A Method for Calculating More Accurate Budget Analyses of "Sigma" Coordinate Model Results

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ABSTRACT

The so-called "sigma" coordinate system has seen increasing use in numerical models developed for general circulation and climate simulation, as well as for weather forecasting. Concurrently, there is an increasing demand for accurate analysis of the interactive physical processes included in these model integrations. However, because of the necessity to transform the model information from sigma levels back to more conventional coordinate surfaces (such as pressure), significant inaccuracies usually result.

To reduce these inaccuracies, an alternative analysis procedure is introduced which avoids the usual ambiguous evaluation of vertical velocity in the transformed coordinate. Tests of this alternative method show that substantial increases in model analysis accuracy can be obtained.

1. Introduction

Since its introduction by Phillips (1957), the so-called "sigma" vertical coordinate system has proven to be very useful for application to numerical models of the atmosphere. In this system, the vertical coordinate is pressure divided by surface pressure ($\sigma = P/P_*$). Thus, the lower boundary is always a coordinate surface. This type of coordinate system is particularly useful when applied to physical problems where orography is important. The sigma system also provides a model framework in which it is relatively easy to guarantee conservation of important integral properties such as mass, energy, etc.

As the models for general circulation, climate and weather forecasting continue to improve, there is an increasing demand that numerical results be carefully analyzed in terms of the relationships among various simulated physical processes. However, it is in such model analysis that the sigma coordinate shows a significant disadvantage. Since the sigma surfaces often deviate significantly from more conventional surfaces (such as height, pressure or potential temperature), it is usually awkward and often misleading to attempt to interpret model results directly in the sigma coordinate. In addition, the verifying data from the real atmosphere are not normally available in this coordinate. Consequently, it becomes necessary to try to interpret the model results in the framework of a simpler system, usually constant pressure surfaces.

As will be shown, the usual approaches to accomplishing the change of coordinates can lead to serious error, even after considerable space and time averaging. In an attempt to alleviate this difficulty, an alternative analysis procedure will be proposed. Test examples will be shown which demonstrate that the alternative procedure can lead to a significant reduction of analysis error.

2. The problem

Although the method introduced here is applicable to the budgets of any quantity, only an arbitrary variable χ will be considered. In sigma coordinates, the local change of χ due to three-dimensional flux convergence of χ is expressed by

$$\frac{\partial \chi P_*}{\partial t} = -\nabla_\sigma \cdot P_* \mathbf{V}_2 \chi - \frac{\partial}{\partial \sigma} P_* \dot{\sigma} \chi, \tag{1}$$

where $\mathbf{V}_2$ is the horizontal vector wind, ∇_σ the "horizontal" gradient operator on a surface of constant σ, $\dot{\sigma} = d\sigma/dt$ (the vertical σ velocity), P_* the surface pressure, and t time. Analysis of the remaining terms in various budgets (such as non-conservative terms) tends to be considerably less troublesome and thus will not be directly considered. In pressure coordinates, the equivalent flux-form expression is

$$\frac{\partial \chi}{\partial t} = -\nabla_p \cdot \mathbf{V}_2 \chi - \frac{\partial \omega \chi}{\partial P}, \tag{2}$$

where ∇_p is the "horizontal" gradient operator on a constant P surface, and $\omega = dP/dt$ (the vertical P velocity).

Unfortunately, unless the P and σ surfaces always coincide (constant P_* in space and time), there is no unique correspondence between the analogous terms in Eqs. (1) and (2). Because of this, the usual procedure is to calculate ω from the sigma coordinate variables, evaluate $\omega\chi$ and $\mathbf{V}_2\chi$, interpolate to constant pressure surfaces, and evaluate the terms in Eq. (2) independently. Also, an evaluation of the term $\chi(\nabla_p\cdot\mathbf{V}_2+\partial\omega/\partial P)$ is usually added to the right-hand side of Eq. (2) because the evaluation of ω and the interpolation from σ to P surfaces leads to computed departures from the exact mass balance which should have been guaranteed in the original sigma coordinate calculation. In addition, such interpolated quantities are often zonally averaged and expressed in terms of convergence of fluxes by zonal-mean motions and by "eddies."

