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The equatorial thermocline*

S. G. H. PHILANDER†

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Abstract—In the temporary absence of surface winds, density gradients in the ocean can give rise to an eastward equatorial current which has a width, depth and maximum speed comparable to that of the equatorial undercurrent. This is offered as an explanation for the deeper of the two eastward currents (the one *in* the thermocline) which HISARD, MERLE and VOITURIEZ (1970) observed at the equator in the western Pacific. It also explains the eastward surface current observed at the equator when the trade winds are weak or absent.

One striking feature of the meridional circulation is the occurrence of downwelling. It is suggested that this accounts for the downward spreading of the equatorial thermocline and for the deep penetration of water of high oxygen and low phosphate concentration at the equator.

1. INTRODUCTION

ACCORDING to constant density models of the equatorial undercurrent, its intensity is proportional to the westward component of the winds so that it disappears when the winds are absent. This is contrary to observations. JONES' (1969) comparison of measurements of the undercurrent at 97°W reveals that the highest maximum eastward velocity was observed when the winds were weakest (143 cm/sec on April 1, 1968, when the winds varied between 0·3 and 3·3 m/sec) and that the lowest maximum eastward velocity was observed when the winds were the most intense (87 cm/sec in October, 1961, when the winds varied between 3·4 and 7·9 m/sec). The observations of BROSIN and NEHRING (1968) at 30°W in the Atlantic also show that the undercurrent was more intense at the time of their second set of measurements when the winds were light, than it was on the occasion of their first set when the wind speed was about 5 m/sec.

Models that neglect density gradients are inadequate in a further respect. They fail to explain the lower of the two eastward cells of the Undercurrent in the western Pacific—the cell in the thermocline—and do not account for the persistence of this eastward flow when the winds have an eastward component (NOËL and MERLE, 1969; HISARD, MERLE and VOITURIEZ, 1970). The homogeneous ocean models may, however, be relevant to the motion in the well-mixed layer above the thermocline in the western Pacific. One source of eastward momentum for the sub-surface eastward current at the base of this homogeneous layer is the eastward pressure gradient which results when a westward wind causes the sea surface to slope upward from east to west. Such a pressure force is present in the constant density models.

MONTGOMERY and PALMÉN (1940) have suggested that the observed slope of the isotherms, upwards from west to east in the tropics, is caused by the trades driving

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†Geophysical Fluid Dynamics Program, Princeton University, Princeton, New Jersey, U.S.A.

low-density water to the west. If the pressure gradients at great depths are assumed small then this slope of the isotherms implies an eastward pressure force near the surface. It may thus be said that because the constant density models include an eastward pressure force, they incorporate the effects of stratification in a crude manner. It is, however, only the long-term interaction between the wind and density field which can be modeled in this way. If the winds should die out for a short period of time then the slope of the isotherms, and the associated pressure gradients will be modified but will not disappear. The models which neglect density gradients cannot accommodate both the short and the long-term effects of the winds and cannot predict changes in the structure of the Undercurrent due to variations in the 'local' winds. This accounts for the earlier mentioned discrepancies between theories and observations.

It is the purpose of this paper to investigate the equatorial flow caused by density gradients (which in turn are caused by the long-term effect of the winds) when the 'local' winds die out for a short period of time. We disregard the local winds by assuming that the ocean surface is stress-free. We incorporate the long-term effects of the winds by assuming that the sea surface temperature increases from east to west in the observed manner. The extra-equatorial thermo-haline circulation due to these density gradients has been studied extensively. The known solutions all have equatorial singularities. In this paper we attempt to remove the singularities.

In the next section we summarize the results to be obtained in the subsequent six sections. The final section (9) is a comparison of these results with some of the observed features of the Undercurrent. The modification of the 'thermally driven' flow to be described here, due to local winds will be the subject of a later paper.

