A Simple, Efficient Solution of Flux–Profile Relationships in the Atmospheric Surface Layer

JONATHAN E. PLEIM*

Atmospheric Sciences Modeling Division, Air Resources Laboratory, National Oceanic and Atmospheric Administration, Research Triangle Park, North Carolina

ABSTRACT

This note describes a simple scheme for analytical estimation of the surface-layer similarity functions from state variables. What distinguishes this note from the many previous papers on this topic is that this method is specifically targeted for numerical models in which simplicity and economic execution are critical. In addition, it has been in use in a mesoscale meteorological model for several years. For stable conditions, a very simple scheme is presented that compares well to the iterative solution. The stable scheme includes a very stable regime in which the slope of the stability functions is reduced to permit significant fluxes to occur, which is particularly important for numerical models in which decoupling from the surface can be an important problem. For unstable conditions, simple schemes generalized for varying ratios of aerodynamic roughness to thermal roughness (z_0/z_{0h}) are less satisfactory. Therefore, a simple scheme has been empirically derived for a fixed z_0/z_{0h} ratio, which represents quasi-laminar sublayer resistance.

1. Background

Interactive linkages between state variables at the earth's surface and in the atmospheric surface layer are essential components of numerical atmospheric models. Surface fluxes of heat, moisture, momentum, and any other modeled quantity (e.g., trace chemical species) are determined by gradients across the surface–atmosphere interface; at the same time, surface fluxes are critical processes determining the time evolution of these gradients. Hence, simultaneous solution of the surface fluxes and the surface layer profiles is required.

Atmospheric models typically use surface-layer similarity theory to describe the flux–profile relationships. In accord with this theory, nondimensional profiles are defined for momentum as

$$\frac{kz}{u_*}\frac{\partial U}{\partial z} = \phi_m\left(\frac{z}{L}\right) \qquad (1)$$

and for potential temperature as

$$\frac{kz}{\theta_*}\frac{\partial \theta}{\partial z} = \phi_h\left(\frac{z}{L}\right), \qquad (2)$$

where the Monin–Obukhov length scale is

$$L = \frac{T_0 u_*^2}{gk\theta_*}. \qquad (3)$$

Here, ϕ_m and ϕ_h are profile functions derived empirically from observed data, k is von Kármán's constant, and T_0 represents the average temperature in the surface layer. Kinematic fluxes of momentum and heat are defined in terms of the friction velocity u_* and the surface temperature scale θ_* as

$$F_m = -u_*^2 \quad \text{and}$$

$$F_h = -u_*\theta_*. \qquad (4)$$

The fluxes can be expressed in terms of state variables by integrating Eqs. (1) and (2) from roughness height z_0 up to z and combining with Eq. (4):

———

* Additional affiliation: On assignment to the National Exposure Research Laboratory, U.S. Environmental Protection Agency, Research Triangle Park, North Carolina.

———

Corresponding author address: Jonathan Pleim, Mail Drop E243-03, Atmospheric Modeling Division/NERL, U.S. EPA, Research Triangle Park, NC 27711.
E-mail: pleim.jon@epa.gov

$$u_* = \frac{kU}{\ln\left(\frac{z}{z_0}\right) - \psi_m\left(\frac{z}{L}, \frac{z_0}{L}\right)} \quad \text{and} \quad (5)$$

$$\theta_* = \frac{k(\theta - \theta_o)}{\phi_{hn}\left[\ln\left(\frac{z}{z_0}\right) - \psi_h\left(\frac{z}{L}, \frac{z_0}{L}\right)\right]}, \quad (6)$$

where ϕ_{hn} is the nondimensional temperature profile constant for neutral conditions [$\phi_{hn} = \phi_h(z/L = 0)$], ψ_m and ψ_h are stability correction functions, and θ_0 is the potential temperature at the aerodynamic roughness height z_0. However, because θ_0 is not generally a known state variable, it is often approximated by θ_g, which is the skin potential temperature. This approximation creates an inconsistency between momentum flux and heat flux because the former is defined from the gradient down to the aerodynamic roughness height z_0, whereas the latter uses the gradient down to the surface.

