Supplementary material for "Estimation of order restricted means from correlated data"

Shyamal Das Peddada * David B. Dunson † Xiaofeng Tan ‡

Material provided in this document supplement the paper "Estimation of order restricted means from correlated data" by Peddada, Dunson and Tan (2005). For notational consistency, in this document we use the same notations as those used in actual paper. Two items described in this document are the asymptotic properties of the proposed estimator and results of a simulation study.

We summarize the simulation results for estimating the smallest parameter when the parameters are subject to simple tree order restriction, and the estimators of individual parameters when the parameters are subject to an umbrella order with location of the peak unknown.

ASYMPTOTIC PROPERTIES OF THE PROPOSED ESTIMATOR

We prove that at every iterate the estimator proposed in the paper is a consistent estimator and under certain conditions it is also asymptotically normal. Before we derive these asymptotic properties we provide some preliminary results as follows.

For a random vector $U = (U_1, U_2, \ldots, U_p)'$, define

$$V_1 = \min\left(U_1, \frac{\omega_{1,1}U_1 + \omega_{1,2}U_2}{\omega_{1,1} + \omega_{1,2}}\right),$$

$$V_i = \frac{1}{2}\left\{\max\left(U_i, \frac{\omega_{i-1,1}V_{i-1} + \omega_{i-1,2}U_i}{\omega_{i-1,1} + \omega_{i-1,2}}\right) + \min\left(U_i, \frac{\omega_{i,1}U_i + \omega_{i,2}U_{i+1}}{\omega_{i,1} + \omega_{i,2}}\right)\right\}, \quad i = 2, \ldots, p-1,$$

$$V_p = \max\left(U_p, \frac{\omega_{p-1,1}V_{p-1} + \omega_{p-1,2}U_p}{\omega_{p-1,1} + \omega_{p-1,2}}\right),$$

*Corresponding author. Biostatistics Branch, National Institute of Environmental Health Sciences, Research Triangle Park, NC 27709. e-mail: peddada@niehs.nih.gov.

†Biostatistics Branch, National Institute of Environmental Health Sciences, Research Triangle Park, NC 27709.

‡China Life Insurance Company, Ltd., Beijing, 100020 P.R. China.

where $\omega_{i,j}$ are constants (not depending upon n) such that $\omega_{i,1} \geq 0, \omega_{i,2} \geq 0, \omega_{i,1} + \omega_{i,2} > 0$, for $i = 1, \ldots, p-1$.

Lemma S1. *Suppose $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_p$ and for $i = 1, 2, \ldots p$, suppose U_i is a consistent estimator of μ_i and asymptotically $\sqrt{n}(U_i - \mu_i) \longrightarrow Z_i$ in distribution, where Z_i has mean 0 and finite variance and whose distribution is independent of the parameters $\mu_1, \mu_2, \ldots, \mu_p$.*

(i) If $\mu_1 < \mu_2$ and if Z_1 is normally distributed then V_1 is a CAN estimator of μ_1.

(ii) If $\mu_1 = \mu_2$ then, for all real x,

$$\lim_{n\to\infty} \Pr\{\sqrt{n}(V_1 - \mu_1) \leq x\} \to \Pr(Z_1 \leq x) + \Pr\left(Z_1 \geq x, \frac{\omega_{1,1} Z_1 + \omega_{1,2} Z_2}{\omega_{1,1} + \omega_{1,2}} \leq x\right).$$

Further, V_1 is a consistent estimator of μ_1.

(iii) For $i = 2, 3, \ldots, p-1$, if $\mu_{i-1} < \mu_i < \mu_{i+1}$ and if Z_i is normally distributed then V_i is a CAN estimator of μ_i.

(iv) For $i = 2, 3, \ldots, p-1$, if $\mu_{i-1} = \mu_i < \mu_{i+1}$ then, for all real x,

$$\lim_{n\to\infty} \Pr\{\sqrt{n}(V_i - \mu_i) \leq x\} \to \Pr(Z_{i-1} \leq Z_i \leq x) + \Pr\left(Z_i + \frac{\omega_{i-1,1} Z_{i-1} + \omega_{i-1,2} Z_i}{\omega_{i-1,1} + \omega_{i-1,2}} \leq 2x, Z_{i-1} \geq Z_i\right).$$

Further, V_i is a consistent estimator of μ_i.