2. SUMMARY OF THE RESULTS

It will be assumed that the flow in the extra-tropical thermocline is in geostrophic balance. The vorticity equation

$$\beta v = f w_z \tag{1.1}$$

follows immediately (see Section 3 for the explanation of notations). In the absence of any wind stress, the thermocline is necessarily diffusive and is maintained by a balance between the downward conduction of heat and the upward advection of cold water. This implies that the vertical velocity at the base of the thermocline (w_∞) is positive (upwards). It then follows from the integration of (1.1) across the thermocline that the meridional transport in the thermocline is equatorward in both hemispheres

$$\int v dz = f/\beta \, w_\infty . \tag{1.2}$$

(Recall that the Coriolis parameter f changes sign when the equator is crossed but that its latitudinal gradient β does not.) There is consequently a convergence of fluid at the equator. Because of the eastward pressure gradient (associated with the increase in sea surface temperature from east to west) this convergence gives rise to an eastward equatorial current.

It should be possible to determine the order of magnitude of the width, depth and downstream velocity of the current from a scale analysis. This scale analysis must take into consideration that the solutions describing the extra-equatorial flow are singular at the equator because certain physical processes which are negligible away from the equator cease to be negligible close to the equator. ROBINSON (1960) who assumed that it is the non-linear advection of zonal momentum which becomes important equatorially, showed that the eastward equatorial current has a width, depth and speed comparable to that of the observed equatorial undercurrent. This current has its downstream flow in geostrophic balance.

Since the equations describing flow in the equatorial thermocline are fully three-dimensional, the problem of solving them is a formidable one. It is, however, possible to reduce the number of independent coordinates from three to two by making the zonal variations appear implicitly rather than explicitly. This may be accomplished by means of a similarity transformation. The motivation for a similarity transformation is the absence of an intrinsic zonal scale in the phenomenon. For example, sections across the current at various longitudes look, apart from an apparent stretching of the coordinates, very similar (see KNAUSS, 1966, Fig. 3). If the similarity transformation is chosen appropriately—so as to be consistent with the increase in sea surface temperature from east to west—then it may be shown that the eastward equatorial current becomes slower, narrower and shallower, and that its transport decreases, downstream.

Since the current is maintained by an influx of fluid from the sides, conservation of mass requires that there be equatorial downwelling if the transport is to decrease downstream. The extra-tropical balance between downward conduction of heat and upward advection of cold water is clearly not possible in the equatorial thermocline since the two processes now transmit heat in the same direction. An advection of water from colder regions is required. Since isotherms slope upwards towards the east, the direction from which the cold water must be advected is the east. Hence a westward current below the eastward surface current is an integral feature of this model.

The numerical solutions of the quasi-two-dimensional (similarity transformed) equations confirm the above inferences about the nature of the flow in the equatorial thermocline.

3. EQUATIONS OF MOTION

The natural parameters for the problem are R, L, D, K_v, K_H, ν_v, ν_H, α, g, Ω and ΔT which respectively denote the radius of the Earth, a characteristic horizontal length scale, the depth of the ocean, the vertical and horizontal thermal eddy diffusivities, the vertical and horizontal eddy viscosities, the coefficient of thermal expansion, the gravitational acceleration, the rate of the Earth's rotation and the difference in apparent temperature between two points either a horizontal distance L apart or a vertical distance D apart.

Let x', y', z' be westward, upward and northward directions, respectively. The origin of the y' axis is at the equator, that of z' lies in the ocean surface and that of x' is at the eastern edge of the ocean. (Note that this is a left-handed coordinate system.) The velocity components in the x', y' and z' directions are u', v' and w', respectively. T' is the apparent temperature.

