A Generalized Method of Resolving Transient Disturbances into Standing and Traveling Waves by Space-Time Spectral Analysis

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ABSTRACT

Space-time spectral formulas are generalized to partition the time power spectrum of transient disturbances consisting of multiple wavenumbers into standing and traveling parts by assuming that these parts are incoherent with each other.

This technique is useful in interpreting the spatial variation of wave amplitude in terms of standing and traveling waves. An example of its application to the analysis of transient planetary waves is given.

1. Introduction

In previous papers (Hayashi, 1971, 1973, 1977a,b, 1979) spectral formulas have been developed to compute space-time (wavenumber-frequency) spectra defined by Kao (1968) by use of time spectral techniques such as the lag correlation method, direct Fourier transform method and maximum entropy method. These formulas are generalizations of the formula given by Deland (1964) who found that the time quadrature spectrum between the zonal cosine and sine coefficients[1] gives the power spectrum of traveling waves. These formulas are also analogous to those of the rotary spectra of vector time series (see Hayashi, 1979).

The above space-time spectral formulas have been extensively applied to wave analysis of a GFDL general circulation model (Hayashi, 1974; Hayashi and Golder, 1977, 1978) and observational analysis (Gruber, 1974; Zangvil, 1975a,b; Hartmann, 1976; Sato, 1977; Fraedrich and Böttger, 1978; Depradine, 1978; Krishnamurti, 1978). However, these space-time spectral analyses do not properly isolate traveling waves from standing wave oscillations which consist of both progressive and retrogressive components interfering with each other to form nodes and antinodes. If standing and traveling waves are generated by different mechanisms, it is important to separate these waves. For this purpose, Hayashi (1977a) derived spectral formulas to partition a space-time power spectrum into "standing" and "traveling" parts which depend on the coherence (a measure of interference) between progressive and retrogressive components,[2] by assuming that these parts are incoherent with each other. These formulas give the zonal mean power spectra of a single wavenumber component.

In practice, Hayashi (1974) and Hayashi and Golder (1977) analyzed the spatial variation of the time amplitude of simulated transient disturbances by computing the time-power spectrum of a space-time series which is filtered in space by a zonal Fourier decomposition. Subsequently, a similar analysis of observed transient disturbances was made by Blackmon (1976), who computed a time variation of space-time series which are filtered both in space and time by spherical harmonics and time filtering. In both these analyses the zonal variation of the time-power spectrum is to some extent due to the nodes and antinodes of standing waves as well as the wave-wave interference of traveling waves consisting of multiple wavenumbers. In order to separate these two effects, we shall generalize the space-time spectral formulas of Hayashi (1977a) to partition the local (rather than zonal mean) power spectrum of transient waves consisting of multiple wavenumbers into "standing" and "traveling" parts.

In Section 2 standing and traveling waves are defined. In Section 3 formulas are derived to partition time power spectra into progressive and retrogressive parts or alternatively standing and traveling parts. In Section 4 formulas are given to compute time cross spectra

[1] These coefficients are 90° out of phase in time, if disturbances are either progressive or retrogressive waves. However, the reverse is not always true. In the presence of both traveling and standing waves, the phase difference can be 90° out of phase depending on the choice of the origin of the zonal coordinate. The quadrature spectrum, however, is invariant with a zonal translation (see Hayashi, 1979).

[2] Pratt (1976) found that the coherence between the zonal cosine and sine components gives a measure of the interference between the progressive and retrogressive component. However, this coherence is not a proper measure, since it generally depends on the origin of the zonal coordinate unlike the coherence between the progressive and retrogressive components formulated by Hayashi (1977a).

between progressive and retrogressive components. In Section 5 an example of its application is given. Appendix A derives spectral formulas given in Section 4. Appendix B describes a partition of power spectra into "wave" and "noise" parts.

2. Definitions of standing and traveling waves

a. Standing waves

Standing waves w^s are defined as transient waves with their nodes and antinodes fixed in space. If they are associated with time-modulation in amplitude and phase, they take the form

$$w^s(x,t)=A_k(t)\cos[kx+\phi_k]\cos[\omega t+\psi_k(t)], \quad (2.1a)$$

$$=2\sum_{\Delta\omega} A_{k,\omega}\cos(kx+\phi_k)\cos(\omega t+\psi_{k,\omega}), \quad (2.1b)$$

where the space phase ϕ_k is constant with time and frequency, while the amplitude $A_k(t)$ and time phase $\psi_k(t)$ vary slowly with time. The summation is taken over a narrow frequency band width $\Delta\omega$ ($\Delta\omega \ll \omega$).

These waves can be decomposed into progressive and retrogressive components with the same amplitude and the same frequency as

$$w^s(x,t)=\sum_{\Delta\omega} A_{k,\omega}\cos(kx+\omega t+\phi_k+\psi_{k,\omega})$$

$$+\sum_{\Delta\omega} A_{k,\omega}\cos(-kx+\omega t-\phi_k+\psi_{k,\omega}). \quad (2.2)$$

The time-coherence squared between the progressive and retrogressive components (Hayashi, 1977a) over this frequency band is reduced to

$$\mathrm{coh}_\omega^2(w^s)=\frac{[\sum_{\Delta\omega} A_{k,\omega}^2\cos(2\phi_k)]^2+[\sum_{\Delta\omega} A_{k,\omega}^2\sin(2\phi_k)]^2}{(\sum_{\Delta\omega} A_{k,\omega}^2)^2}. \quad (2.3)$$

This coherence is actually 1.0 since the space phase ϕ_k does not depend on frequencies.

On the other hand, if there is no physical cause for fixing nodes and antinodes in space, the space phase ϕ_k varies randomly with time in (2.1a) and with frequency in (2.1b). In this case the above coherence is close to zero if the length of time series is sufficiently large.

More generally, standing waves consist of multiple wavenumbers as well as multiple frequencies.

b. Traveling waves

Traveling waves w^t are defined as either progressive or retrogressive waves which are incoherent with each other. For example, if one wave travels back and forth or if one wave travels eastward in the Eastern Hemisphere and another wave travels westward in the Western Hemisphere at the same time, the coherence between progressive and retrogressive components will approach zero.

If the wave energy is dissipated away from its source or the basic state is not spatially uniform, traveling waves have nonuniform amplitude and are associated with multiple wavenumbers for the same frequency.

For a simple example, let us assume that traveling waves in a spatially nonuniform medium consist of two wavenumbers $k-\Delta k/2$ and $k+\Delta k/2$ as

$$w_{k,\omega}^t(x,t)=\cos\left[\left(k-\frac{\Delta k}{2}\right)x+\omega t\right]$$

$$+\cos\left[\left(k+\frac{\Delta k}{2}\right)x+\omega t\right], \quad (2.4a)$$

$$=2\cos\frac{\Delta k}{2}x\cos(kx+\omega t). \quad (2.4b)$$

The time power spectrum varies in space as

$$P_\omega(w_{k,\omega}^t)=2\cos^2\left(\frac{\Delta k}{2}x\right). \quad (2.5)$$

More generally, even in a uniform and stationary medium, traveling waves take the form of moving wave packets consisting of wavenumber-frequency pairs (k_1,ω_1) and (k_2,ω_2) which fall on a dispersion curve. If the energy source or medium is not spatially uniform, each pair is further associated with a broadening (Δk) in wavenumber independent of frequency, as illustrated by Fig. 1a for a simple example (Δk should not be confused with k_1-k_2).