(v) For $i = 2, 3, \ldots, p-1$, if $\mu_{i-1} < \mu_i = \mu_{i+1}$ then, for all real x,

$$\lim_{n\to\infty} \Pr\{\sqrt{n}(V_i - \mu_i) \leq x\} \to \Pr(Z_i \leq x, Z_i \leq Z_{i+1}) + \Pr\left(Z_i + \frac{\omega_{i,1} Z_i + \omega_{i,2} Z_{i+1}}{\omega_{i,1} + \omega_{i,2}} \leq 2x, Z_i \geq Z_{i+1}\right).$$

Further, V_i is a consistent estimator of μ_i.

(vi) For $i = 2, 3, \ldots, p-1$, if $\mu_{i-1} = \mu_i = \mu_{i+1}$ then, for all real x,

$$\begin{aligned}
\lim_{n\to\infty} \Pr\{\sqrt{n}(V_i - \mu_i) \leq x\} &\to \Pr(Z_i \leq x, Z_{i-1} \leq Z_i \leq Z_{i+1}) \\
&+ \Pr\left(Z_i + \frac{\omega_{i,1} Z_i + \omega_{i,2} Z_{i+1}}{\omega_{i,1} + \omega_{i,2}} \leq 2x, Z_{i-1} \leq Z_i, Z_i \geq Z_{i+1}\right) \\
&+ \Pr\left(Z_i + \frac{\omega_{i-1,1} Z_{i-1} + \omega_{i-1,2} Z_i}{\omega_{i-1,1} + \omega_{i-1,2}} \leq 2x, Z_{i-1} \geq Z_i, Z_i \geq Z_{i+1}\right) \\
&+ \Pr\left(\frac{\omega_{i-1,1} Z_{i-1} + \omega_{i-1,2} Z_i}{\omega_{i-1,1} + \omega_{i-1,2}} + \frac{\omega_{i,1} Z_i + \omega_{i,2} Z_{i+1}}{\omega_{i,1} + \omega_{i,2}} \leq 2x, Z_{i-1} \geq Z_i \geq Z_{i+1}\right).
\end{aligned}$$

Further, V_i is a consistent estimator of μ_i.

(vii) If $\mu_{p-1} < \mu_p$ and if Z_p is normally distributed then V_p is a CAN estimator of μ_p.

(viii) If $\mu_{p-1} = \mu_p$ then, for all real x,

$$\lim_{n\to\infty} \Pr(\sqrt{n}(V_p - \mu_p) \leq x) \to \Pr(Z_p \leq x) - \Pr\left(Z_p \leq x, \frac{\omega_{p-1,1} Z_1 + \omega_{p-1,2} Z_2}{\omega_{p-1,1} + \omega_{p-1,2}} \geq x\right).$$

Further, V_p *is a consistent estimator of* μ_p.

Proof of (i):

We first derive the asymptotic distribution of V_1. If $\omega_{1,2} = 0$ then the proof is trivial, as $V_1 = U_1$. Thus, we focus on the case where $\omega_{1,2} > 0$.

Let $W_i = \sqrt{n}(U_i - \mu_i)$, $i = 1, 2$ with $\lim_{n\to\infty} \Pr(W_i \le x) = \Pr(Z_i \le x)$, for all x. Then

$$\Pr(\sqrt{n}(V_1 - \mu_1) \le x) = 1 - \Pr(\sqrt{n}(V_1 - \mu_1) \ge x)$$
$$= 1 - \Pr\left(W_1 \ge x, \frac{\omega_{1,1}W_1 + \omega_{1,2}W_2 + \sqrt{n}\omega_{1,2}(\mu_2 - \mu_1)}{\omega_{1,1} + \omega_{1,2}} \ge x\right)$$
$$= 1 - \Pr(W_1 \ge x) + \Pr\left(W_1 \ge x, \frac{\omega_{1,1}W_1 + \omega_{1,2}W_2 + \sqrt{n}\omega_{1,2}(\mu_2 - \mu_1)}{\omega_{1,1} + \omega_{1,2}} \le x\right)$$

As $n \to \infty$ we therefore have

$$\Pr(\sqrt{n}(V_1 - \mu_1) \le x) \to \Pr(Z_1 \le x).$$

Thus, if Z_1 is normally distributed then $\sqrt{n}(V_1 - \mu_1)$ is asymptotically normal.