Non-dimensionalize as follows

$$(x', y', z') = (xL, yL, zD); \; p' = \frac{2\Omega\, L^3 K_v \rho_0}{D^2\, R}\, p \qquad (2.1a, b)$$

$$(u', v', w') = \left(u, v, \frac{D}{L} w\right) \frac{K_v L}{D^2}; \; T' = \Delta T \, T. \tag{2.1c, d}$$

If we assume a hydrostatic balance for the vertical momentum equation, then the equations of motion expressing conservation of mass, momentum and heat (assumed valid in Boussinesq form) are

$$-E(u_{zz} + \sigma_1 \lambda^2 [u_{xx} + u_{yy}]) + \frac{E}{\sigma}(uu_x + vu_y + wu_z) + yv + p_x = 0 \tag{2.2a}$$

$$-E\{v_{zz} + \sigma_1 \lambda^2 (v_{xx} + v_{yy})\} + \frac{E}{\sigma}(uv_x + vv_y + wv_z) - yu + p_y = 0 \tag{2.2b}$$

$$-\frac{1}{\varepsilon} T + p_z = 0 \tag{2.2c}$$

$$u_x + v_y + w_z = 0 \tag{2.2d}$$

$$-T_{zz} - \sigma_1 \lambda^2 (T_{xx} + T_{yy}) + uT_x + vT_y + wT_z = 0 \tag{2.2e}$$

The Ekman number $E = v_v R / 2\Omega \, D^2 \, L$. (2.3a)

The Rayleigh number $R_a = \alpha g \, \Delta T \, D^3 / v_v \, K_v$. (2.3b)

The aspect ratio $\lambda = D/L$. (2.3c)

$$\varepsilon = 1/R_a \, E \, \lambda^2. \tag{2.3d}$$

It will be assumed that the value of the Prandtl number $\sigma = v_v/K_v$ is one, that the ratios v_H/v_v and K_H/K_v have the same value σ_1, and that the non-dimensional numbers ε, λ and E are much smaller than one. The analysis will also require, and it will be assumed, that

$$E \ll \varepsilon^{2/3} \rightarrow R^5 \, v_v^3 \, (\alpha g \, \Delta T)^2 / \Omega^5 \, L^9 \, K_v^2 \ll 1. \tag{2.4a}$$

The horizontal Coriolis terms will be neglected throughout. This assumption is contingent on

$$[\Omega^5 \, K_v^3 \, L / v_v^2 \, (\alpha g \, \Delta T)^3]^{1/5} \ll 1. \tag{2.4b}$$

Reasonable oceanic values for the various parameters are

$$\begin{aligned} &\Omega = 0{\cdot}7 \times 10^{-4} \text{ sec}^{-1} \qquad && D = 5 \times 10^5 \text{ cm} \\ &L = 10^8 \text{ cm} && \alpha g \, \Delta T = 3 \text{ cm sec}^{-2}. \end{aligned} \tag{2.5}$$

It follows that $\varepsilon \sim 0(v_v \, 10^{-5})$, $E \sim 0(v_v \, 10^{-7})$, $\lambda \sim 0(2 \times 10^{-2})$ so that (2.4a and b) are satisfied provided v_v does not exceed 10^3 cm^2 sec^{-1}.

For the sake of mathematical expediency, it will be assumed that K_v, K_H, v_v and v_H are functions of x. This assumption is no less arbitrary than the more usual one that these coefficients have constant values. In the rest of this paper K_v, K_H, v_v and v_H will denote non-dimensionalized coefficients of diffusivity whenever they appear in the equations of motion.

4. THE EXTRA-EQUATORIAL FLOW

It is assumed that the flow in the extra-equatorial thermocline is hydrostatic and is such that the horizontal velocities are in geostrophic balance. It is also assumed that the role of the horizontal diffusion of heat is of secondary importance. If the variables are scaled as follows

$$u(x, y, z) = \varepsilon^{-2/3}\, \tilde{u}(x, y, \tilde{z}); \qquad v(x, y, z) = \varepsilon^{-2/3}\, \tilde{v}(x, y, \tilde{z})$$
$$w(x, y, z) = \varepsilon^{1/3}\, \tilde{w}(x, y, \tilde{z}); \qquad z = \varepsilon^{1/3}\, \tilde{z}, \tag{3.1}$$