There are two common approaches for correcting this inconsistency. One approach is to define thermal roughness height z_{0h} differently from aerodynamic roughness height z_0 to account for the difference in the source/sink heights of heat and momentum. Using this approach, z_0 in Eq. (6) would be replaced by z_{0h}. A difficulty with this approach is that additional roughness parameters would need to be specified for every quantity to be modeled. Thus, z_{0q} would need to be specified for water vapor fluxes and specific z_{0c} values would be needed for dry deposition fluxes of each chemical species.

Another approach is to add a quasi-laminar boundary layer resistance to the turbulent aerodynamic resistance such that when θ_0 is replaced by θ_g, Eq. (6) becomes

$$\theta_* = \frac{(\theta - \theta_g)}{u_*(R_a + R_b)}, \quad (7)$$

where

$$R_a = \frac{\phi_{hn}}{ku_*}\left[\ln\left(\frac{z}{z_0}\right) - \psi_m\left(\frac{z}{L}, \frac{z_0}{L}\right)\right], \quad (8)$$

which is the aerodynamic resistance, and R_b is the quasi-laminar boundary layer resistance. A general form of R_b for any scalar quantity was recommended by Wesely and Hicks (1977) as

$$R_b = \frac{B^{-1}}{u_*}\left(\frac{\text{Sc}}{\text{Pr}}\right)^{2/3}, \quad (9)$$

where Sc is the Schmidt number (Sc = ν/D), with ν representing the kinematic molecular viscosity and D

representing the molecular diffusivity of the scalar quantity. The quantity Pr is the Prandtl number, which is the analogous quantity for heat (Pr = ν/ν_θ), where ν_θ is the molecular thermal diffusivity. The quantity B^{-1} is the inverse Stanton number, a dimensionless heat transfer coefficient. Note that for heat Sc = Pr and thus $R_b = B^{-1}/u_*$.

Conceptually, R_a accounts for turbulent diffusion, whereas R_b accounts for molecular diffusion across a very thin quasi-laminar boundary layer adjacent to the surface. An advantage of this approach is that different quantities, such as moisture and trace chemical species, can be similarly treated by using the appropriate molecular or Brownian (for aerosol particles) diffusivity to define Sc.

The two approaches can be reconciled for heat as

$$\ln\frac{z_0}{z_{0h}} = ku_*R_b = kB^{-1}. \quad (10)$$

Although the value of B^{-1} may depend on the Reynolds number as well as the type of surface roughness, we follow the recommendation of Garratt and Hicks (1973) by assuming a constant value of $kB^{-1} = 2$ for fully turbulent flow over fibrous vegetative canopies.

Although Eqs. (1)–(9) represent a closed system of equations, solution is not possible without iteration. Therefore, parameterizations in which the stability parameter (z/L) is estimated from state variables before the fluxes are determined are usually used. The bulk Richardson number is a convenient parameter for this purpose:

$$R_B = \frac{gz_1(\theta_1 - \theta_0)}{T_0(U_1^2 + V_1^2)}, \quad (11)$$

where the subscript 1 indicates values at the lowest model level. In a numerical model, however, the potential temperature at z_0 is not known prior to the flux–profile calculations. Therefore, the bulk Richardson number is often approximated as

$$R_B = \frac{gz_1(\theta_1 - \theta_g)}{T_0(U_1^2 + V_1^2)}. \quad (12)$$

2. Analytical approximation

Many techniques have been proposed for an approximate analytic solution to the flux–profile equations, such as Yang et al. (2001), De Bruin et al. (2000), Holtslag and Ek (1996), Launiainen (1995), and Byun (1990), to name a few. Van den Hurk and Holtslag (1997) compared and assessed several techniques. They found that some of the techniques do well for the simplified case in which $z_0 = z_{0h}$ (i.e., $R_b = 0$) but have

more difficulty for the general case in which $z_0 \neq z_{0h}$. The goal of the work presented here is to develop an analytical solution to the above equations, including the quasi-laminar boundary layer resistance, that is simple enough for efficient use in numerical meteorological and air-quality models.