The proof of consistency of V_1 can be deduced along the above lines as follows. For all $x \ge 0$

$$\Pr(|V_1 - \mu_1| \le x) = \Pr(V_1 - \mu_1 \le x) - \Pr(V_1 - \mu_1 \le -x).$$

Note that

$$\Pr(V_1 - \mu_1 \le x) = 1 - \Pr(\sqrt{n}(V_1 - \mu_1) \ge \sqrt{n}x)$$
$$= 1 - \Pr\left(W_1 \ge \sqrt{n}x, \frac{\omega_{1,1}W_1 + \omega_{1,2}W_2 + \sqrt{n}\omega_{1,2}(\mu_2 - \mu_1)}{\omega_{1,1} + \omega_{1,2}} \ge \sqrt{n}x\right)$$
$$= 1 - \Pr(W_1 \ge \sqrt{n}x)$$
$$+\Pr\left(W_1 \ge \sqrt{n}x, \frac{\omega_{1,1}W_1 + \omega_{1,2}W_2 + \sqrt{n}\omega_{1,2}(\mu_2 - \mu_1)}{\omega_{1,1} + \omega_{1,2}} \le \sqrt{n}x\right). \quad (S1)$$

Similarly, note that

$$\Pr(V_1 - \mu_1 \le -x) = 1 - \Pr(W_1 \ge -\sqrt{n}x) +$$
$$\Pr\left(W_1 \ge -\sqrt{n}x, \frac{\omega_{1,1}W_1 + \omega_{1,2}W_2 + \sqrt{n}\omega_{1,2}(\mu_2 - \mu_1)}{\omega_{1,1} + \omega_{1,2}} \le -\sqrt{n}x\right). \quad (S2)$$

Since $x > 0$ and $\mu_2 > \mu_1$ we therefore observe that the right hand side of (S1) converges to 1 and similarly the right hand side of (S2) converges to 0. Hence combining the two we note that

$$\lim_{n\to\infty} \Pr(|V_1 - \mu_1| \le x) \to 1.$$

Thus V_1 is a consistent estimator of μ_1. Consequently, if Z_1 is normally distributed then V_1 is a CAN estimator.

Proof of (ii):

We first derive the asymptotic distribution of V_1. Note that

$$\begin{aligned}\Pr(\sqrt{n}(V_1 - \mu_1) \leq x) &= 1 - \Pr(\sqrt{n}(V_1 - \mu_1) \geq x) \\ &= 1 - \Pr\Big(W_1 \geq x, \frac{\omega_{1,1}W_1 + \omega_{1,2}W_2}{\omega_{1,1} + \omega_{1,2}} \geq x\Big) \\ &= 1 - \Pr(W_1 \geq x) + \Pr\Big(W_1 \geq x, \frac{\omega_{1,1}W_1 + \omega_{1,2}W_2}{\omega_{1,1} + \omega_{1,2}} \leq x\Big)\end{aligned}$$

As $n \to \infty$ we therefore have

$$\Pr(\sqrt{n}(V_1 - \mu_1) \leq x) \to \Pr(Z_1 \leq x) + \Pr\Big(Z_1 \geq x, \frac{\omega_{1,1}Z_1 + \omega_{1,2}Z_2}{\omega_{1,1} + \omega_{1,2}} \leq x\Big)$$

To prove the consistency of V_1 we note that for all $x \geq 0$

$$\Pr(|V_1 - \mu_1| \leq x) = \Pr(V_1 - \mu_1 \leq x) - \Pr(V_1 - \mu_1 \leq -x).$$

