NOTES AND CORRESPONDENCE

A Method of Analyzing Transient Waves by Space-Time Cross Spectra

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ABSTRACT

Spectral formulas for analyzing transient waves are presented. Cross-spectral analysis of space-Fourier coefficients isolates travelling waves and standing wave oscillations, and provides statistical information concerning their structure and energetics.

1. Introduction

In a previous paper (Hayashi, 1971), the author presented a space-time cross-spectral method of resolving transient disturbances into progressive and retrogressive wave components. If one component dominates the other, the disturbances are regarded as travelling waves.[1] If both the components are comparable, they are regarded as quasi-standing wave oscillations.

In the latter case we shall further be interested in where the nodes and antinodes occur and what the maximum and minimum local amplitudes are.

The aim of this note is to summarize the previous paper and to present additional formulas for the above purposes.

2. Space-time cross spectra of travelling waves

For readers who are not familiar with the previous papers, we shall recapitulate the formulas for space-time cross spectra of travelling waves.

We express disturbances $W(x,t)$ by sinusoidal waves as

$$W(x,t)=\sum_k W_k(x,t), \tag{2.1}$$

$$W_k(x,t)=\sum_{\omega} R_{k,\omega}\cos(kx+\omega t+\varphi_{k,\omega}) + \sum_{-\omega} R_{k,-\omega}\cos(kx-\omega t+\varphi_{k,-\omega}), \tag{2.2}$$

where $R_{k,\omega}$, $\varphi_{k,\omega}$ and $R_{k,-\omega}$, $\varphi_{k,-\omega}$ are amplitudes and phase angles of retrogressive and progressive waves[2], respectively.

We define space-time cross spectra between two travelling waves W and W' as

$$P_{k,\pm\omega}(W)\equiv\frac{1}{2\Delta\omega}\sum_{\omega-\Delta\omega}^{\omega+\Delta\omega}\tfrac{1}{2}R^2_{k,\pm\omega}, \tag{2.3}$$

$$K_{k,\pm\omega}(W,W')\equiv\frac{1}{2\Delta\omega}\sum_{\omega-\Delta\omega}^{\omega+\Delta\omega}\tfrac{1}{2}R_{k,\pm\omega}R'_{k,\pm\omega}\times\cos(\varphi'_{k,\pm\omega}-\varphi_{k,\pm\omega}), \tag{2.4}$$

$$Q_{k,\pm\omega}(W,W')\equiv\frac{1}{2\Delta\omega}\sum_{\omega-\Delta\omega}^{\omega+\Delta\omega}\tfrac{1}{2}R_{k,\pm\omega}R'_{k,\pm\omega}\times\sin(\varphi'_{k,\pm\omega}-\varphi_{k,-\omega}), \tag{2.5}$$

where $P_{k,\pm\omega}$, $K_{k,\pm\omega}$, $Q_{k,\pm\omega}$ are, respectively, the space-time power spectrum, the co-spectrum, and the quadrature spectrum, with the plus and minus signs indicating the directions of propagation.

Since the variables are cyclic in longitude and infinite in time, we shall define these spectra discrete in wavenumber and continuous in frequency by averaging[3] over a frequency band of $2\Delta\omega$.

On the other hand, Fourier analysis of the space-time

[1] Deland (1964, 1972) presented formulas for travelling waves defined as residues between progressive and retrogressive waves.

[2] In the previous paper the terms progressive and retrogressive were interchanged.