The problem can be separated into stable and unstable regimes. For stable conditions, we follow the scheme outlined by Blackadar (1976) that has been used in the fifth-generation Pennsylvania State University–National Center for Atmospheric Research Mesoscale Model (MM5) for many years. For stable conditions, z/L is estimated from R_B as

$$\frac{z}{L} = \ln\left(\frac{z}{z_0}\right) \frac{R_B}{1 - R_B/R_{\text{crit}}}, \tag{13}$$

where R_{crit} is the critical Richardson number ($=0.25$). Note that Eq. (13) is an exact relationship only where $z_0 = z_{0h}$, but it is a good approximation to the more general case in which $z_0 \neq z_{0h}$. The estimated z/L is then used with the linear ψ functions as proposed by Webb (1970) and Dyer (1974):

$$\psi_{(m,h)} = -\beta_{(m,h)} \frac{z}{L}. \tag{14}$$

Note that Eq. (14) neglects the dependence of ψ on z_0/L because $z_0 \ll z$. Equation (13) results directly from the definitions of L, u_*, θ_*, R_B, and ψ [Eqs. (3), (5), (6), (11), and (14)] if $\beta_m = \beta_h = 1/R_{\text{crit}}$.

Högström (1988) found that the linear form fits experimental data very well for $0 < z/L < 0.5$ but results in underestimated surface fluxes under more strongly stable conditions. He suggested that the slope should begin to flatten somewhere in the range of $0.5 < z/L < 1.0$. Parameterization of strongly stable conditions is particularly important in numerical models when decoupling of the surface and air occurs when the ψ functions get too negative, causing runaway surface cooling. To alleviate this problem, Beljaars and Holtslag (1991) developed ψ curves for stable conditions that mimic Eq. (14) for mild stability ($0 < z/L < 0.5$) while flattening out at greater stabilities. We have adopted a simpler solution using Eq. (14) for z/L between 0 and 1 and a linear function with reduced slope at greater stabilities such that

$$\psi_{(m,h)} = 1 - \beta_{(m,h)} - \frac{z}{L}, \quad \text{for} \quad z/L > 1. \tag{15}$$

To account for the reduced slope of the ψ functions for very stable conditions ($z/L > 1$), Eq. (13) becomes

$$\frac{z}{L} = \ln\left(\frac{z}{z_0}\right) \frac{R_B}{1 - R_{\text{cut}}/R_{\text{crit}}}, \quad \text{when} \quad R_B > R_{\text{cut}}, \tag{16}$$

where

$$R_{\text{cut}} = \left[\ln\left(\frac{z}{z_0}\right) + \frac{1}{R_{\text{crit}}}\right]^{-1}, \tag{17}$$

which is the solution of Eq. (13) for R_B when $z/L = 1$.

For unstable conditions, a simple linear function of R_B gives a good estimate of z/L:

$$\frac{z}{L} = \left[a \ln\left(\frac{z}{z_0}\right) - b\right] R_B, \tag{18}$$

where a and b are constants. However, the introduction of a quasi-laminar boundary layer results in a more nonlinear relationship, which becomes more pronounced at lower wind speeds (larger R_B) and greater roughness lengths. Numerical comparison studies by van den Hurk and Holtslag (1997) and by Lo (1996) show that application of schemes that do not consider the quasi-laminar boundary layer ($z_0 = z_{0h}$), such as Eq. (18) or more complicated schemes such as described by Byun (1990), to cases in which $z_0 \neq z_{0h}$ will result in serious errors. Launiainen (1995) proposed a simple scheme that allows for $z_0 \neq z_{0h}$:

$$\frac{z}{L} = \left[\frac{\ln(z/z_0)^2}{\ln(z/z_{0h})} - 0.55\right] R_B. \tag{19}$$

However, for z_0/z_{0h} constant over varying stabilities, Eq. (19) is just a specific form of Eq. (18); therefore, this expression still has difficulty at large roughness lengths.

Another difficulty for unstable conditions is the arithmetic complexity of the Dyer functions. Therefore, De Bruin et al. (2000) derived simplified expressions that approximate the Dyer functions in the form

$$\psi\left(\frac{z}{L}\right) = a \ln\left(1 - b \frac{z}{L}\right), \tag{20}$$

where a and b are constants that differ for heat and momentum. Instead of using one empirical expression to approximate z/L and then another to approximate ψ, a single empirical expression could be derived that relates ψ directly to R_B:

$$\psi_{h,m} = a_{h,m} \ln\left\{1 - b_{h,m}\left[\ln\left(\frac{z}{z_0}\right)\right]^{1/2} R_B\right\}, \tag{21a}$$

where

$$a_{h,m} = c_{h,m} + d_{h,m} \ln\left[\ln\left(\frac{z}{z_0}\right)\right]. \qquad (21b)$$

For our application in MM5, using the Dyer (1974) functions for ψ with Högström's (1988) recommended γ values, we empirically determined values of $b_h = 15.7$, $b_m = 13.0$, $c_h = 0.04$, $c_m = 0.031$, $d_h = 0.355$, and $d_m = 0.276$. Note that we use the quasi-laminar boundary layer resistance R_b [Eq. (9)], which is equivalent to $z_0/z_{0h} = 7.4$. For other values of z_0/z_{0h}, the b, c, and d coefficients would need to be rederived.