Note that

$$\begin{aligned}\Pr(V_1 - \mu_1 \leq x) &= 1 - \Pr(\sqrt{n}(V_1 - \mu_1) \geq \sqrt{n}x) \\ &= 1 - \Pr\Big(W_1 \geq \sqrt{n}x, \frac{\omega_{1,1}W_1 + \omega_{1,2}W_2}{\omega_{1,1} + \omega_{1,2}} \geq \sqrt{n}x\Big) \\ &= 1 - \Pr(W_1 \geq \sqrt{n}x) + \Pr\Big(W_1 \geq \sqrt{n}x, \frac{\omega_{1,1}W_1 + \omega_{1,2}W_2}{\omega_{1,1} + \omega_{1,2}} \leq \sqrt{n}x\Big). \qquad (S3)\end{aligned}$$

Similarly, note that

$$\begin{aligned}\Pr(V_1 - \mu_1 \leq -x) &= 1 - \Pr(\sqrt{n}(V_1 - \mu_1) \geq -\sqrt{n}x) \\ &= 1 - \Pr\Big(W_1 \geq -\sqrt{n}x, \frac{\omega_{1,1}W_1 + \omega_{1,2}W_2}{\omega_{1,1} + \omega_{1,2}} \geq -\sqrt{n}x\Big) \\ &= 1 - \Pr(W_1 \geq -\sqrt{n}x) + \Pr\Big(W_1 \geq -\sqrt{n}x, \frac{\omega_{1,1}W_1 + \omega_{1,2}W_2}{\omega_{1,1} + \omega_{1,2}} \leq -\sqrt{n}x\Big). \qquad (S4)\end{aligned}$$

Since $x > 0$ we therefore observe that the right hand side of (S3) converges to 1 and similarly the right hand side of (S4) converges to 0. Hence combining the two we note that

$$\lim_{n\to\infty} \Pr(|V_1 - \mu_1| \leq x) \to 1.$$

Thus V_1 is a consistent estimator of μ_1.

Proof of (iii):

We assume $\omega_{i-1,1} > 0$, and $\omega_{i,2} > 0$. The cases when either $\omega_{i-1,1} = 0$, or $\omega_{i,2} = 0$ can be deduced similarly.

For $j \geq i$, let $W_j = \sqrt{n}(U_j - \mu_j)$, and let $W_{i-1} = \sqrt{n}(V_{i-1} - \mu_{i-1})$. Further, let $\mathcal{A} = \{W_i - W_{i-1} \geq \sqrt{n}(\mu_{i-1} - \mu_i)\}$ and $\mathcal{B} = \{W_i - W_{i+1} \leq \sqrt{n}(\mu_{i+1} - \mu_i)\}$. We derive the asymptotic distribution of V_i by induction. By virtue of (i) and (ii) we assume that the asymptotic distribution of W_{i-1} is not dependent upon $\mu_1, \mu_2, \ldots, \mu_p$.

Note that for each x,

$$
\begin{aligned}
&\Pr(\sqrt{n}(V_i - \mu_i) \leq x) \\
&= \Pr\Big[\sqrt{n}\Big\{\frac{1}{2}\Big(\max\Big(U_i, \frac{\omega_{i-1,1}V_{i-1} + \omega_{i-1,2}U_i}{\omega_{i-1,1} + \omega_{i-1,2}}\Big) + \min\Big(U_i, \frac{\omega_{i,1}U_i + \omega_{i,2}U_{i+1}}{\omega_{i,1} + \omega_{i,2}}\Big)\Big) - \mu_i\Big\} \leq x\Big] \\
&= \Pr\Big[\Big\{\frac{1}{2}\Big(\max\Big(W_i, \frac{\omega_{i-1,1}W_{i-1} + \omega_{i-1,2}W_i + \sqrt{n}\omega_{i-1,1}(\mu_{i-1} - \mu_i)}{\omega_{i-1,1} + \omega_{i-1,2}}\Big) \\
&\quad + \min\Big(W_i, \frac{\omega_{i,1}W_i + \omega_{i,2}W_{i+1} + \sqrt{n}\omega_{i,2}(\mu_{i+1} - \mu_i)}{\omega_{i,1} + \omega_{i,2}}\Big)\Big)\Big\} \leq x\Big] \\
&= \Pr\Big(W_i \leq x, \mathcal{A}, \mathcal{B}\Big) + \Pr\Big(W_i + \frac{\omega_{i,1}W_i + \omega_{i,2}W_{i+1} + \sqrt{n}\omega_{i,2}(\mu_{i+1} - \mu_i)}{\omega_{i,1} + \omega_{i,2}} \leq 2x, \mathcal{A}, \bar{\mathcal{B}}\Big) \\
&\quad + \Pr\Big(W_i + \frac{\omega_{i-1,1}W_{i-1} + \omega_{i-1,2}W_i + \sqrt{n}\omega_{i-1,1}(\mu_{i-1} - \mu_i)}{\omega_{i-1,1} + \omega_{i-1,2}} \leq 2x, \bar{\mathcal{A}}, \mathcal{B}\Big) \\
&\quad + \Pr\Big(\frac{\omega_{i-1,1}W_{i-1} + \omega_{i-1,2}W_i + \sqrt{n}\omega_{i-1,1}(\mu_{i-1} - \mu_i)}{\omega_{i-1,1} + \omega_{i-1,2}} \\
&\quad + \frac{\omega_{i,1}W_i + \omega_{i,2}W_{i+1} + \sqrt{n}\omega_{i,2}(\mu_{i+1} - \mu_i)}{\omega_{i,1} + \omega_{i,2}} \leq 2x, \bar{\mathcal{A}}, \bar{\mathcal{B}}\Big)
\end{aligned}
$$

