.4°N in January.

5. An application

As reviewed by Madden (1979) and Walterscheid (1980), there is considerable observational evidence of westward moving pressure waves associated with wavenumber 1 and periods near 5 and 15 days. These waves are characterized by little phase variation in the vertical and attain their maximum and minimum amplitudes in the mid-latitudes and tropics, respectively. They are interpreted as external Rossby waves. These waves are likely to be a source of error in some numerical forecast models.

Hayashi (1974, Fig. 6f) found, among other equatorial planetary waves, spectral peaks at wavenumber 1 and westward moving periods of 5 and 15 days in the geopotential height over the equator in an 11-layer GFDL grid type general circulation model. However, these westward moving waves were not clearly detected in the mid-latitudes of the same model (Hayashi and Golder, 1977; Pratt, 1979).[3] It is of interest here to examine whether these waves are better simulated by a current 9-layer GFDL spectral model (Manabe *et al.* 1979) with 30 wavenumber components.

Space-time power spectra are estimated by use of the single-variate maximum entropy method,

[3] This negative conclusion was based on the lag correlation method. It has, however, been confirmed by the MEM.

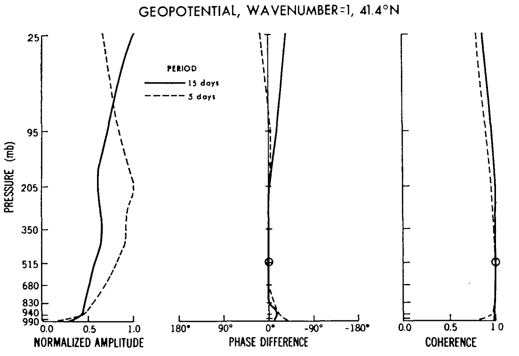

Fig. 3. Vertical structure of simulated geopotential height (wavenumber 1) at 41.4°N in January. Normalized amplitude (left), phase difference (middle), coherence (right) for westward moving periods of 15 days (solid) and 5 days (dashed). The reference level (515 mb) is indicated by open circles.

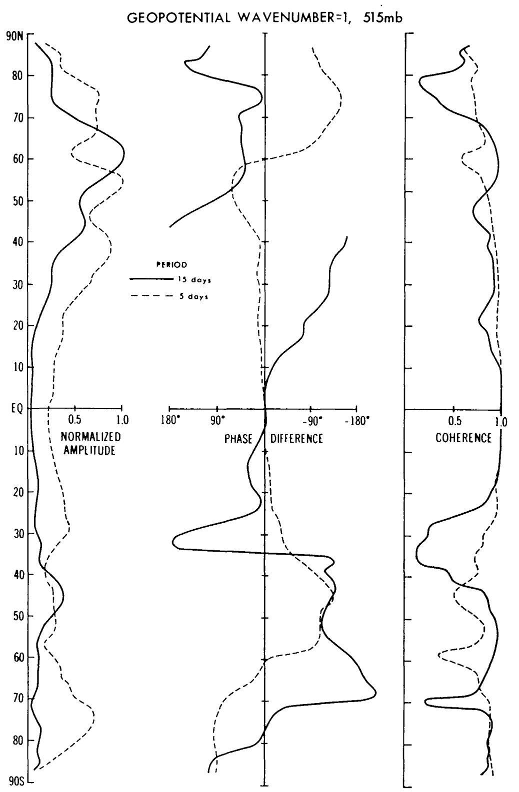

Fig. 4. Meridional structure of simulated geopotential height (wavenumber 1) at 515 mb in January. Normalized space-time amplitude (left), phase difference (middle), coherence (right) for westward moving periods of 15 days (solid) and 5 days (dashed). The reference latitude is 1.1°N.

while phase difference and coherence are estimated by the bivariate maximum entropy method.

Fig. 2 shows a power spectral density (wavenumber 1) of geopotential height at 515 mb, 41.4°N during January. Distinct spectral peaks are seen near 15 and 5 days, which correspond to those observed.

The vertical structure of these waves at 41.4°N is shown in Fig. 3. Both 15 and 5 day period waves exhibit little phase variation in the vertical in agreement with those observed.

The meridional structure of these waves at 515 mb is shown in Fig. 4. Both 15 and 5 day period waves attain their maximum and minimum amplitudes in the mid-latitudes and tropics, respectively. They are nearly symmetric in the tropics with respect to the equator.

6. Remarks

The MEM cross spectral analysis can be used to replace the conventional composite analysis to find the phase distribution of waves based on a short time record. The advantage of this spectral analysis is that it gives the additional information of coherence. Another advantage of space-time spectral analysis is that eastward and westward moving waves can be distinguished from each other.

The confidence limits of the MEM power spectra have been discussed by Reid (1979). However, his estimate of the upper confidence limit at times become infinite even if the estimated spectra are close to the known true values, while the lower confidence limit is always finite. One should regard an MEM analysis as preliminary until a statistical significance test is made of the conventional spectra of a longer time series.

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Maximum Entropy Method による時空間クロス・スペクトル解析法

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多変数最大エントロピー法により試験的に与えた正弦波動の時空間クロス・スペクトルを計算した。この方法は短い時系列から振動数分解能の良いパワ・スペクトルだけでなくコー・スペクトル, 位相差, コヒーレンスも与える。応用例として GFDL スペクトル大循環モデルの外部ロスビー波の時空間スペクトル解析を行った。