3. Testing

The new method for approximation of the ψ functions computed from bulk Richardson number without iteration has different stable and unstable parts. For stable conditions, z/L is approximated by analytical functions of bulk Richardson number according to Eq. (13) for slightly stable conditions ($0 < z/L < 1$) and Eqs. (16) and (17) for strongly stable conditions ($z/L > 1$). Then $\psi(z/L)$ is computed from Eqs. (14) or (15), depending on the degree of stability. For unstable conditions, ψ functions are estimated directly from R_B according to Eq. (21). Heat and momentum fluxes are then computed using Eqs. (4), (5), (7), (8), and (9).

Figure 1 shows normalized heat flux as functions of bulk Richardson number as computed by the new method in comparison with the method proposed by Launiainen (1995) and the exact solution produced by 10 iterations, for z/z_0 of 10, 100, 1000, and 10 000 and wind speed of 1 and 5 m s^{-1}. Note that a height of $z = 10z_0$ is within the roughness sublayer where the Monin–Obukhov similarity equations are not valid (Garratt 1992). However, we include these results because such calculations are often required in numerical models. Thus, it is important to show how the scheme performs under these conditions.

Following De Bruin et al. (2000), the sensible heat flux is normalized by

$$H_0 = \frac{k^2(\theta - \theta_g)U}{\phi_{hn} \ln^2\left(\frac{z}{z_0}\right) + 2\ln\left(\frac{z}{z_0}\right)}, \qquad (22)$$

where U is the wind speed at z and ϕ_{hn} is given the value 0.95 as recommended by Högström (1988). Here H_0 represents the heat flux without the effect of the stability functions. Note that because H_0 is zero when R_B is zero the normalized heat flux is undefined at that point. Thus, in the plots shown in Fig. 1, the lines are interpolated linearly across the $R_B = 0$ point. The exact solutions are produced using the Dyer (1974) functions with the Högström (1988) coefficients and with the

modified ψ function for very stable conditions [Eq. (15)].

For stable conditions, we compare our method versus the iterative solution only. The Launiainen (1995) method for unstable conditions compares well to the iterative solution for small roughness lengths but becomes increasingly inaccurate for rougher surfaces, especially at greater instabilities. This result agrees with the assessment of van den Hurk and Holtslag (1997) and the limitations discussed by Launiainen (1995). Our new scheme, however, performs well over the entire range of conditions. The worst agreement is for high instability, where $z/z_0 = 10$ and wind speed = 1 m s^{-1}, which are extreme conditions for numerical meteorological models. This error results from the difficulty of specifying general functions for a_h and a_m [Eq. (21b)].

On the stable side, the analytical method follows the double-linear form of Eqs. (14) and (15). The biggest differences from the iterative solutions are due to the slower drop toward zero heat flux under very stable light wind conditions. As noted above, this feature is desirable for numerical modeling. The other relatively large difference is for stable, high wind speed conditions, which are unrealistic.

A comparison of friction velocity computed by the same three methods (Fig. 2) shows very similar results. The Launiainen (1995) method agrees well with the iterative solution for small roughness lengths but shows increasing deviation for rougher surfaces. The largest deviation of the analytical method from the iterative solution is for light wind, stable conditions. For all roughness lengths, the iterative solution goes to zero for $R_B > 1$ while the analytical solution remains nonzero out to $R_B = 3$. Such large values of R_B are not unrealistic at these low wind speeds. For example, $R_B = 2$ represents a temperature difference between the ground and air of about 3 K when wind speed = 1 m s^{-1} ($z = 20$ m for this example). Most models ensure nonzero surface fluxes for all possible conditions by imposing arbitrary minimum values for u_* and $\psi_{h,m}$. With this analytical method, such limitations are not necessary, as demonstrated by application of this method in MM5.