Since $\mu_1 < \mu_2 < \ldots < \mu_p$, therefore $\bar{\mathcal{A}}$, $\bar{\mathcal{B}}$ are impossible events as $n \to \infty$. Thus the above expressions converge to $P(Z_i \leq x)$. Thus, if Z_i is normally distributed then V_i is asymptotically normally distributed. Using arguments similar to those made in the proof of (a) it can also be deduced that V_i is consistent for μ_i.

Proof of (iv):

Suppose $\mu_{i-1} = \mu_i$ and $\mu_i < \mu_{i+1}$. Let $\mathcal{A} = \{W_i \geq W_{i-1}\}$ and $\mathcal{B} = \{W_i - W_{i+1} \leq \sqrt{n}(\mu_{i+1} - \mu_i)\}$. Then following the above calculations we have, for all x,

$$
\begin{aligned}
&\Pr(\sqrt{n}(V_i - \mu_i) \leq x) = \\
&P\Big(W_i \leq x, \mathcal{A}, \mathcal{B}\Big) + \Pr\Big(W_i + \frac{\omega_{i,1}W_i + \omega_{i,2}W_{i+1} + \sqrt{n}\omega_{i,2}(\mu_{i+1} - \mu_i)}{\omega_{i,1} + \omega_{i,2}} \leq 2x, \mathcal{A}, \bar{\mathcal{B}}\Big)
\end{aligned}
$$

$$+\Pr\left(W_i + \frac{\omega_{i-1,1}W_{i-1} + \omega_{i-1,2}W_i + \sqrt{n}\omega_{i-1,1}(\mu_{i-1}-\mu_i)}{\omega_{i-1,1}+\omega_{i-1,2}} \leq 2x, \bar{\mathcal{A}}, \mathcal{B}\right)$$

$$+\Pr\left(\frac{\omega_{i-1,1}W_{i-1} + \omega_{i-1,2}W_i + \sqrt{n}\omega_{i-1,1}(\mu_{i-1}-\mu_i)}{\omega_{i-1,1}+\omega_{i-1,2}} + \frac{\omega_{i,1}W_i + \omega_{i,2}W_{i+1} + \sqrt{n}\omega_{i,2}(\mu_{i+1}-\mu_i)}{\omega_{i,1}+\omega_{i,2}}\right) \leq 2x, \bar{\mathcal{A}}, \bar{\mathcal{B}}\Big)$$

Since, as $n \to \infty$, $\bar{\mathcal{B}}$ is an impossible event hence we deduce that

$$\Pr(\sqrt{n}(V_i - \mu_i) \leq x) \to \Pr\Big(Z_{i-1} \leq Z_i \leq x\Big) + \Pr\left(Z_i + \frac{\omega_{i-1,1}Z_{i-1} + \omega_{i-1,2}Z_i}{\omega_{i-1,1}+\omega_{i-1,2}} \leq 2x, Z_{i-1} \geq Z_i\right)$$

Following arguments very similar to those in the proof of (ii) we can prove that V_i is a consistent estimator of μ_i.