The traveling waves take the form

$$w_{k_1,\omega_1}^t(x,t)+w_{k_2,\omega_2}^t(x,t)=4\cos\left(\frac{\Delta k}{2}x\right)$$

$$\times\cos\left(\frac{k_1-k_2}{2}x+\frac{\omega_1-\omega_2}{2}t\right)\cos\left(\frac{k_1+k_2}{2}x+\frac{\omega_1+\omega_2}{2}t\right). \quad (2.6)$$

The envelope function of these wave packets is given by

$$E(x,t)=4\cos\left(\frac{\Delta k}{2}x\right)\cos\left(\frac{k_1-k_2}{2}x+\frac{\omega_1-\omega_2}{2}t\right). \quad (2.7)$$

The group velocity is given by

$$C_g=-\frac{\omega_1-\omega_2}{k_1-k_2}. \quad (2.8)$$

The time power spectrum varies in space as

$$P_\omega(w_{k_1,\omega_1}+w_{k_2,\omega_2})=4\cos^2\left(\frac{\Delta k}{2}x\right), \quad \text{for} \quad C_g\neq 0. \quad (2.9)$$

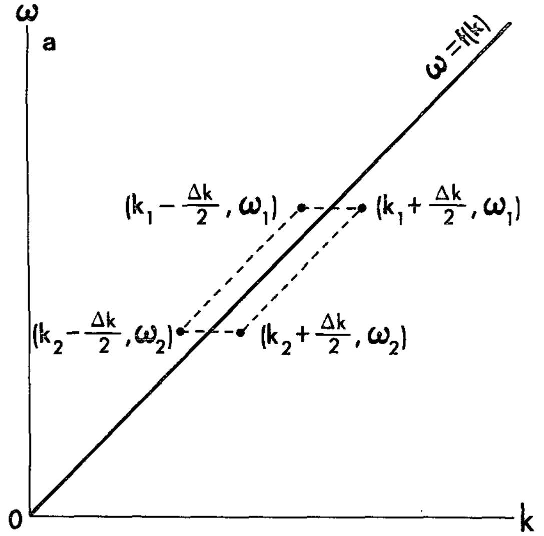

FIG. 1a. Schematic diagram of wavenumber-frequency pairs (dots) in a spatially non-uniform medium. Straight line represents a dispersion curve in a uniform medium. On a spatially nonuniform medium, a broadening (Δk) occurs only in wavenumber.

Thus, the time-power spectrum can also be interpreted as the "envelope" of the envelope given by (2.7) as illustrated by Fig. 1b, for a fixed time, provided that

$$\Delta k \ll |k_1 - k_2| \ll |k_1 + k_2|. \tag{2.10}$$

This condition, however, is not very well satisfied by planetary waves with small discrete wavenumbers such as $\Delta k = 1$, $k_1 = 1$, $k_2 = 3$. In this case, the time-power spectrum is interpreted as merely the time-amplitude squared of the oscillation caused by traveling waves.

c. Partition into standing and traveling waves

In order to partition waves into standing and traveling waves uniquely some assumptions are necessary as discussed by Deland (1972) and Tsay (1974).

In the present paper we make the assumptions (i) and the definitions (ii) and (iii):

(i) It is assumed that the space-time series w over some wavenumber and frequency range can be partitioned into standing w^s and traveling w^t components as

$$w(x,t) = w^s(x,t) + w^t(x,t). \tag{2.11}$$

It is further assumed that these components are incoherent with each other as

$$\mathrm{coh}_f(w^s, w^t) = 0. \tag{2.12}$$

This is true if these components are generated by different causes. For example, standing planetary waves are forced waves, while traveling planetary waves may be free or unstable waves. This assumption may not be true if traveling waves appear as a result of different vertical propagation of the progressive and retrogressive components of standing waves forced from below as discussed theoretically by Hirota (1971). In this case it may not be meaningful to isolate traveling waves from standing waves, since they are a single phenomenon. However, it is still meaningful to partition these waves into progressive and retrogressive components, since they are associated with different vertical propagation characteristics.

(ii) The standing component w^s is defined as consisting of progressive and retrogressive components as

$$w^s(x,t) = w_+^s(x,t) + w_-^s(x,t), \tag{2.13}$$

where these components are of the same magnitude and

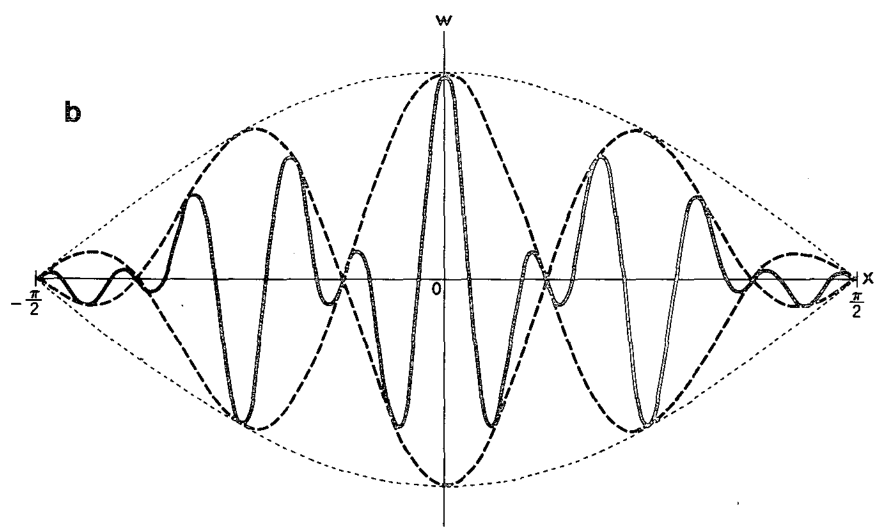

FIG. 1b. Wave packet (solid curve) at $t=0$ of the form $w(x,0) = \cos(x)\cos(4x)\cos(16x)$. The dashed curve is the envelope, while the dotted curve is the "envelope" of the envelope.

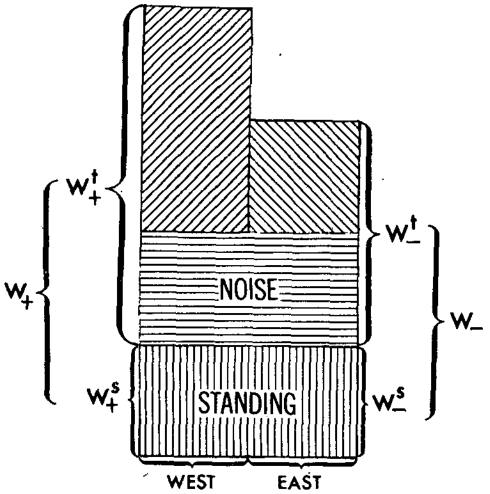

FIG. 2. Schematic diagram of standing ($w^s_\pm$) and traveling ($w^t_\pm$) components. See text for explanation.

coherent with each other as

$$P_f(w^s_+) = P_f(w^s_-), \tag{2.14}$$

$$\mathrm{coh}_f(w^s_+, w^s_-) = 1. \tag{2.15}$$

(iii) The traveling components w^t is defined as consisting of progressive and retrogressive components as

$$w^t(x,t) = w^t_+(x,t) + w^t_-(x,t), \tag{2.16}$$

where these components are incoherent with each other as

$$\mathrm{coh}_f(w^t_+, w^t_-) = 0. \tag{2.17}$$

It should be remembered that the traveling components may in part be due to random noise which contributes to both the progressive and retrogressive components equally as shown schematically by Fig. 2.

3. Partition of time power spectrum

In this section, spectral formulas are derived to partition the time-power spectrum of transient disturbances into "progressive," "retrogressive" and "interference" parts or alternatively, "standing" and "traveling" parts.

a. Progressive and retrogressive parts

It is assumed that $w(x,t)$ is a stationary random space-time series which is cyclic in longitude x and extends infinitely in time t.

In principle, this space-time series can be resolved into the progressive (w_-), retrogressive (w_+) and zonal mean (w_0) components as follows, although this decomposition is not necessary for actual computation of their power spectra:

$$w(x,t) = w_+(x,t) + w_-(x,t) + w_0(t), \tag{3.1}$$

where

$$w_\pm(x,t) = 2\ \mathrm{Re} \sum_{k=1}^{\infty} \int_0^\infty e^{i(\pm kx + 2\pi ft)} d\hat{W}_{\pm k}(f). \tag{3.2}$$

Here, $W_{\pm k}(t)$ is the space-Fourier transform of $w(x,t)$, while $\hat{W}_{\pm k}(f)$ denotes the Fourier-Stieltjes transform of $W_{\pm k}(t)$ (see Yaglom, 1962, Lumley and Panofsky, 1964). The increment $d\hat{W}_{\pm k}(f)$ is interpreted as the complex space-time amplitude associated with an infinitesimal frequency increment df. In practice, the summation is taken over a wavenumber range of interest.