then the equation of motion may then be written

$$0(E\varepsilon^{-2/3}) + y\tilde{v} + \tilde{p}_x = 0 \tag{3.2}$$

$$0(E\varepsilon^{-2/3}) - y\tilde{u} + \tilde{p}_y = 0 \tag{3.3}$$

$$-T + \tilde{p}\tilde{z} = 0 \tag{3.4}$$

$$\tilde{u}_x + \tilde{v}_y + \tilde{w}_{\tilde{z}} = 0 \tag{3.5}$$

$$-K_v\, T_{\tilde{z}\tilde{z}} + \tilde{u}\, T_x + \tilde{v}\, T_y + \tilde{w}\, T\tilde{z} = 0. \tag{3.6}$$

To reduce this set of partial differential equations to a single ordinary differential equation, we follow ROBINSON and WELANDER (1963) and write

$$\tilde{u} = x^{t+2r+1}\, y^{2s}\, [2s + 2)F' + s\xi F''] \tag{3.7}$$

$$\tilde{v} = -\, x^{2r+t}\, y^{2s+1}\, [(2r + 1 + t)F' + r\xi F''] \tag{3.8}$$

$$\tilde{w} = -\, x^{r+t}\, y^{s}\, [(1 + r + t)F + r\xi F'] \tag{3.9}$$

$$T = x^{3r+1+t}\, y^{3s+2}\, F'', \tag{3.10}$$

where F is a function of ξ only and

$$\xi = x^r\, y^s\, \tilde{z}. \tag{3.11}$$

F satisfies the equation

$$0 = -\, F^{iv} + (r - s - ts)\xi F'F'' + (2r - s - ts)(F'F'' - \xi F''^2) - (1 + r + t)FF'''. \tag{3.12}$$

It has been assumed that the coefficient of vertical eddy diffusivity has an x dependence of the form x^t. The range of values which the constants r, s and t may assume will be discussed in Section 6. The appropriate boundary conditions are the following:

(1) The non-dimensional sea surface temperature is prescribed

$$F'' = 1 \text{ on } \xi = 0. \tag{3.13a}$$

(2) The vertical velocity vanishes at the surface

$$F = 0 \text{ on } \xi = 0. \tag{3.13b}$$

(3) The temperature and horizontal velocities vanish at great depth

$$F', F'' \to 0 \text{ as } \xi \to -\infty. \tag{3.14a}$$

Should the model have a rigid bottom, then

(4) the temperature and velocity normal to the bottom vanish at $\xi = -H$. If the bottom is flat,

$$F' = F'' = 0 \text{ on } \xi = -H. \tag{3.14b}$$

Equation (3.12) may be solved either by the Runga Kutta method or by converting the equation into a time-dependent one. [The introduction of the term F_t'' into (3.12) corresponds to the introduction of a time-dependent term into the heat equation (3.6.) Solutions are obtained by iterating in time until a steady state is reached]. The latter method was adopted.

For the sea surface temperature to decrease with increase in latitude it is necessary that $s < -2/3$. Unless $s = -2/3$ the surface temperature, a given boundary condition, is singular at the equator. Therefore, we put $s = -2/3$ and note that equation (3.12) cannot be valid close to the equator because the various fields have singularities as $y \to 0$. As a result, some of the terms which have been neglected in equations (3.2)–(3.6) need to be taken into consideration equatorially.

5. THE EQUATIONS DESCRIBING FLOW IN THE EQUATORIAL THERMOCLINE

We make the following assumptions about the flow in the equatorial thermocline:

(1) the downstream flow is in geostrophic balance.

(2) The zonal pressure gradient is non-zero at the equator and, in the x momentum equation, is balanced by the non-linear terms when the vertical Coriolis force vanishes at the equator.

(3) The flow is hydrostatic.

(4) The vertical conduction of heat is balanced by the advection of heat.

(5) The terms v_y and w_z in the continuity equation are equally important.

(6) The equatorial flow matches smoothly with the extra-equatorial flow when $y \ll 1$ but $Y \gg 1$.