The proofs of (v), (vi), (vii) and (viii) follow very similarly and hence we omit them.

□

We now re-state and prove Theorem 3 stated in the accompanying paper. Theorem 3 assumes that $\hat{\mu}^{(0)} \equiv \bar{X}$ has a multivariate normal distribution.

Theorem 3. *Suppose* $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_p$ *and, for* $i = 1, 2, \ldots p$ *and* $s = 1, 2, 3, \ldots$, $\hat{\mu}_i^{(t-1)}$ *is a consistent estimator of* μ_i *and asymptotically* $\sqrt{n}(\hat{\mu}_i^{(t-1)} - \mu_i) \longrightarrow Z_i$ *in distribution, where* $E(Z_i) = 0$, $var(Z_i) < \infty$ *and the distribution of* Z_i *is independent of* $\mu_1, \mu_2, \ldots, \mu_p$. *Then, for each* i, $i = 1, 2, \ldots, p$, $\hat{\mu}_i^{(t)}$ *is a consistent estimator.*

(i) If $\mu_1 < \mu_2$ *then* $\hat{\mu}_1^{(t)}$ *is asymptotically normally distributed.*

(ii) If $\mu_1 = \mu_2$ *then, for all real* x,

$$\lim_{n\to\infty} \Pr\{\sqrt{n}(\hat{\mu}_1^{(t)} - \mu_1) \leq x\} \to \Pr(Z_1 \leq x) + \Pr\left(Z_1 \geq x, \frac{\omega_{1,1}Z_1 + \omega_{1,2}Z_2}{\omega_{1,1}+\omega_{1,2}} \leq x\right).$$

(iii) For $i = 2, 3, \ldots, p-1$, *if* $\mu_{i-1} < \mu_i < \mu_{i+1}$ *then* $\hat{\mu}_i^{(t)}$ *is asymptotically normally distributed.*

(iv) For $i = 2, 3, \ldots, p-1$, *if* $\mu_{i-1} = \mu_i < \mu_{i+1}$ *then, for all real* x,

$$\lim_{n\to\infty} \Pr\{\sqrt{n}(\hat{\mu}_i^{(t)} - \mu_i) \leq x\} \to \Pr(Z_{i-1} \leq Z_i \leq x)$$
$$+\Pr\left(Z_i + \frac{\omega_{i-1,1}Z_{i-1} + \omega_{i-1,2}Z_i}{\omega_{i-1,1}+\omega_{i-1,2}} \leq 2x, Z_{i-1} \geq Z_i\right).$$

(v) For $i = 2, 3, \ldots, p-1$, *if* $\mu_{i-1} < \mu_i = \mu_{i+1}$ *then, for all real* x,

$$\lim_{n\to\infty} \Pr\{\sqrt{n}(\hat{\mu}_i^{(t)} - \mu_i) \leq x\} \to \Pr(Z_i \leq x, Z_i \leq Z_{i+1})$$

$$+\Pr\left(Z_i + \frac{\omega_{i,1}Z_i + \omega_{i,2}Z_{i+1}}{\omega_{i,1}+\omega_{i,2}} \le 2x, Z_i \ge Z_{i+1}\right).$$

(vi) For $i = 2, 3, \ldots, p-1$, if $\mu_{i-1} = \mu_i = \mu_{i+1}$ then, for all real x,

$$\lim_{n\to\infty} \Pr\{\sqrt{n}(\hat{\mu}_i^{(t)} - \mu_i) \le x\} \to P(Z_i \le x, Z_{i-1} \le Z_i \le Z_{i+1})$$
$$+\Pr\left(Z_i + \frac{\omega_{i,1}Z_i + \omega_{i,2}Z_{i+1}}{\omega_{i,1}+\omega_{i,2}} \le 2x, Z_{i-1} \le Z_i, Z_i \ge Z_{i+1}\right)$$
$$+\Pr\left(Z_i + \frac{\omega_{i-1,1}Z_{i-1} + \omega_{i-1,2}Z_i}{\omega_{i-1,1}+\omega_{i-1,2}} \le 2x, Z_{i-1} \ge Z_i, Z_i \ge Z_{i+1}\right)$$
$$+\Pr\left(\frac{\omega_{i-1,1}Z_{i-1} + \omega_{i-1,2}Z_i}{\omega_{i-1,1}+\omega_{i-1,2}} + \frac{\omega_{i,1}Z_i + \omega_{i,2}Z_{i+1}}{\omega_{i,1}+\omega_{i,2}} \le 2x, Z_{i-1} \ge Z_i \ge Z_{i+1}\right).$$