The time-power spectrum at longitude x of transient waves consisting of both progressive and retrogressive components can be partitioned into three parts by the identity

$$P_f(w_+ + w_-) = P_f(w_+) + P_f(w_-) + 2K_f(w_+, w_-), \tag{3.3}$$

where P_f and K_f are the power spectrum and cospectrum, respectively.

Hereafter the above three parts are called the retrogressive, progressive and interference parts, respectively.

If a zonal mean of (3.3) is taken, the interference part vanishes as

$$\overline{P_f(w_+ + w_-)}^x = \sum_{k=1}^{\infty} P_{f,k}(w) + \sum_{k=1}^{\infty} P_{f,-k}(w), \tag{3.4}$$

where $P_{f,\pm k}$ are the space-time power spectrum of individual wavenumber components.

Similarly, the time-power spectrum of multiple wavenumbers can be partitioned into single wave and wave-wave interference parts. These wave-wave interference parts vanish if they are zonally averaged.

The cospectrum K_f and quadrature spectrum Q_f are interpreted as

$$K_f(w_+, w_-) = P_f^{\frac{1}{2}}(w_+) P_f^{\frac{1}{2}}(w_-)\ \mathrm{coh}_f(w_+, w_-) \times \cos[\mathrm{Ph}_f(w_+, w_-)], \tag{3.5}$$

and

$$Q_f(w_+, w_-) = P_f^{\frac{1}{2}}(w_+) P_f^{\frac{1}{2}}(w_-)\ \mathrm{coh}_f(w_+, w_-) \times \sin[\mathrm{Ph}_f(w_+, w_-)], \tag{3.6}$$

where the coherence coh_f and phase difference Ph_f are defined by

$$P_f^{\frac{1}{2}}(w_+) P_f^{\frac{1}{2}}(w_-)\ \mathrm{coh}_f(w_+, w_-) = [K_f^2(w_+, w_-) + Q_f^2(w_+, w_-)]^{\frac{1}{2}}, \tag{3.7}$$

$$\tan[\mathrm{Ph}_f(w_+, w_-)] = Q_f(w_+, w_-)/K_f(w_+, w_-). \tag{3.8}$$

Thus, the interference part $2K_f(w_+, w_-)$ in (3.3) represents an interference between w_+ and w_- and vanishes when the coherence is zero or the phase difference is 90°. It takes positive (negative) values at the antinodes (nodes).

b. Standing and traveling parts

In the above partition, the progressive and retrogressive parts involve contributions from standing waves and the interference part takes negative values at nodes. Alternatively, the power spectrum of transient disturbances can be partitioned into standing and traveling parts by assuming that these parts are incoherent with each other as

$$P_f(w^s+w^t)=P_f(w^s)+P_f(w^t)+2K_f(w^s,w^t), \quad (3.9a)$$

$$=P_f(w^s)+P_f(w^t). \quad (3.9b)$$

These parts are further partitioned into progressive and retrogressive parts as

$$P_f(w^s)=P_f(w^s_+)+P_f(w^s_-)+2K_f(w^s_+,w^s_-), \quad (3.10)$$

$$P_f(w^t)=P_f(w^t_+)+P_f(w^t_-). \quad (3.11)$$

By use of the definitions and assumptions of standing and traveling components discussed in Section 2c, the matrix of cross spectra between progressive and retrogressive components can be partitioned into standing and traveling parts, respectively as

$$\begin{bmatrix} P_+ & R \\ R^* & P_- \end{bmatrix}=\begin{bmatrix} |R| & R \\ R^* & |R| \end{bmatrix}+\begin{bmatrix} P_+-|R| & 0 \\ 0 & P_--|R| \end{bmatrix}, \quad (3.12)$$

where the asterisk denotes the complex conjugate. Here the diagonal elements $P_\pm$ represent power spectra as

$$P_\pm=P_f(w_\pm), \quad (3.13)$$

while the off-diagonal elements R and R^* represent complex cross spectrum as

$$R=K_f(w_+,w_-)+iQ_f(w_+,w_-). \quad (3.14)$$

Since the coherence between the progressive and retrogressive components of standing waves is equal to 1.0, the determinant the matrix of standing waves vanishes as

$$P_f^{\frac{1}{2}}(w^s_+)P_f^{\frac{1}{2}}(w^s_-)-[K_f^2(w^s_+,w^s_-)+Q_f^2(w^s_+,w^s_-)]^{\frac{1}{2}}=0. \quad (3.15)$$

Since the coherence between the progressive and retrogressive components of traveling waves is zero, the off-diagonal elements of the traveling part vanish as

$$K_f^2(w^t_+,w^t_-)+Q_f^2(w^t_+,w^t_-)=0. \quad (3.16)$$

The standing and traveling parts are analogous to the rectilinear and non-rectilinear parts of rotary spectra (see Appendix D of Hayashi, 1979). These partitions are somewhat similar to the empirical orthogonal decomposition of space-time cross spectrum proposed by Pratt and Wallace (1976).

The matrix representation (3.12) with (3.15) and (3.16) gives

$$P_f(w^s_\pm)=[K_f^2(w_+,w_-)+Q_f^2(w_+,w_-)]^{\frac{1}{2}}, \quad (3.17a)$$

$$=P_f^{\frac{1}{2}}(w_+)P_f^{\frac{1}{2}}(w_-)\,\mathrm{coh}_f(w_+,w_-), \quad (3.17b)$$

$$P_f(w^t_\pm)=P_f(w_\pm)-P_f(w^s_\pm), \quad (3.18a)$$

$$=P_f(w_\pm)-P_f^{\frac{1}{2}}(w_+)P_f^{\frac{1}{2}}(w_-)\,\mathrm{coh}_f(w_+,w_-). \quad (3.18b)$$

Thus, formulas for computing traveling and standing parts are given by use of (3.18b) as

$$P_f(w^t)=P_f(w^t_+)+P_f(w^t_-), \quad (3.19a)$$

$$=[P_f^{\frac{1}{2}}(w_+)-P_f^{\frac{1}{2}}(w_-)]^2+2P_f^{\frac{1}{2}}(w_+)P_f^{\frac{1}{2}}(w_-)[1-\mathrm{coh}_f(w_+,w_-)], \quad (3.19b)$$

$$P_f(w^s)=P_f(w_++w_-)-P_f(w^t), \quad (3.20a)$$

$$=2P_f^{\frac{1}{2}}(w_+)P_f^{\frac{1}{2}}(w_-)\,\mathrm{coh}_f(w_+,w_-)\times\{1+\cos[\mathrm{Ph}_f(w_+,w_-)]\}, \quad (3.20b)$$

where (3.3), (3.5) and (3.19b) have been used to derive (3.20b). It should be noted that w need not be explicitly partitioned into w^t and w^s or w_+ and w_- in order to compute the power spectra of these components. For a single wavenumber, formulas (3.17b) and (3.18b) are reduced to those derived by Hayashi (1977a).

The standing part as expressed by (3.20b) is similar to the interference part given by (3.5) except that the former becomes zero at the nodes, while the latter takes a negative value. The traveling part as expressed by (3.19b) vanishes when the progressive and retrogressive components are of the same magnitude and coherent with each other. Both these parts are non-negative. However, the progressive and retrogressive components of traveling parts (3.18b) are not always non-negative as will be discussed below.