[3] This frequency-averaging was implicit in the previous paper. Kao's (1968) two-dimensional Fourier analysis defines space-time cross spectra as line spectra both in wavenumber and frequency.

series with respect to space yields the time series of cosine and sine coefficients as

$$W_k(x,t)=C_k(t)\cos kx+S_k(t)\sin kx. \quad (2.6)$$

Then space-time cross spectra are estimated by applying conventional time cross-spectral analysis to these coefficients as follows:

$$4P_{k,\pm\omega}(W)=P_\omega(C_k)+P_\omega(S_k)\pm 2Q_\omega(C_k,S_k), \quad (2.7)$$

$$4K_{k,\pm\omega}(W,W')=K_\omega(C_k,C_k')+K_\omega(S_k,S_k')\pm Q_\omega(C_k,S_k')\mp Q_\omega(S_k,C_k'), \quad (2.8)$$

$$4Q_{k,\pm\omega}(W,W')=\pm Q_\omega(C_k,C_k')\pm Q_\omega(S_k,S_k')-K_\omega(C_k,S_k')+K_\omega(S_k,C_k'), \quad (2.9)$$

$$\text{Phase}_{k,\pm\omega}(W,W')=\tan^{-1}[Q_{k,\pm\omega}(W,W')/K_{k,\pm\omega}(W,W')], \quad (2.10)$$

$$\text{Coherence}_{k,\pm\omega}=\left[\frac{K_{k,\pm\omega}^2(W,W')+Q_{k,\pm\omega}^2(W,W')}{P_{k,\pm\omega}(W)P_{k,\pm\omega}(W')}\right]^{\frac{1}{2}}, \quad (2.11)$$

where P_ω, K_ω, Q_ω are, respectively, the power spectrum, the co-spectrum, and the quadrature spectrum which can be computed either by the lag method (see Panofsky and Brier, 1958) or by the fast Fourier transform method (see Bingham *et al.*, 1967).

3. Determination of nodes of standing wave oscillations by cross-spectral analysis

We shall now derive a formula which locates the position of the nodes of quasi-standing wave oscillations.

First, we express local oscillations caused by quasi-standing wave component with wavenumber k at a given location $x=x_1$ by

$$W_k(x_1,t)=C_k(t)\cos kx_1+S_k(t)\sin kx_1. \quad (3.1)$$

Power spectra of this time series can be expressed in terms of cross spectra of $C_k(t)$ and $S_k(t)$ as

$$\begin{aligned}P_\omega[W_k(x_1,t)]&=K_\omega[W_k(x_1,t),W_k(x_1,t)],\\&=K_\omega(C_k\cos kx_1,C_k\cos kx_1)+K_\omega(S_k\sin kx_1,S_k\sin kx_1)\\&\qquad+K_\omega(C_k\cos kx_1,S_k\sin kx_1)+K_\omega(S_k\sin kx_1,C_k\cos kx_1),\\&=P_\omega(C_k)\cos^2 kx_1+P_\omega(S_k)\sin^2 kx_1+2K_\omega(C_k,S_k)\cos kx_1\sin kx_1,\\&=\tfrac{1}{2}[P_\omega(C_k)-P_\omega(S_k)]\cos 2kx_1+K_\omega(C_k,S_k)\sin 2kx_1+\tfrac{1}{2}[P_\omega(C_k)+P_\omega(S_k)],\\&=\{\tfrac{1}{4}[P_\omega(C_k)-P_\omega(S_k)]^2+K_\omega^2(C_k,S_k)\}^{\frac{1}{2}}\cos(2kx_1-\alpha)+\tfrac{1}{2}[P_\omega(C_k)+P_\omega(S_k)],\end{aligned} \quad (3.2)$$