(vii) If $\mu_{p-1} < \mu_p$ then $\hat{\mu}_p^{(t)}$ is asymptotically normally distributed.

(viii) If $\mu_{p-1} = \mu_p$ then, for all real x,

$$\lim_{n\to\infty} \Pr\{\sqrt{n}(\hat{\mu}_p^{(t)} - \mu_p) \le x\} \to \Pr(Z_p \le x) - \Pr\left(Z_p \le x, \frac{\omega_{p-1,1}Z_{p-1} + \omega_{p-1,2}Z_p}{\omega_{p-1,1}+\omega_{p-1,2}} \ge x\right).$$

Proof:

We sketch the proof for the case when Σ is known. If Σ is unknown then one may appeal to Slutsky's theorem. Further, it is enough to prove the theorem for the estimators produced after one iteration because the same argument can be applied for every iteration.

For each new iterate t, the proof follows by appealing to Lemma S1 with $U \equiv \hat{\mu}^{(t-1)}$, $V \equiv \hat{\mu}^{(t)}$ and by noting that $\hat{\mu}^{(0)} \equiv \bar{X}$ is a normally distributed random variable.

□

SIMULATION STUDY

Umbrella order with unknown location of the peak

In §4 of the accompanying paper, we compared the performance of $\bar{X}$, $\hat{\mu}^{HP}$ and $\hat{\mu}$, in terms of $E(\hat{\theta}-\theta)'(\hat{\theta}-\theta)$. In this document we provide the comparisons between the three estimators in terms of the mean squared error (MSE) of the individual components, namely, $E(\hat{\theta}_i - \theta_i)^2$, $i = 1, 2, \ldots, p$, where $\hat{\theta}_i$ is an estimator of θ_i.

Simulations were conducted under the same set of configurations as in the accompanying paper. All results are based on 10,000 simulation runs. Results summarised in Table 2 indicate that $\hat{\mu}$ generally performs better than $\bar{X}$ and it competes very well with $\hat{\mu}^{HP}$.

Simple tree order restriction

We compared the performance of the following estimators of μ_1 under the simple tree order restriction, $\mu_1 \leq \mu_i$, $i = 2, 3, \ldots, p$. As before, the simulation results are based on 10,000 simulation runs.

- The restricted maximum likelihood estimator (RMLE) of μ_1 under the assumption that Σ is diagonal, with weights $1/\sigma_{i,i}$, $i = 1, 2, \ldots, p$.

  This estimator was considered only in the case when Σ is diagonal.

- $\hat{\mu}_1$, the proposed estimator.

- $\hat{\mu}_1^{HP}$, the estimator introduced in Hwang & Peddada (1994).

- Unrestricted maximum likelihood estimator $\bar{X}_1$.

For our simulation study, we generated the sample mean vector $\bar{X}$ from a p-variate normal distribution with $\mu' = (0, 1, 1, 0, \ldots, 0)$ and a tri-diagonal covariance matrix Σ whose elements are defined as follows:

$$\sigma_{1,1} = c_1, \sigma_{1,2} = d\sigma c_1, \sigma_{i,i} = \sigma^2(c_{i-1} + c_i), \sigma_{i,i+1} = d\sigma c_i, \text{for all } i \geq 2,$$

with $d = 0, \pm 1$ and all remaining off-diagonal elements of Σ being zero. Thus we considered 3 different covariance structures; $d = 0$ corresponds to a diagonal Σ, $d = 1$ corresponds to the case where the components of $\bar{X}$ are non-negatively correlated and $d = -1$ corresponds to the case where the components of $\bar{X}$ are non-positively correlated. The covariance matrix Σ is assumed to be known.