In a special case where the traveling component consists of only a retrogressive (or progressive) component as

$$P_f(w^t_-)=0, \quad (3.21)$$

we have from (3.18a) and (2.14)

$$P_f(w^s_+)=P_f(w^s_-)=P_f(w_-), \quad (3.22)$$

$$\mathrm{coh}_f(w_+,w_-)=P_f^{\frac{1}{2}}(w_-)/P_f^{\frac{1}{2}}(w_+). \quad (3.23)$$

Then the formulas (3.18a) and (3.20b) are reduced to

$$P_f(w^t_+)=P_f(w_+)-P_f(w_-), \quad (3.24)$$

$$P_f(w^s)=2P_f(w_-)\{1+\cos[\mathrm{Ph}_f(w_+,w_-)]\}. \quad (3.25)$$

These formulas coincide with their conventional definitions of traveling and standing parts.[3]

[3] In order that the conventional definitions hold, it is not necessary that the coherence between the progressive and retrogressive component (3.23) be 1.0, since standing and traveling waves are not necessarily coherent with each other.

On the other hand, it follows from (3.18b) that if

$$\mathrm{coh}_f^2(w_+,w_-) > P_f(w_\pm)/P_f(w_\mp), \text{ then } P_f(w^t_\pm) < 0. \quad (3.26)$$

This negative value of power spectrum occurs when the assumptions (i) do not hold. In this case additional terms appear in (3.17) and (3.18) due to the interference between standing and traveling components. For example, a negative value occurs when the traveling component consists of only one component which is not completely incoherent with the standing component. Also it occurs for a short record, if the frequency band is taken to be too narrow, resulting in overestimation of the true coherence [See Foster and Guinzy (1967) and Julian (1975) for the statistical significance of coherence]. However, if this negative value is small compared to the other parts it can be regarded as zero. If it is not negligible, the present partition is not physically meaningful.

4. Cross spectrum between progressive and retrogressive components

a. Formulas for the cross spectrum

The time cross spectra between progressive and retrogressive components with multiple wavenumbers which are used in the previous sections can be computed by use of the following formulas without explicitly decomposing w into w_+ and w_-. These formulas are generalizations of those given by Hayashi (1971) for a single wavenumber and are analogous to those of rotary spectra (see Appendix A of Hayashi, 1979). The derivation is given in Appendix A.

$$P_f(w_+ + w_-) = P_f(c), \quad (4.1)$$

$$P_f(w_\pm) = \tfrac{1}{4}[P_f(c) + P_f(s) \pm 2Q_f(c,s)], \quad (4.2)$$

$$K_f(w_+,w_-) = \tfrac{1}{4}[P_f(c) - P_f(s)], \quad (4.3)$$

$$Q_f(w_+,w_-) = \tfrac{1}{2}K_f(c,s), \quad (4.4)$$

$$\mathrm{coh}_f^2(w_+,w_-) = \frac{K_f^2(w_+,w_-) + Q_f^2(w_+,w_-)}{P_f(w_+)P_f(w_-)}, \quad (4.5)$$

$$\mathrm{Ph}_f(w_+,w_-) = \tan^{-1}[Q_f(w_+,w_-)/K_f(w_+,w_-)], \quad (4.6)$$

where

$$w(x,t) = \sum_{k=0}^{\infty} C_k(t)\cos kx + S_k(t)\sin kx, \quad (4.7a)$$

$$= \sum_{k=0}^{\infty} A_k(t)\cos[-kx + \phi_k(t)], \quad (4.7b)$$

$$c(x,t) = \sum_{k=1}^{\infty} C_k(t)\cos kx + S_k(t)\sin kx, \quad (4.8a)$$

$$= \sum_{k=1}^{\infty} A_k(t)\cos[-kx + \phi_k(t)], \quad (4.8b)$$

$$s(x,t) = \sum_{k=1}^{\infty} -C_k(t)\sin kx + S_k(t)\cos kx, \quad (4.9a)$$

$$= \sum_{k=1}^{\infty} A_k(t)\sin\{-kx + \phi_k(t)\}. \quad (4.9b)$$

b. Computational procedure

The computational procedure for the partition of the time power spectrum is as follows:

1) Compute the zonal cosine and sine coefficients C_k and S_k as defined by (4.7a).

2) Compute $c(x,t)$ and $s(x,t)$ by use of (4.8a) and (4.9a) over wavenumber range of interest.

3) Compute the time cross spectra between $c(x,t)$ and $s(x,t)$ by the lag correlation method or the direct Fourier transform method (see Bendat and Piersol, 1971). These cross spectra should be integrated over a sufficiently wide frequency band in order to avoid overestimation of coherence.

4) Compute the cross spectra between progressive and retrogressive components by use of formulas (4.1)–(4.6).

5) The time power spectrum is partitioned into "progressive," "retrogressive" and "interference" parts by use of (3.3), or alternatively into "standing" and "traveling" parts by use of (3.20b) and (3.19b). The traveling part is further partitioned into progressive and retrogressive parts by use of (3.18b).

5. Example of application

As an example, the present method is applied to an analysis of transient planetary waves appearing in a GFDL general circulation model (Manabe and Mahlman, 1976). The three-dimensional structure of these simulated waves has been analyzed in detail by Hayashi and Golder (1977). The output data analyzed in the present paper are the geopotential height at 38 mb level during the period October–March.

Fig. 3 shows a wavenumber-frequency diagram of the space-time power spectrum. It is seen that transient planetary waves consist typically of wavenumbers 1–3 and periods of 10–60 days. The eastward moving component has a larger amplitude than the westward moving component and is associated with a spectral peak at wavenumber 1 and 30 days. In the following, wavenumbers 1–3 and periods 20–30 days were chosen. It was confirmed that a choice of wider period range does not significantly alter the ratio between the standing and traveling parts.

Fig. 4 compares the stationary (6-month mean) planetary waves with the power spectrum of transient planetary waves consisting of both eastward and westward moving components. It is seen that the power spectrum attains its major and minor maxima where the high and low of the stationary pattern are situated, respectively. This comparison suggests the possibility

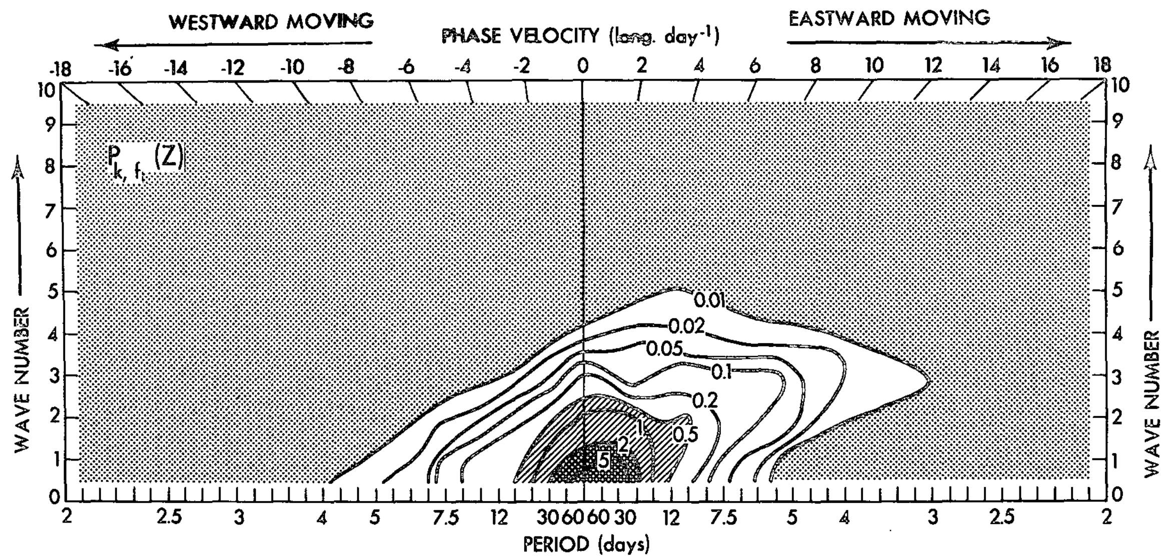

FIG. 3. Wavenumber-frequency diagram of space-time power spectrum density (10^5 m² day) of geopotential height at 38 mb at 60°N of a GFDL general circulation model during the period October–March.

that the geographical distribution of the power spectrum of the transient planetary waves are caused by the pulsation of quasi-stationary planetary waves. However, there is another possibility that this distribution is due to traveling waves consisting of multiple wavenumbers. In order to clarify these two possibilities, the above power spectrum is first partitioned into eastward, westward and interference parts.