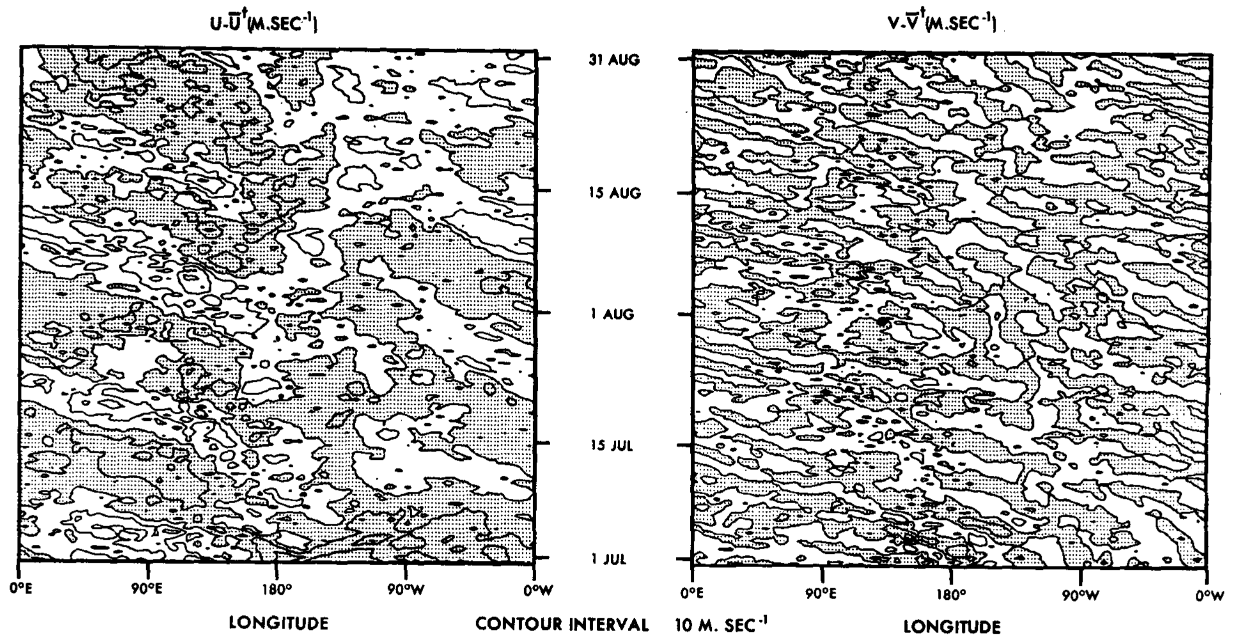

FIG. 1. Time-longitude section at 110 mb over the equator of wind disturbances appearing in the GFDL general circulation model. Two-month mean is subtracted out.

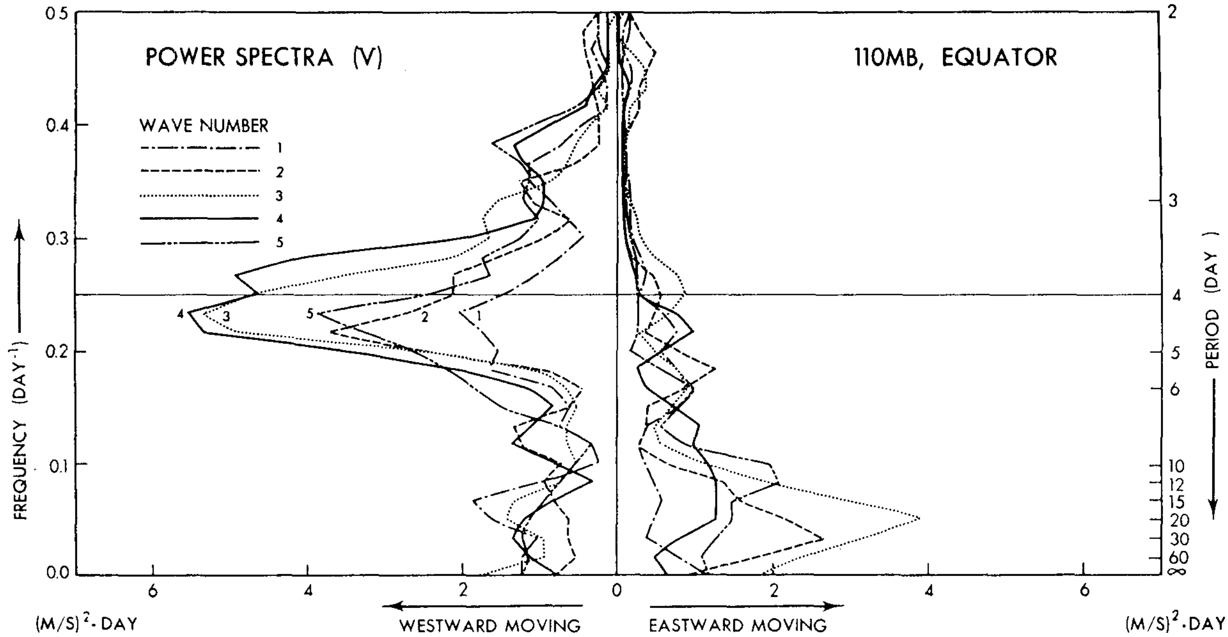

FIG. 2. Space-time power spectra of the meridional component of the wind disturbances shown by Fig. 1 during the period July through October, 1964.