4. Summary

The methods described here provide economical, accurate estimations of the surface flux–profile relationships for use in numerical models. These methods are designed for specific stability functions. For stable conditions, we use the linear functions of Webb (1970) and Dyer (1974) with the coefficients recommended by Högström (1988). For very stable conditions, however,
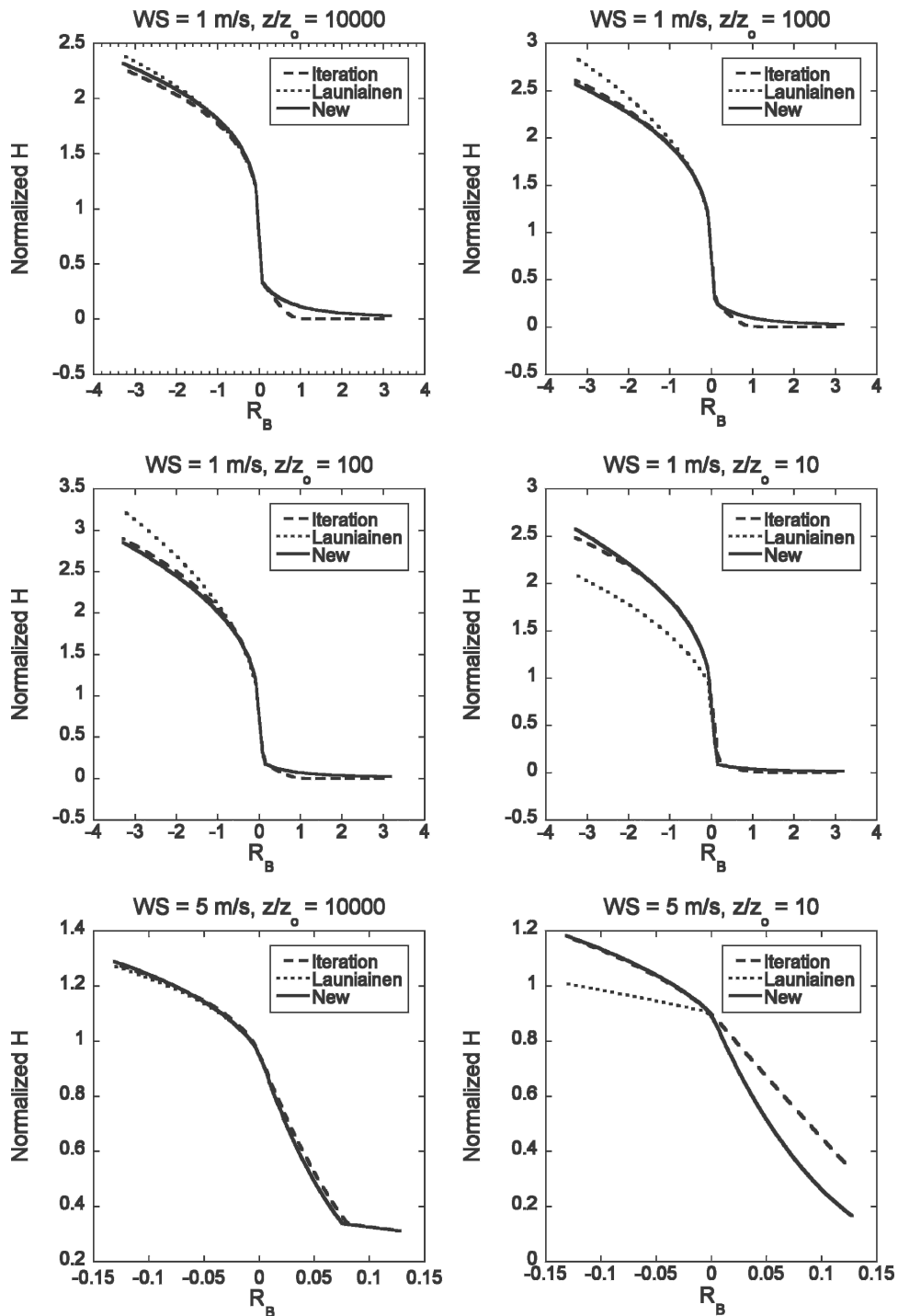
FIG. 1. Normalized kinematic heat flux computed using the new analytical scheme in comparison with the scheme of Launiainen (1995) and an iterative solution, for various ratios of z/z_0 and two different wind speeds (WS). Note that the Launiainen scheme is shown only for unstable conditions.