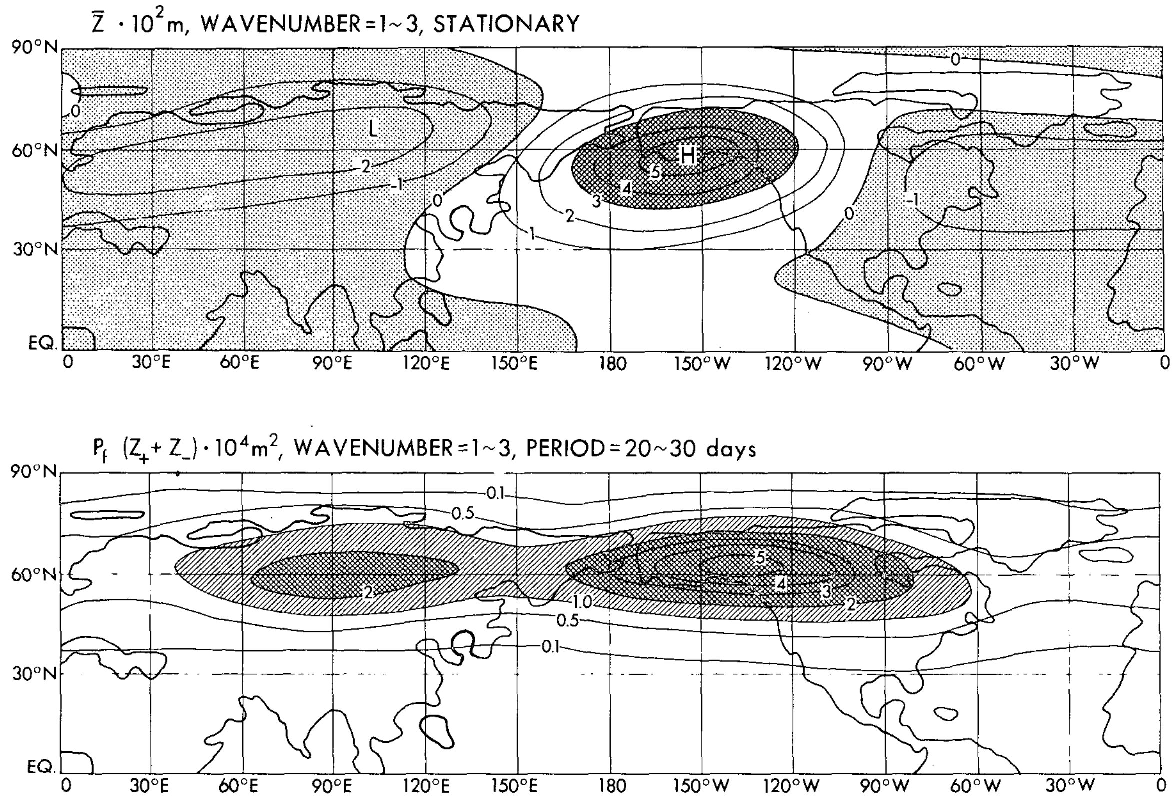

FIG. 4. Longitude-latitude section of time mean (upper) 10^2 m and time power spectrum (10^4 m²) with periods 20–30 days (lower) of geopotential height consisting of wavenumbers ~1–3.

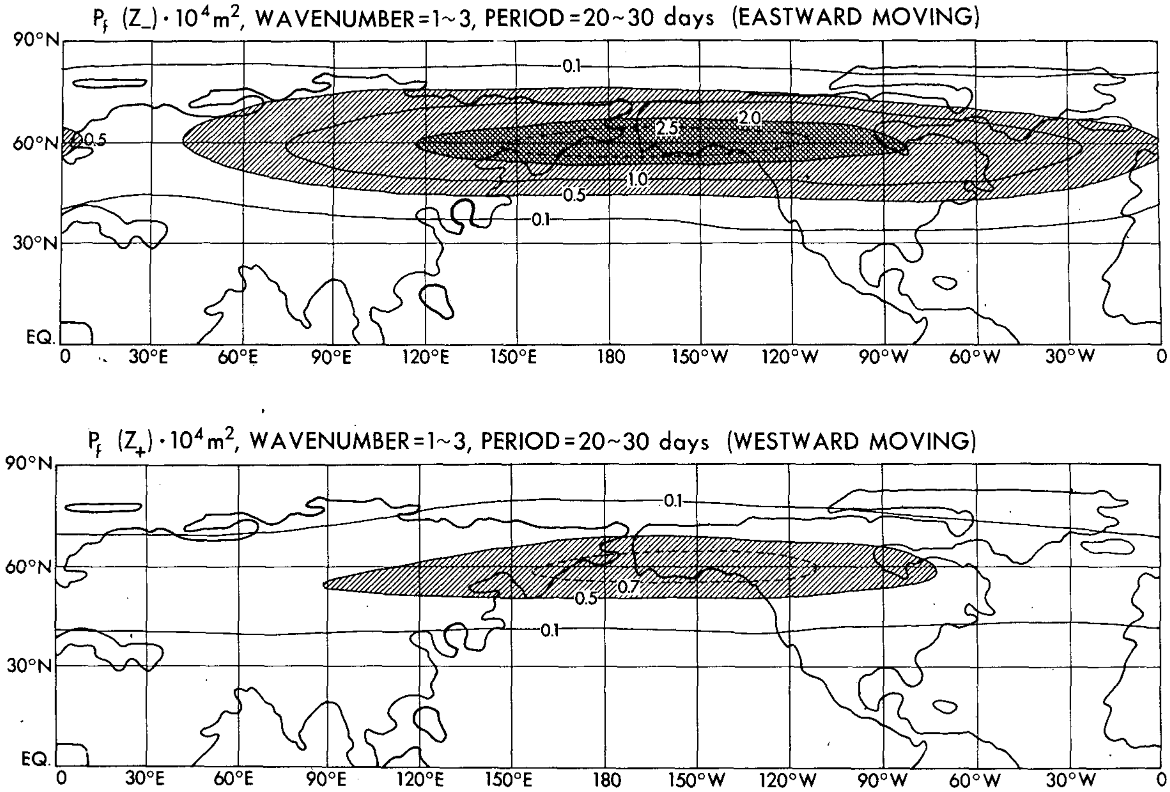

FIG. 5. Longitude-latitude section of time power spectrum (10^4 m^2) with periods 20–30 days of eastward (upper) and westward (lower) moving components.

a. Progressive and retrogressive parts

Fig. 5. compares the power spectra of the eastward and westward moving components. Both these power spectra attain only one major maximum near 180° longitude which is about 30° to the west of the major maximum in Fig. 4. The eastward moving component is larger than the westward moving component. It should be remembered that these components are not isolated from those of standing wave oscillations.

Fig. 6 shows the interference part (cospectrum×2) and the coherence between eastward and westward moving components. It is seen that the negative (nodes) and positive (antinodes) values of the interference part occur approximately where the minima and maxima of the power spectrum of transient planetary waves in Fig. 4 are situated. The coherence exceeds 0.5 poleward of 45°N.

b. Standing and traveling parts

In order to isolate traveling waves from standing waves, the power spectrum is repartitioned into standing and traveling parts.

Fig. 7 shows that the standing part is associated with one major and one minor maximum and is larger than the traveling part with only one maximum. Thus, the major maximum and minor maximum of the power spectrum of transient planetary waves in Fig. 4 is due more to the standing waves than the traveling waves.