where

$$\alpha \equiv \frac{2K_\omega(C_k,S_k)}{P_\omega(C_k)-P_\omega(S_k)}. \tag{3.3}$$

We see from (3.2) that maximum and minimum local amplitudes of quasi-standing wave oscillations occur, respectively, at

$$x_{\max}=\frac{\alpha+2m\pi}{2k}, \quad x_{\min}=\frac{\alpha+(2m+1)\pi}{2k}, \tag{3.4}$$

where $m=0, \pm1, \pm2 \cdots$. Maximum and minimum

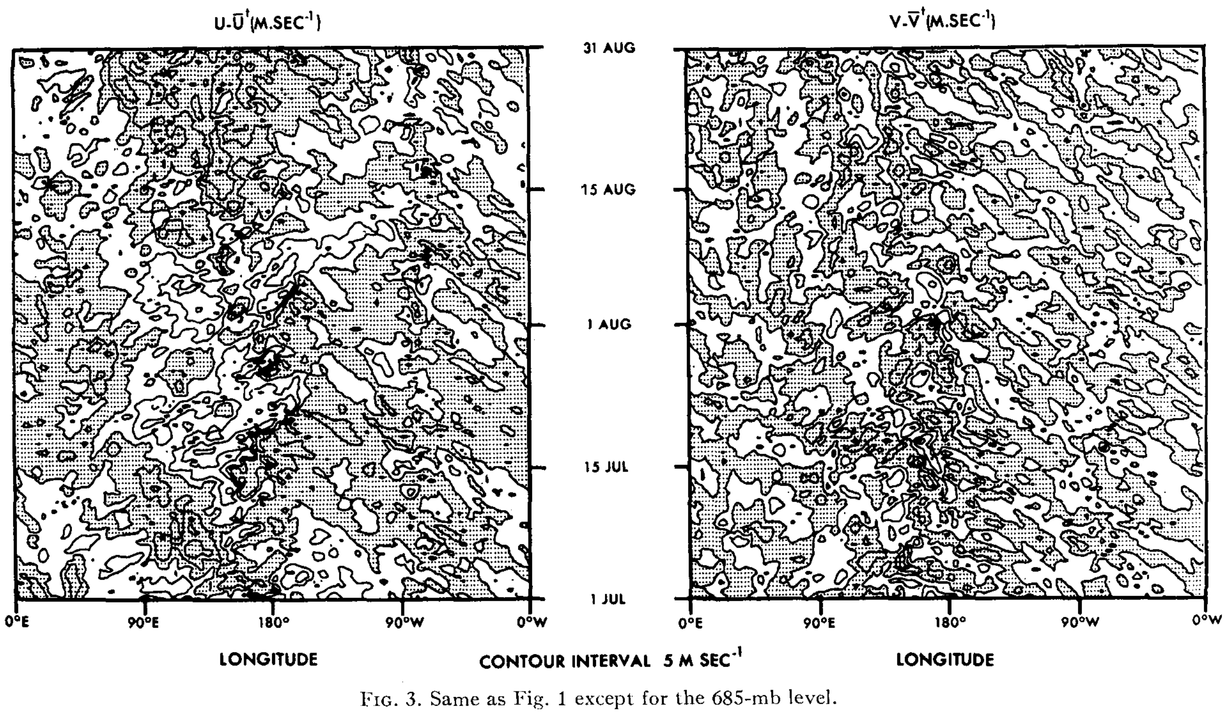

FIG. 3. Same as Fig. 1 except for the 685-mb level.