a reduced slope is adopted to avoid decoupling with the surface. For unstable conditions, we approximate the ψ functions directly from R_B. We account for the difference between momentum and scalar fluxes by inclusion of a quasi-laminar sublayer resistance for scalar fluxes that is a function of molecular diffusivity, which is equivalent to a fixed z_0/z_{0h} ratio for heat.
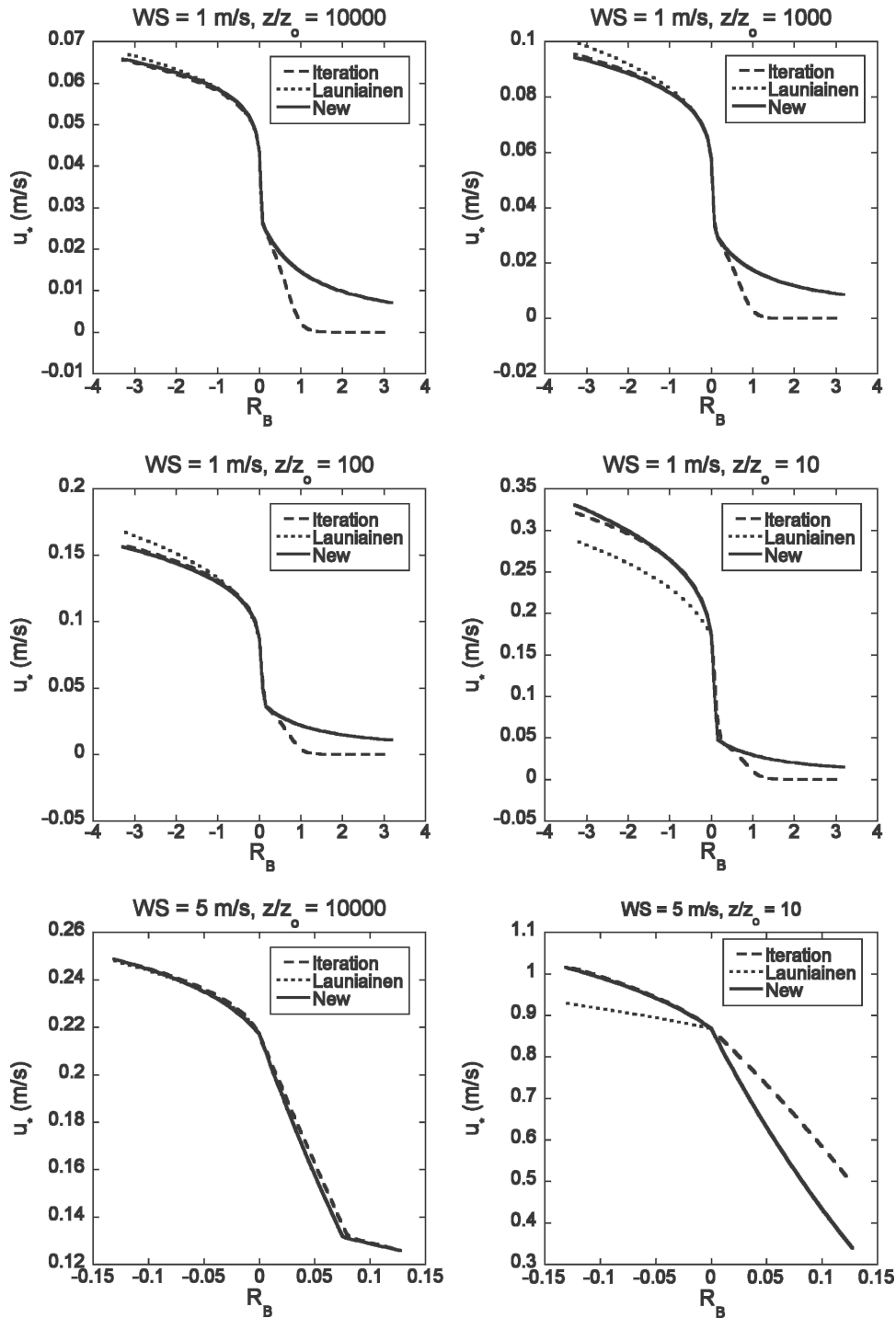
This method has been applied to the fifth-generation

FIG. 2. As in Fig. 1, but for friction velocity.