The traveling part is further partitioned into eastward and westward moving parts. Fig. 8 shows that the positive and negative values of the westward moving part are negligible compared to the eastward moving part which is positive everywhere. This result does not reject the basic assumption that standing and traveling waves are incoherent. This small negative value occurs due to an overestimation of coherence [see Eq. (3.26)], since it is further reduced by taking a wider frequency range (not illustrated). Thus in the present example, the traveling and standing parts defined by (3.18) and (3.17) coincide with the conventional interpretations (3.24) and (3.25), respectively.

The above spectral results are visualized by longitude-time sections (Fig. 9) at 60°N consisting of wavenumbers 1–3. The stationary ridge and trough are seen in Fig. 9a, while standing wave oscillations and eastward moving waves are more clearly seen in Fig. 9b, without time mean.

In concluding this section, it is found that the model's stratospheric transient planetary waves with periods around 30 days consist of standing and eastward moving waves. The major and minor maxima of the time power spectrum of the transient planetary waves are due mainly to standing waves and coincide with the

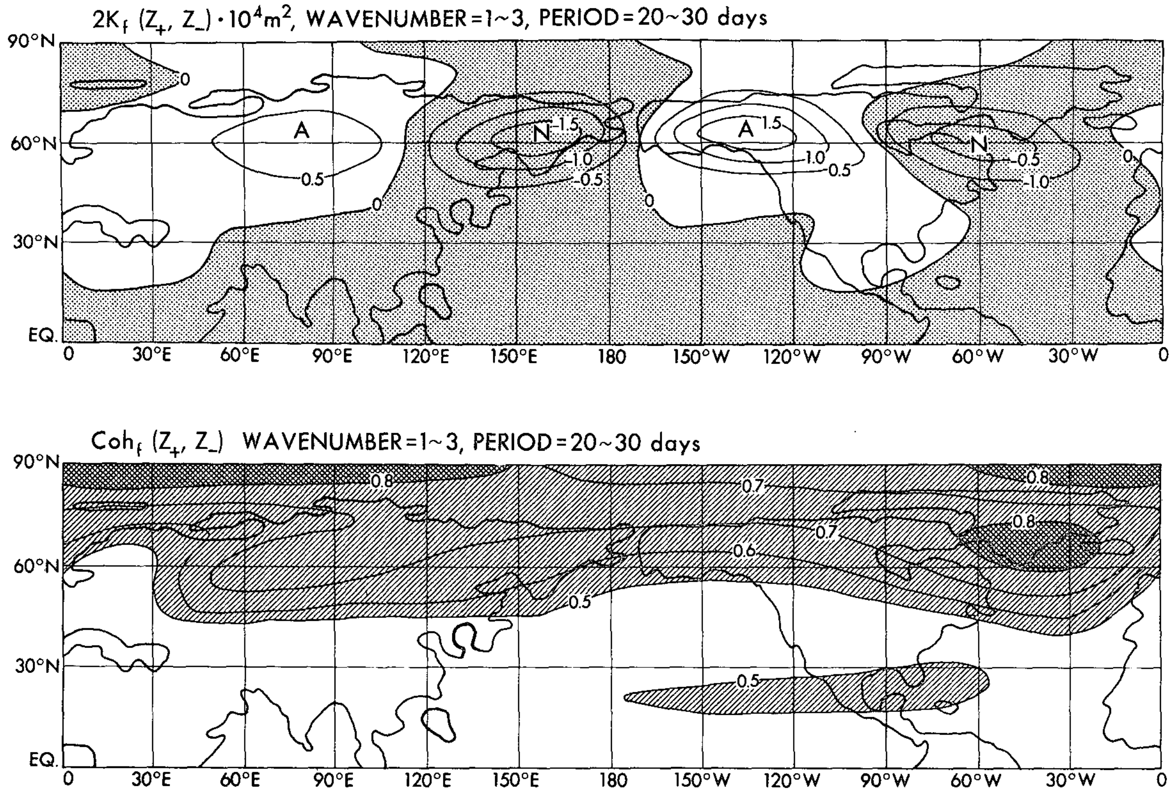

FIG. 6. As in Fig. 5 except for cospectrum ×2.0 (upper) and coherence (lower) between eastward and westward moving components.

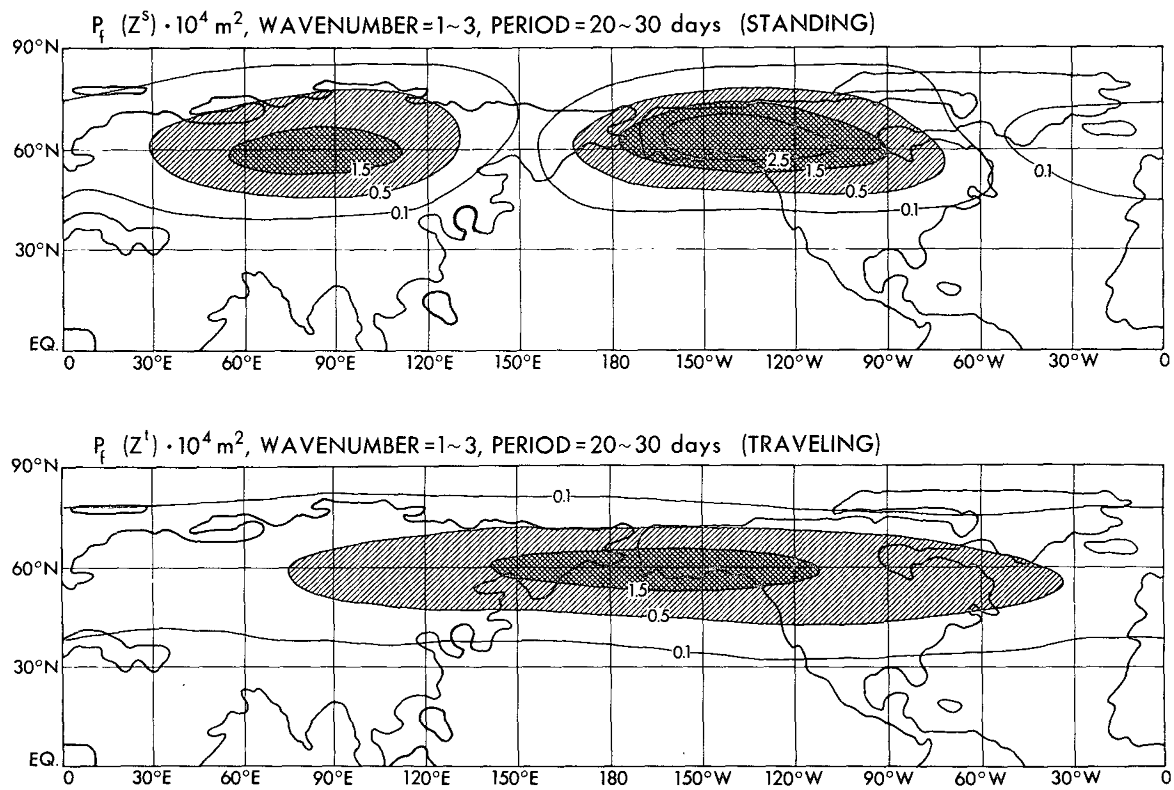

FIG. 7. As in Fig. 5 except for power spectrum of standing (upper) and traveling (lower) components.