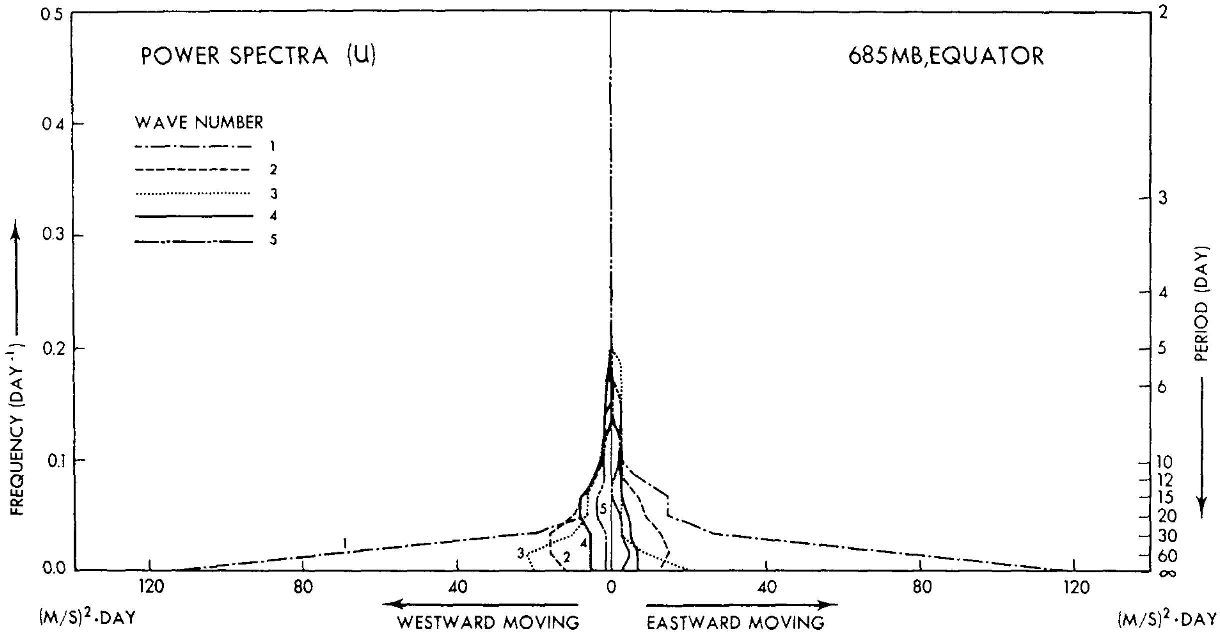

FIG. 4. Space-time power spectra of the zonal component of the wind disturbances shown by Fig. 3 during the period July through October, 1964.

values of local power spectra of quasi-standing wave oscillations are given by

$$\pm\{\tfrac{1}{4}[P_\omega(C_k)-P_\omega(S_k)]^2+K_\omega^2(C_k,S_k)\}^{\frac{1}{2}} +\tfrac{1}{2}[P_\omega(C_k)+P_\omega(S_k)], \quad (3.5)$$

where plus and minus signs correspond to maximum and minimum, respectively.

Space-averaged power spectra of standing wave oscillations are given by

$$\text{Space-mean power}=\tfrac{1}{2}[P_\omega(C_k)+P_\omega(S_k)]. \quad (3.6)$$

On the other hand, from (2.7) we have

$$P_{k,\omega}(W)+P_{k,-\omega}(W)=\tfrac{1}{2}[P_\omega(C_k)+P_\omega(S_k)]. \quad (3.7)$$

Thus, we see that a summation of space-time spectra of progressive and retrogressive waves gives a space-mean local power spectrum of standing wave oscillations.

4. Local cross spectra of quasi-standing wave oscillations

In this section we shall generalize Section 3 and obtain local cross spectra of quasi-standing wave oscillations.