An example of an application of the above "traditional-method" analysis procedure is given in Fig. 1. These results are obtained from the general-circulation/tracer model experiment described by Mahlman (1973). This experiment is a three-dimensional simulation of tracer behavior following an instantaneous release of inert tracer in the lower stratosphere. In Fig. 1, the zonal-mean balance analysis at 110 mb shows the presence of zonal-mean tracer time changes (net tendencies) being effected by partially compensating "eddy" and zonal-mean flux convergences. In addition, the computational residual term (the fictitious term which must be added to the calculation to give an exact balance) can be seen to be generally comparable in magnitude to the net tendency term. When this occurs, the capability of the analysis to "explain" the net tracer tendency is significantly reduced.

In an effort to determine the basic cause of the large computational residuals as shown in Fig. 1, a large number of linear correlations were calculated between the computational residual and the other individual terms in the zonal-mean analysis of Eq. (2). These calculations showed that the sum of the calculated mean and eddy vertical flux convergence terms were strongly negatively correlated with the computational residual term (~ -0.65). All other correlations with the residual term were much smaller (< -0.20), suggesting that the calculation of ω produces more serious analysis errors than does the interpolation from σ to p surfaces. This seems to occur because of the difficulty in formulating a numerical procedure which calculates ω in a manner completely consistent with the sigma coordinate variables.

Examples of these difficulties can be given by first starting with the definition $P\equiv\sigma P_*$ and differentiating to get

$$\omega=P_*\dot{\sigma}+\sigma\dot{P}_*=P_*\dot{\sigma}+\sigma\left(\frac{\partial P_*}{\partial t}+\mathbf{V}_2\cdot\nabla_\sigma P_*\right). \quad (3)$$

In Eq. (3), P_*, σ and $\partial P_*/\partial t$ can be calculated directly from the sigma coordinate information. However, no completely consistent calculation of the advective term can be made directly in the flux-form sigma coordinate

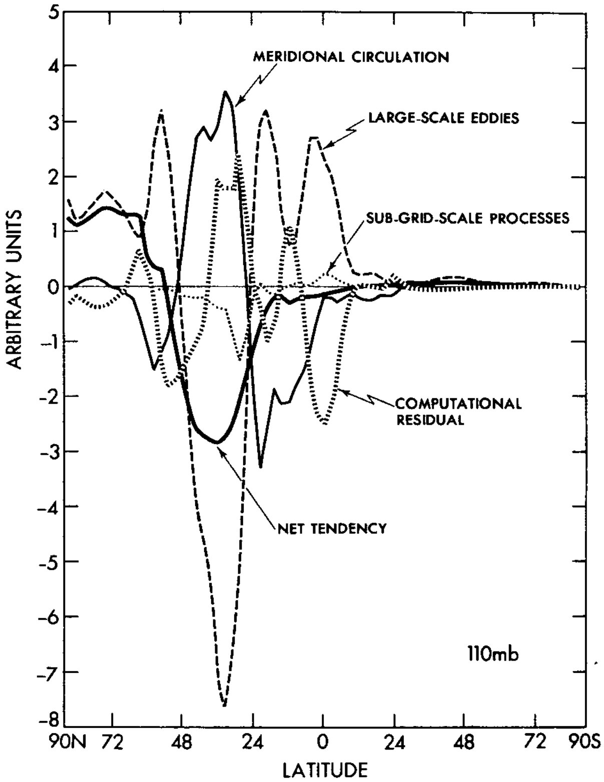

FIG. 1. "Traditional method" analysis of mechanisms leading to 110 mb changes in the zonal-mean mixing ratio of a simulated instantaneous inert tracer release in mid-latitudes of the lower stratosphere. Results are taken from the fourth month (April) of the "sigma" coordinate inert tracer experiment described by Mahlman (1973). Various terms included are net tracer tendency or time rate of change (heavy solid line); flux convergence by the meridional circulation (thin solid line); flux convergence by large-scale eddies (heavy dashed line); flux convergence due to sub-grid-scale processes (thin dashed line); and the computational residual (wide dashed line). Arbitrary units.

models. As a result, there are a large number of possible choices for an approximate numerical evaluation of $\mathbf{V}_2\cdot\nabla_\sigma P_*$. Calculating ω directly from Eq. (3) has also been found to produce significant numerical agitation when applied to the thermodynamic equation in a model calculation (e.g., Holloway and Manabe, 1971).