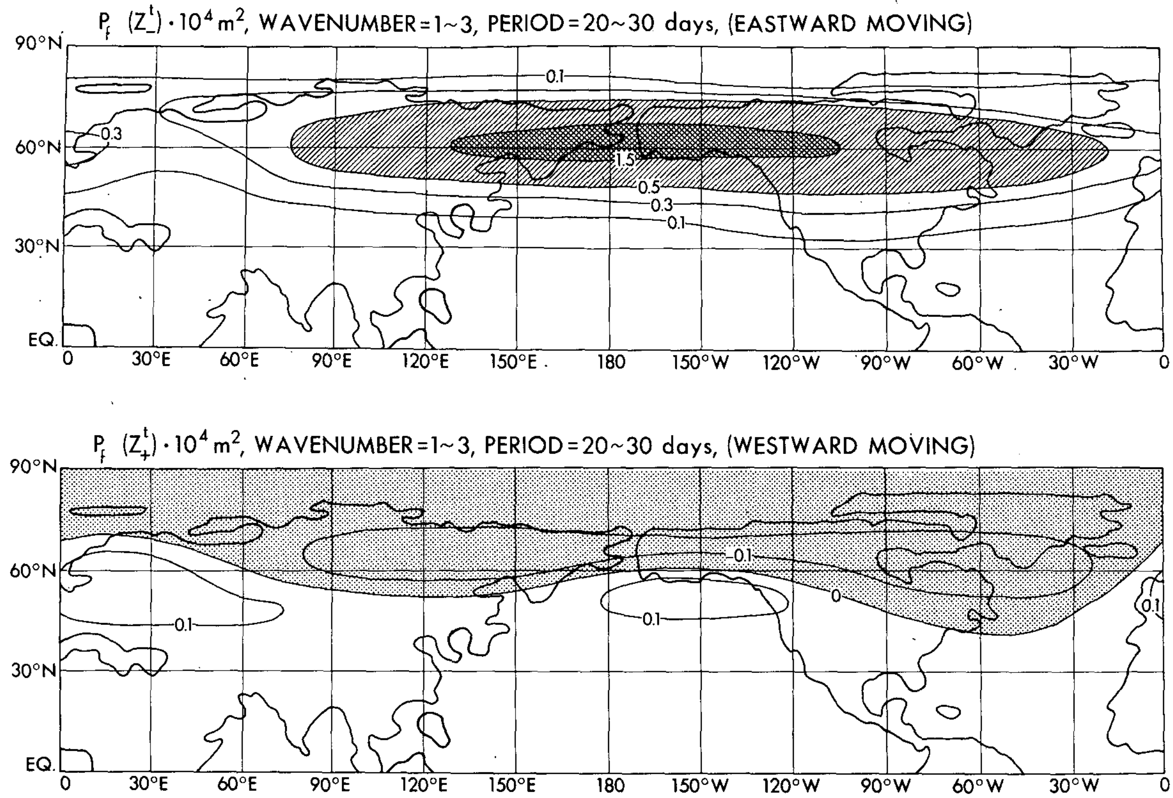

FIG. 8. As in Fig. 5 except for the power spectrum of eastward (upper) and westward (lower) moving components from which standing component is substracted.

high and low of stationary (6-month mean) waves. Thus it is likely that these standing waves and stationary waves are interpreted as mainly due to quasi-stationary waves which pulsate with periods around 30 days as well as an annual period. However, it is unlikely that eastward moving waves and standing waves are of the same origin.

6. Remarks

Space-time spectral formulas are generalized to partition the time-power spectrum of transient disturbances consisting of multiple wavenumbers into standing and traveling parts by assuming that these parts are incoherent with each other. If the present method results in a large *negative* value of the power spectrum (3.26), this assumption must be rejected and the present partition is not physically meaningful. It should also be noted that the present method does not explicitly partition transient waves themselves, although it partitions their power spectra.

The advantage of the generalized method is that it gives the local amplitudes of standing and traveling waves which vary zonally due to interference between multiple wavenumbers. If the local amplitude of an individual wavenumber component is computed, the antinodes of standing waves have equal amplitudes and there is no zonal variation in the amplitude of traveling waves. Unlike the zonal mean (space-time) power spectrum, the local (time) power spectra of the individual wavenumber components do not sum up to the power spectrum of multi-wavenumbers due to the wave-wave interference terms.

The present method was derived without assuming that the power spectrum of traveling waves is given by the difference between the power spectrum of progressive and retrogressive components. It was not assumed that the position of antinodes of standing waves coincides with that of quasi-stationary (time-mean) waves as was assumed by Iwashima and Yamamoto (1971). Although these assumptions are shown by the present example to hold in the model stratosphere, this is not so in the model troposphere (see Hayashi and Golder, 1977; Hayashi, 1977a).

The traveling part defined in the present paper contains, in part, random noise. However, this random noise does not exceed the smaller value of either progressive or retrogressive components of traveling waves, since random noise should appear in both progressive and retrogressive components equally as shown schematically by Fig. 2. If the traveling part consists of progressive and retrogressive components which are of equal amplitudes and are not characterized by a statistically significant spectral peak, this part should be interpreted as a random noise. On the other hand, if the progressive and retrogressive components of waves

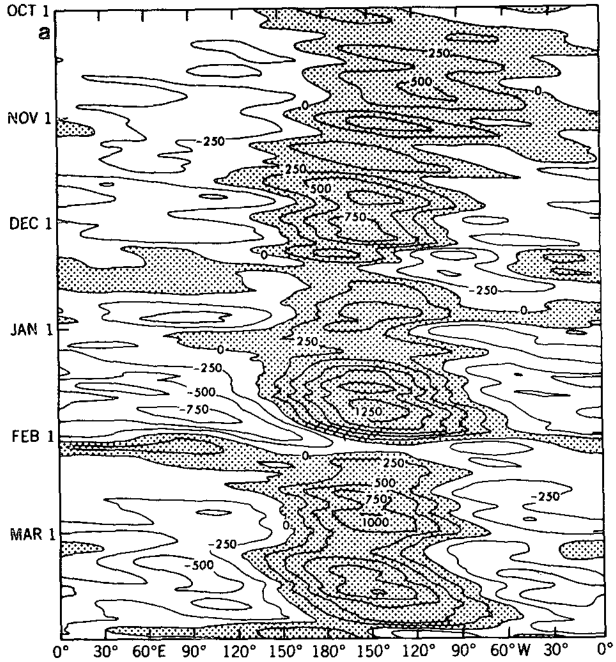

FIG. 9a. Longitude-time section of geopotential height composed of wavenumbers 1–3 at 38 mb, 60°N. Contour interval 250 m. Positive areas are shaded.

are assumed to be coherent in contrast to the present basic assumption, it is possible to exclude noise from waves by the method given in Appendix B, although it is not possible to isolate traveling waves from standing waves without additional assumptions. However, if this method results in a statistically significant spectral peak in the "noise" part, the basic assumption must be rejected and this partition is not physically meaningful.

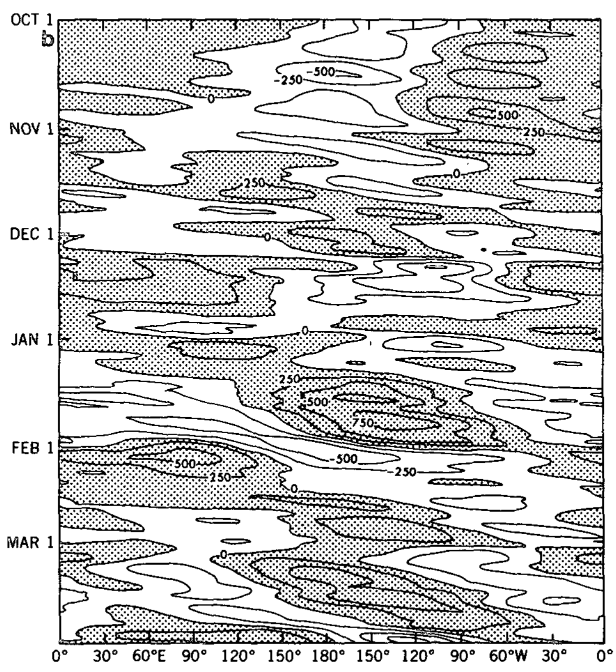

FIG. 9b. As in Fig. 9a except that the time-mean is substracted out.

APPENDIX A

Derivation of Space-Time Spectral Formulas (4.1)–(4.4)

The Fourier-Stieltjes representation of the time series $w_\pm$ is given by

$$w_\pm = 2\,\mathrm{Re}\int_0^\infty e^{i2\pi ft}d\hat{w}_\pm(f), \tag{A1}$$

where $\hat{w}_\pm(f)$ denotes the Fourier-Stieltjes transform of $w_\pm$.