Pennsylvania State University–National Center for Atmospheric Research Mesoscale Model, where it is part of the Pleim–Xiu land surface model (PX LSM) option (Pleim and Xiu 1995, 2003; Xiu and Pleim 2001). The PX LSM has been in MM5 since version 3.4, which was first released in 2001. Thus, the method has proven to be robust under extensive use. We are currently working to incorporate the PX LSM into the Weather Research and Forecast (WRF) Model. Note that in WRF the surface-layer schemes are in a separate module

from the planetary boundary layer (PBL) and LSM modules. Therefore, the surface layer method described here will be available for use with any other PBL or LSM schemes.

REFERENCES

Beljaars, A. C. M., and A. A. M. Holtslag, 1991: Flux parameterization over land surfaces for atmospheric models. *J. Appl. Meteor.,* **30,** 327–341.

Blackadar, A. K., 1976: Modeling the nocturnal boundary layer. Preprints, *Third Symp. on Atmospheric Turbulence and Air Quality,* Raleigh, NC, Amer. Meteor. Soc., 46–49.

Byun, D. W., 1990: On the analytical solution of the flux-profile relationships for the atmospheric surface layer. *J. Appl. Meteor.,* **29,** 652–657.

De Bruin, H. A. R., R. J. Ronda, and B. J. H. van de Wiel, 2000: Approximate solutions for the Obukhov length and the surface fluxes in terms of bulk Richardson numbers. *Bound.-Layer Meteor.,* **95,** 145–157.

Dyer, A. J., 1974: A review of flux-profile relationships. *Bound.-Layer Meteor.,* **7,** 363–372.

Garratt, J. R., 1992: *The Atmospheric Boundary Layer.* Cambridge University Press, 316 pp.

——, and B. B. Hicks, 1973: Momentum, heat and water vapour transfer to and from natural and artificial surfaces. *Quart. J. Roy. Meteor. Soc.,* **99,** 680–687.

Högström, U., 1988: Non-dimensional wind and temperature profiles in the atmospheric surface layer: A re-evaluation. *Bound.-Layer Meteor.,* **42,** 55–78.

Holtslag, A. A. M., and M. Ek, 1996: Simulation of surface fluxes and boundary layer development over the pine forest in HAPEX-MOBILHY. *J. Appl. Meteor.,* **35,** 2396–2409.

Launiainen, J., 1995: Derivation of the relationship between the Obukhov stability parameter and the bulk Richardson number for flux-profile studies. *Bound.-Layer Meteor.,* **76,** 165–179.

Lo, A. K.-F., 1996: On the role of roughness lengths in the flux parameterizations of boundary-layer models. *Bound.-Layer Meteor.,* **80,** 403–413.

Pleim, J. E., and A. Xiu, 1995: Development and testing of a surface flux and planetary boundary layer model for application in mesoscale models. *J. Appl. Meteor.,* **34,** 16–32.

——, and ——, 2003: Development of a land surface model. Part II: Data assimilation. *J. Appl. Meteor.,* **42,** 1811–1822.

van den Hurk, B. J. J. M., and A. A. M. Holtslag, 1997: On the bulk parameterization of surface fluxes for various conditions and parameter ranges. *Bound.-Layer Meteor.,* **82,** 119–134.

Webb, E. K., 1970: Profile relationships: The log-linear range and extension for strong stability. *Quart. J. Roy. Meteor. Soc.,* **96,** 67–90.

Wesely, M. L., and B. B. Hicks, 1977: Some factors that affect the deposition rates of sulfur dioxide and similar gases on vegetation. *J. Air Pollut. Control Assoc.,* **27,** 1110–1116.

Xiu, A., and J. E. Pleim, 2001: Development of a land surface model. Part I: Application in a mesoscale meteorological model. *J. Appl. Meteor.,* **40,** 192–209.

Yang, K., N. Tamai, and T. Koike, 2001: Analytical solution of the surface layer similarity equations. *J. Appl. Meteor.,* **40,** 1647–1653.