Nearly 100 different patterns of $c_1, c_2, \ldots, c_8$ and σ were considered. A representative sample of the simulation study is summarised in Table 3. We compared the above estimators in terms of MSE, $E(\hat{\mu}_1 - \mu_1)^2$, and the coverage probability, $P(|\hat{\theta}_1 - \mu_1| \leq 1.96\sqrt{\sigma_{11}})$, where $\hat{\theta}_1$ is an estimator of μ_1. Note that, theoretically the MSE of $\bar{X}_1$ is $\sigma_{11} = c_1$ and its coverage probability is 0.95.

In the case when Σ is diagonal (i.e. $d = 0$) and $c_1 = .1, c_2 = 0.005$, the MSE of RMLE exceeds that of $\bar{X}$ and also its coverage probability drops well below 0.95. Thus when Σ is diagonal, the RMLE performs very poorly. Further, in view of the theoretical result obtained in Hwang & Peddada (1994), it may be avoided when the covariance matrix is nondiagonal. Hence one should be very careful in using RMLE as the method of choice.

As expected, from a theoretical result of Tan & Peddada (2000), the performance of $\hat{\mu}_1^{HP}$ is poor for certain covariance structures. For example when the components of $\bar{X}$ are non-negatively correlated (i.e. $d = 1$) and $c_1 = .1, c_2 = 0.005$ the coverage probability is almost zero! In contrast, the proposed estimator $\hat{\mu}_1$ performs best in every case considered in this simulation study.

Table 2: Umbrella Order

Component-wise comparison of the MSE, with $n = 10$.

Parameter	Variance pattern								
	(i)			(ii)			(iii)		
	$\hat{\mu}$	$\hat{\mu}^{HP}$	$\bar{X}$	$\hat{\mu}$	$\hat{\mu}^{HP}$	$\bar{X}$	$\hat{\mu}$	$\hat{\mu}^{HP}$	$\bar{X}$
μ_1	0.361	0.359	0.394	0.363	0.349	0.390	1.596	3.376	1.615
μ_2	0.343	0.333	0.398	0.661	0.622	0.909	4.218	6.418	8.122
μ_3	0.336	0.330	0.393	1.089	1.108	1.605	13.072	16.002	25.613
μ_4	0.363	0.368	0.402	1.746	1.894	2.470	34.081	38.756	62.588
μ_5	0.392	0.395	0.405	2.686	2.898	3.610	68.376	75.535	128.346
μ_6	0.357	0.359	0.393	1.788	1.930	2.530	33.127	37.309	61.834
μ_7	0.348	0.338	0.403	1.088	1.108	1.618	12.515	14.796	25.349
μ_8	0.337	0.329	0.399	0.641	0.621	0.907	4.099	5.873	8.260
μ_9	0.340	0.328	0.392	0.357	0.343	0.405	1.057	2.430	1.606
μ_{10}	0.371	0.369	0.401	0.100	0.100	0.100	0.100	0.823	0.098

Table 3: Simple Tree Order

Comparison of estimators in terms of MSE (first row) and coverage probability (second row). Note that in this simulation experiment $c_2 = c_3 = c_4$ and $c_5 = c_6 = c_7 = c_8 = 1$. Results are based on 10,000 simulation runs. Here $\sigma = 2$.

c_1, c_2	$\bar{X}_1$	Uncorrelated ($d = 0$)			Positive correlation ($d = 1$)		Negative correlation ($d = -1$)	
	$\bar{X}$	RMLE	$\hat{\mu}$	$\hat{\mu}^{HP}$	$\hat{\mu}$	$\hat{\mu}^{HP}$	$\hat{\mu}$	$\hat{\mu}^{HP}$
1, 0.1	1.00	0.653	0.608	0.626	0.721	1.413	0.739	1.149
	0.951	0.975	0.975	0.975	0.974	0.938	0.974	0.931
0.2, 0.1	0.200	0.178	0.164	0.177	0.179	0.703	0.177	0.212
	.946	0.960	0.968	0.969	0.961	0.655	0.962	0.940
0.1, 0.005	0.100	0.192	0.055	0.092	0.064	0.920	0.064	0.099
	0.951	0.845	0.975	0.974	0.975	0.008	0.975	0.951