Another approach to the calculation of ω begins by substituting the mass continuity equation

$$\frac{\partial P_*}{\partial t}=-\nabla_\sigma\cdot P_*\mathbf{V}_2-\frac{\partial}{\partial\sigma}P_*\dot{\sigma} \quad (4)$$

into Eq. (3) to obtain

$$\omega=P_*\dot{\sigma}-P_*\sigma\left(\frac{\partial\dot{\sigma}}{\partial\sigma}+\nabla_\sigma\cdot\mathbf{V}_2\right) \quad (5)$$

(Holloway and Manabe, 1971; attributed originally to W. E. Sangster and J. Smagorinsky). This form would be consistent with the model equations if applied to results from a model in which all variables and derivatives appear at the same model level in the vertical.

However, for a number of reasons, this is not done in the current sigma coordinate models. Usually, $\dot{\sigma}$ is located halfway between model levels and all other dependent variables are located at model levels. Thus, in the evaluation of Eq. (5), $\dot{\sigma}$ is readily obtained between model levels, while $\partial\dot{\sigma}/\partial\sigma$ and $\nabla_\sigma \cdot \mathbf{V}_2$ are consistently calculated at model levels. Consequently, the numerical calculation of Eq. (5) requires some form of interpolation whether ω is chosen to be evaluated at model levels or at half-levels. For analysis purposes, it is generally better to evaluate ω at model half-levels to insure greater consistency with the model calculation of $\dot{\sigma}$.

Both Eqs. (3) and (5) were used for evaluation of ω in the usual analysis scheme. The computational residuals of Fig. 1, obtained by evaluating ω from Eq. (5), were slightly improved over those from the analysis scheme which used Eq. (3). However, the relatively large error in both schemes suggests that the most important analysis difficulty is due to the fundamental ambiguity in the evaluation of ω.

3. An alternative procedure

As noted above, the ambiguity in calculating ω appears to be a larger source of analysis error than the error produced by interpolation from σ to P surfaces. This suggests that significant improvements in analysis accuracy might be obtained in the various budgets if a direct calculation of ω were to be avoided.

This can be accomplished by a manipulation of Eqs. (1), (2) and (4). Thus, we multiply Eq. (2) by P_*, multiply Eq. (4) by χ, and add to obtain

$$\frac{\partial}{\partial t}P_*\chi = -\chi\left(\nabla_\sigma \cdot P_*\mathbf{V}_2 + \frac{\partial}{\partial\sigma}P_*\dot{\sigma}\right) - P_*\nabla_p \cdot \mathbf{V}_2\chi - P_*\frac{\partial}{\partial P}\omega\chi. \quad (6)$$

We now use Eq. (1) to eliminate $\partial P_*\chi/\partial t$ and solve for the P coordinate flux divergences to get

$$\frac{\partial}{\partial P}\omega\chi + \nabla_p \cdot \mathbf{V}_2\chi = \left[\frac{1}{P_*}\left(\nabla_\sigma \cdot P_*\mathbf{V}_2\chi + \frac{\partial}{\partial\sigma}P_*\dot{\sigma}\chi\right) - \frac{\chi}{P_*}\left(\nabla_\sigma \cdot P_*\mathbf{V}_2 + \frac{\partial}{\partial\sigma}P_*\dot{\sigma}\right)\right] \quad (\sigma \text{ surfaces}). \quad (7)$$