We are interested in two local oscillations, $W_k(x_1,t)$ and $W_k'(x_2,t)$, which are caused by two quasi-standing wave oscillations with the same number k at $x=x_1$ and $x=x_2$, respectively. Time cross spectra between these two local oscillations can be expressed in terms of time cross spectra of space-Fourier coefficients as follows:

$$K_\omega[W_k(x_1,t),W_k'(x_2,t)] =K_\omega(C_k,C_k')\cos kx_1\cos kx_2+K_\omega(S_k,S_k')\sin kx_1\sin kx_2 +K_\omega(C_k,S_k')\cos kx_1\sin kx_2+K_\omega(S_k,C_k')\sin kx_1\cos kx_2, \quad (4.1)$$

$$Q_\omega[W_k(x_1,t),W_k'(x_2,t)] =Q_\omega(C_k,C_k')\cos kx_1\cos kx_2+Q_\omega(S_k,S_k')\cos kx_1\sin kx_2+Q_\omega(C_k,S_k')\cos kx_1\sin kx_2+Q_\omega(S_k,C_k')\sin kx_1\cos kx_2, \quad (4.2)$$

$$\text{Phase}_\omega[W_k(x_1,t),W_k'(x_2,t)]=\tan^{-1}\left\{\frac{Q_\omega[W_k(x_1,t),W_k'(x_2,t)]}{K_\omega[W_k(x_1,t),W_k'(x_2,t)]}\right\}, \quad (4.3)$$

$$\text{Coherence}_\omega[W_k(x_1,t),W_k'(x_2,t)]=\left\{\frac{K_\omega^2[W_k(x_1,t),W_k'(x_2,t)]+Q_\omega^2[W_k(x_1,t),W_k'(x_2,t)]}{P_\omega[W_k(x_1,t)]P_\omega[W_k'(x_2,t)]}\right\}^{\frac{1}{2}}. \quad (4.4)$$

5. Remarks

Application of our formulas to space-time series data will enable us to isolate travelling waves and standing wave oscillations and to discuss their three-dimensional structure and energetics. We shall also be able to find the nodes and antinodes of quasi-standing oscillations. It will be interesting if one of the nodes is located at a

TABLE 1. Position of antinode (wavenumber 1).

	Period (days)						
	∞	60	30	20	15	12	10
$P_{k,+\omega}(U)$ [(m sec^{-1})^2 day]	116	78	19	7	7	3	2
$P_{k,-\omega}(U)$ [(m sec^{-1})^3 day]	116	76	27	13	13	7	2
Antinode (E longitude)	94	95	131	115	108	103	90

physical barrier or one of the antinodes is located in the region of external excitation.

Our method is powerful in the analysis of transient ultra-long waves in middle latitudes and also in the analysis of tropical disturbances using satellite observations. The author is currently applying this technique to tropical disturbances appearing in a numerical model of the general circulation at the Geophysical Fluid Dynamics Laboratory at Princeton University. Some examples of the results are given in the Appendix.

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APPENDIX

Applications of Space-Time Cross Spectra

In order to illustrate application of space-time cross spectra we shall present some examples obtained in the course of a wave analysis of the GFDL general circulation model.

1. Example of travelling wave

Fig. 1 shows a time-longitude section of wind disturbances at 110 mb over the equator of the model. The meridional component (right) exhibits westward phase velocity.

Fig. 2 shows space-time power spectra of the meridional component of the above disturbances. We see that they are westward-moving waves with wavenumber 4 and periods of around 4 days. This is quite in agreement with the large-scale equatorial waves discovered by Yanai and Maruyama (1966).

2. Example of standing wave oscillation

Fig. 3 is a time-longitude section of wind disturbances at 685 mb over the equator of the model. The zonal component (left) shows a large-scale standing wave oscillation.

Fig. 4 shows space-time power spectra of the zonal component during the period July through October. We see that this standing wave oscillation is composed of westward- and eastward-moving waves with wavenumber 1 and very long periods.

Table 1 shows the position of one of the antinodes of this standing wave oscillation estimated by use of Eq. (3.4). We see that a maximum amplitude occurs around 100E. This location is consistent with the Indian monsoon in summer.

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