The time power spectrum and cross spectrum of $w_\pm$ are defined as

$$P_f(w_+ + w_-)df = 2\langle |d\hat{w}_+(f) + d\hat{w}_-(f)|^2\rangle, \tag{A2}$$

$$P_f(w_\pm)df = 2\langle |d\hat{w}_\pm(f)|^2\rangle, \tag{A3}$$

$$K_f(w_+,w_-)df = 2\,\mathrm{Re}\langle d\hat{w}_+^*(f)d\hat{w}_-(f)\rangle, \tag{A4}$$

$$Q_f(w_+,w_-)df = 2\,\mathrm{Im}\langle d\hat{w}_+^*(f)d\hat{w}_-(f)\rangle. \tag{A5}$$

Here the asterisk denotes the complex conjugate. The angle braces denote an ensemble average which can be replaced by a frequency average over an infinitesimal frequency band for an ergodic time series of infinite length (see Beran and Parrent, 1964, p. 23). In some papers, the sign of quadrature spectrum is reversed.

On the other hand, the space-time series w can be represented by a space-Fourier series as

$$w(x,t) = \tilde{w}_+(x,t) + \tilde{w}_-(x,t) + w_0(t), \tag{A6}$$

where

$$\tilde{w}_\pm(x,t) = \sum_{k=1}^{\infty} W_{\pm k}(t)e^{\pm ikx}. \tag{A7}$$

Here the complex space series $\tilde{w}_\pm$ corresponds to the "analytic signal" in optics (see Born and Wolf, 1975, p. 494). It can also be interpreted as the "space-rotary component" of rotary vector series (see Hayashi, 1979).

Fourier-Stieltjes transform of (A7) gives

$$\hat{\tilde{w}}_\pm(f) = \sum_{k=1}^{\infty} \hat{W}_{\pm k}(f)e^{\pm ikx}. \tag{A8}$$

The space-time series $w_\pm$ defined by (3.2) can be rewritten by inserting (A8) into (3.2) as

$$w_\pm = 2\,\mathrm{Re}\int_0^\infty e^{i2\pi ft}d\hat{\tilde{w}}_\pm(f). \tag{A9}$$

It should be remarked that this integral is *not* equivalent to the Fourier-Stieltjes representation of the complex time series $\tilde{w}_\pm$ given by

$$\tilde{w}_\pm = \int_{-\infty}^{\infty} e^{i2\pi ft}d\hat{\tilde{w}}_\pm(f), \tag{A10}$$

where the negative frequency does *not* denote the complex conjugate of $d\tilde{w}_\pm(f)$ for $f>0$, since $\tilde{w}_\pm$ is not real.

Comparing (A1) with (A9), we have

$$\hat{w}_\pm(f)=\hat{\tilde{w}}_\pm(f) \quad \text{for} \quad f>0. \tag{A11}$$

For convenience in computation, the complex series $\tilde{w}_\pm$ is rewritten as

$$\tilde{w}_\pm(x,t)=\tfrac{1}{2}[c(x,t)\mp is(x,t)]. \tag{A12}$$

The explicit expressions of $c(x,t)$ and $s(x,t)$ are given by (4.8) and (4.9). The space series $s(x,t)$ can be interpreted as the Hilbert transform of $c(x,t)$ (see Born and Wolf, 1975, p. 495).

Inserting (A12) into (A11) gives

$$\hat{w}_\pm(f)=\tfrac{1}{2}[\hat{c}(f)\mp i\hat{s}(f)] \quad \text{for} \quad f>0. \tag{A13}$$

Inserting (A13) into (A3) gives the formula (4.2) as follows:

$$\begin{aligned}2\langle|d\hat{w}_\pm(f)|^2\rangle &=\tfrac{1}{2}\langle|d\hat{c}(f)|^2\rangle+\tfrac{1}{2}\langle|d\hat{s}(f)|^2\rangle\pm\mathrm{Im}\langle d\hat{c}^*(f)d\hat{s}(f)\rangle \\ &=\tfrac{1}{4}P_f(c)df+\tfrac{1}{4}P_f(s)df\pm\tfrac{1}{2}Q_f(c,s)df.\end{aligned} \tag{A14}$$

Inserting (A13) into (A4) and (A5) gives the formulas (4.3) and (4.4), respectively as follows:

$$\begin{aligned}2\langle d\hat{w}_+^*(f)d\hat{w}_-(f)\rangle &=\tfrac{1}{2}\langle|d\hat{c}(f)|^2\rangle-\tfrac{1}{2}\langle|d\hat{s}(f)|^2\rangle+i\,\mathrm{Re}\langle d\hat{c}^*(f)d\hat{s}(f)\rangle, \\ &=\tfrac{1}{4}[P_f(c)-P_f(s)]df+i\tfrac{1}{2}K_f(c,s)df.\end{aligned} \tag{A15}$$

APPENDIX B

Partition of Power Spectra into Wave and Noise Parts

The time-power spectra of transient waves consisting of multiple wavenumbers can be partitioned into the wave (W) and noise (N) parts by assuming that these parts are incoherent with each other as

$$P_f(w^{\mathrm{W}}+w^{\mathrm{N}})=P_f(w^{\mathrm{W}})+P_f(w^{\mathrm{N}}), \tag{B1}$$

$$P_f(w_\pm^{\mathrm{W}}+w_\pm^{\mathrm{N}})=P_f(w_\pm^{\mathrm{W}})+P_f(w_\pm^{\mathrm{N}}). \tag{B2}$$

These parts can be determined by further assuming that 1) the progressive and retrogressive components of the wave part are coherent with each other and 2) the progressive and retrogressive components of the noise part are incoherent and have equal amplitudes.

These parts can be computed by the following formulas:

$$\begin{aligned}P_f(w^{\mathrm{N}})=P_f(w_+)+P_f(w_-)-\{[P_f(w_+)+P_f(w_-)]^2 \\ -4P_f(w_+)P_f(w_-)[1-\mathrm{coh}^2(w_+,w_-)]\}^{\frac{1}{2}},\end{aligned} \tag{B3}$$

$$P_f(w^{\mathrm{W}})=P_f(w_++w_-)-P_f(w^{\mathrm{N}}), \tag{B4}$$

$$P_f(w_\pm^{\mathrm{W}})=P_f(w_\pm)-\tfrac{1}{2}P_f(w^{\mathrm{N}}). \tag{B5}$$

The wave and noise parts are analogous to the polarized and unpolarized parts (see Appendix C of Hayashi, 1979). For a single wavenumber (B5) is equivalent to a formula derived by Schäfer (1979).

DERIVATION OF (B3)–(B5)

The cross-spectrum matrix of progressive and retrogressive components is partitioned into wave and noise parts as

$$\begin{pmatrix}P_+ & R \\ R^* & P_-\end{pmatrix}=\begin{pmatrix}P_+-N_+ & R \\ R^* & P_--N_-\end{pmatrix}+\begin{pmatrix}N_+ & 0 \\ 0 & N_-\end{pmatrix}, \tag{B6}$$

where

$$P_\pm=P_f(w_\pm), \tag{B7}$$

$$R=K_f(w_+,w_-)+iQ_f(w_+,w_-), \tag{B8}$$

$$N_\pm=P_f(w_\pm^{\mathrm{N}}). \tag{B9}$$

The assumptions 1) and 2) give

$$(P_+-N)(P_--N)-|R|^2=0, \tag{B10}$$

where

$$N_+=N_-=N. \tag{B11}$$

Thus N is given by the smaller root of (B10), since the larger root results in a negative value of $P_\pm$.

The noise and wave parts are given by

$$P_f(w^{\mathrm{N}})=2N, \tag{B12}$$

$$P_f(w^{\mathrm{W}})=P_f(w^{\mathrm{W}}+w^{\mathrm{N}})-2N, \tag{B13}$$

$$P_f(w_\pm^{\mathrm{W}})=P_f(w_\pm)-N. \tag{B14}$$

It can be also proven that both the noise and wave parts are non-negative.

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