This is now in a form where the local isobaric flux divergence can be calculated in a manner which is exactly consistent with the method of calculation in the sigma coordinate framework. Note that the method does not allow for exact evaluation of the isobaric flux convergence terms individually, but only guarantees that their sum is accurately calculated. However, if these indirect calculations of isobaric flux convergence on the right-hand side of Eq. (7) are interpolated to P surfaces and zonally averaged, an evaluation is obtained for

$$\left[\frac{\partial}{\partial P}\overline{\omega\chi}^\lambda + \frac{1}{a\cos\theta}\frac{\partial}{\partial\theta}\overline{(v\chi\cos\theta)}^\lambda\right] = B \quad (P \text{ surfaces}), \quad (8)$$

where the λ overbar is a zonal mean, a the earth's radius, v the northward wind component and θ latitude. In Eq. (8), B is defined as

$$B \equiv \left[\overline{\frac{1}{P_*}\left(\nabla_\sigma \cdot P_*\mathbf{V}_2\chi + \frac{\partial}{\partial\sigma}P_*\dot{\sigma}\chi\right)}^\lambda - \overline{\frac{\chi}{P_*}\left(\nabla_\sigma \cdot P_*\mathbf{V}_2 + \frac{\partial}{\partial\sigma}P_*\dot{\sigma}\right)}^\lambda\right] \quad (\text{interpolated to } P \text{ surfaces}). \quad (9)$$

However, Eqs. (8) and (9) only allow an accurate calculation of the sum of the zonal-mean isobaric flux convergences. It is still desirable to provide an accurate evaluation of the individual components. One can begin by the usual separation of mean and eddy quantities, i.e.

$$\left.\begin{aligned} \overline{\omega\chi}^\lambda &= \overline{\omega}^\lambda\overline{\chi}^\lambda + \overline{\omega'\chi'}^\lambda \\ \overline{v\chi}^\lambda &= \overline{v}^\lambda\overline{\chi}^\lambda + \overline{v'\chi'}^\lambda \end{aligned}\right\}, \quad (10)$$

where $(\ )' = (\ ) - \overline{(\ )}^\lambda$.

However, a direct calculation of ω is still to be avoided. One can proceed by evaluating $\overline{v}^\lambda$, $\overline{\chi}^\lambda$ and $\overline{v\chi}^\lambda$ on P surfaces directly from the sigma coordinate data and assume that the interpolation error is not too serious. Fortunately, there is an independent check on this because the zonal-mean pressure coordinate continuity equation

$$\frac{1}{a\cos\theta}\frac{\partial}{\partial\theta}\overline{v}^\lambda\cos\theta + \frac{\partial}{\partial P}\overline{\omega}^\lambda = 0 \quad (11)$$

should be satisfied. This can be directly checked using the calculated values of $\overline{v}^\lambda$ by numerically integrating Eq. (11) from the top of the model atmosphere (where $\overline{\omega}^\lambda = 0$) to the surface. If the calculated values of $\overline{\omega}^\lambda$ at the surface are found to be consistent with the surface boundary condition, one can be reasonably certain that the interpolation error in $\overline{v}^\lambda$ is very small and that, as a consequence, $\overline{\omega}^\lambda$ is also nearly correct. The results show this to be true to an excellent approximation. Such a calculation provides a strong necessary test of interpolation skill because of the well-known error growth associated with numerical integration of the mass continuity equation when small errors are present in the data (e.g., O'Brien, 1970).

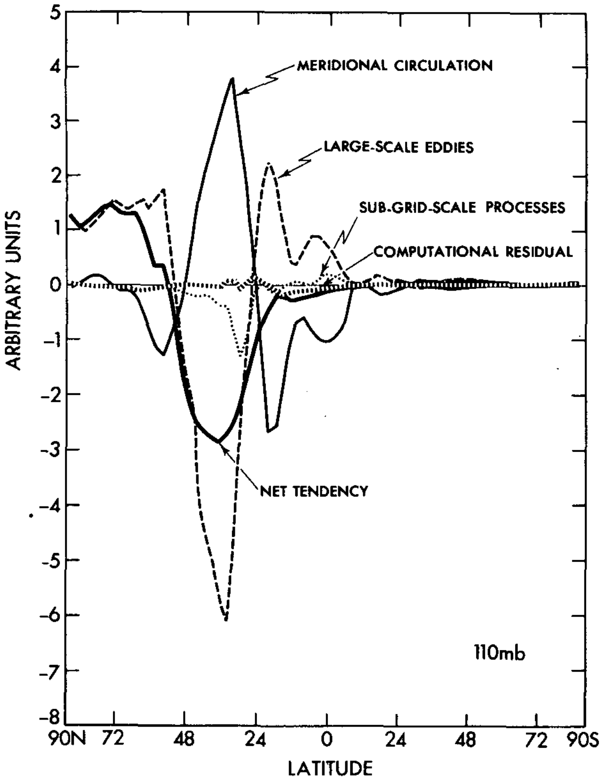

FIG. 2. "Alternative method" analysis of the same basic data set used to obtain the results shown in Fig. 1. Notations and units are the same as in Fig. 1.

Once $\bar{v}^{\lambda}$ can be calculated with confidence, it follows that $\bar{\chi}^{\lambda}$ and $\overline{v\chi}^{\lambda}$ can probably also be calculated with confidence. In addition $\bar{\omega}^{\lambda}$ is now inferred from $\bar{v}^{\lambda}$ so that it is exactly consistent with Eq. (11). Consequently, the last remaining term, the isobaric vertical eddy flux convergence, can be calculated from

$$\frac{\partial}{\partial P}\overline{\omega'\chi'}^{\lambda} = B - \frac{1}{a\cos\theta}\frac{\partial}{\partial\theta}\bar{v}^{\lambda}\bar{\chi}^{\lambda}\cos\theta - \frac{\partial}{\partial P}\bar{\omega}^{\lambda}\bar{\chi}^{\lambda} - \frac{1}{a\cos\theta}\frac{\partial}{\partial\theta}\overline{v'\chi'}^{\lambda}\cos\theta, \quad (12)$$

where B is given by (9). Now that the vertical flux convergences have been obtained, the vertical fluxes can be readily calculated by integrating downward from the top of the model atmosphere.

A test of this alternative scheme was performed by applying it to the same data set as given in Fig. 1. The results of this test are given in Fig. 2 and show that the magnitude of the computational residual term has been significantly reduced. In fact, the ratio of the rms computational residual of Fig. 2 to that of Fig. 1 is 0.067. In addition, the magnitude of the computational residual term is now significantly smaller than that of the net tendency term.

The alternative scheme thus exhibits a much higher potential for "explaining" the basic balances and the time variations. An illustration of this is given in Table 1, which shows the results of a calculation of the correlation coefficient between the net tendency term and all the terms in the original zonally averaged tracer continuity equation (including the convergence of subgrid-scale fluxes and the source-sink term which is zero above 315 mb). Table 1 shows a marked difference in the relative skill of the two analysis methods as applied to this problem. Note that the alternative scheme still does not show perfect skill. The unexplained remainder appears to be due to problems of interpolation and sampling error. Also, note that the skill of the alternative method deteriorates significantly at the bottom two levels. This appears to occur because of aggravated interpolation error when a significant fraction of the points in the P coordinate are below the earth's surface.

4. Summary

Employment of the alternative analysis procedure suggested in the previous section has been shown to produce a substantial improvement in the skill of the "sigma" coordinate budget analysis evaluated on pressure surfaces. This improvement results from avoiding an ambiguous direct calculation of ω from the sigma coordinate information.

The improved method should be applicable to analysis applied to any well-defined coordinate surfaces using variables calculated in the sigma coordinate. However, the only tests to date have been from sigma to pressure coordinates with an inert tracer calculation. The magnitude of improvement should be somewhat smaller when applied to variables, such as momentum,

TABLE 1. Calculation of the correlation coefficient (ρ) between the net tendency term and the remaining terms in the trace constituent continuity equation. The correlations are a monthly mean (April, month 4) of the results from the mid-latitude point source tracer experiment of Mahlman (1973). Values of ρ are presented for each model analysis level.

Pressure (mb)	ρ (traditional scheme)	ρ (alternative scheme)
10	+0.663	+0.965
38	+0.001	+0.989
65	+0.588	+0.998
110	+0.819	+0.994
190	+0.888	+0.997
315	+0.357	+0.988
500	+0.663	+0.983
685	+0.420	+0.980
835	+0.345	+0.986
940	+0.681	+0.931
990	−0.040	+0.654
Entire volume	+0.621	+0.993

which are not as sensitively dependent upon an accurate evaluation of vertical flux convergences.

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