Write

$$u = E^{N_1}\varepsilon^{n_1}[U(x, Y, Z) + E^N\varepsilon^n U^{(1)}(x, Y, Z) + \ldots] \tag{4.1a}$$

$$v = E^{N_2}\varepsilon^{n_2}[V(x, Y, Z) + E^N\varepsilon^n V^{(1)}(x, Y, Z) + \ldots] \tag{4.1b}$$

$$w = E^{N_3}\varepsilon^{n_3}[W(x, Y, Z) + E^N\varepsilon^n W^{(1)}(x, Y, Z) + \ldots] \tag{4.1c}$$

$$p = E^{N_4}\varepsilon^{n_4}[P(x, Y, Z) + E^N\varepsilon^n P^{(1)}(x, Y, Z) + \ldots] \tag{4.1d}$$

$$T = T(x, Y, Z) + E^N\varepsilon^n T^{(1)}(x, Y, Z), \tag{4.1e}$$

where

$$y = E^{N_6}\varepsilon^{n_6}y \tag{4.2a}$$

$$z = E^{N_7}\varepsilon^{n_7}Z. \tag{4.2b}$$

From (3.7) we know that at a reasonable distance from the equator (when $Y > 1$)

$$u = \varepsilon^{-2/3}x^{t+2r+1}y^{2s}G(\xi)$$

$$= \varepsilon^{-2/3+2sn_6}E^{2sN_6}x^{t+2r+1}Y^{2s}G(\varepsilon^{n_7-1/3+sn_6}E^{N_7+sn_6}x^rY^sZ). \tag{4.3}$$

For assumption (6) to be satisfied it is necessary that

$$n_1 = -2/3 + 2sn_6;\ N_1 = 2sN_6. \tag{4.4}$$

From assumptions (1)–(5) we obtain similar equations. A different set of equations would result if, rather than the zonal velocity, one of the other fields was matched to its extra-equatorial counterpart. Since it may be shown that such a new set does not constitute an independent set of equations, the solutions are unique. These are shown in Table 1 for the case $s = -2/3$. The orders of magnitude are seen to be in reasonable agreement with the observed values if $K_v \sim 10$ cm^2 sec^{-1}. Since the equatorial scaling depends on the nature of the singularity of the extra-equatorial flow and since the extra-equatorial solution which has been chosen is not very realistic, better agreement with observation than that depicted in Table 1 should not be expected. The solution to the extra-tropical flow (that of ROBINSON and WELANDER, 1963) is not very realistic because it has the depth of the thermocline decreasing much too rapidly as the equator is approached, by a factor of 1·7 between 10°N and 5°N if the surface temperature is independent of latitude.

Table 1. *Scaling for the equatorial thermocline. The orders of magnitude of various fields are shown for various values of the coefficient of vertical eddy diffusivity. The values for the other parameters are given in* (2.5). *The value of the Prandtl number* ν_v/K_v *has been taken to be one.*

Diffusivity (cm^2/sec)	K_v	1	10	100
Width (cm)	$\left(\frac{\alpha^2 g^2 \Delta T^2 K_v L R^5}{\Omega^5}\right)^{1/10}$	$1{\cdot}8 \times 10^7$	$2{\cdot}2 \times 10^7$	$2{\cdot}8 \times 10^7$
Depth (cm)	$\left(\frac{K_v{}^2 R^2}{\alpha g \Delta T}\right)^{1/5}$	$0{\cdot}28 \times 10^4$	$0{\cdot}7 \times 10^4$	17×10^4
Zonal velocity (cm/sec)	$(\alpha^2 g^2 \Delta T^2 K_v L)^{1/5}$	80	100	125
Meridional velocity (cm/sec)	$\left(\frac{\alpha^6 g^6 \Delta T^6 K_v{}^3 R^5}{\Omega^5 L^7}\right)^{1/10}$	10	21·8	43
Vertical velocity (cm/sec)	$(\alpha g \Delta T K_v{}^3/L^2)^{1/5}$	8×10^{-4}	$31{\cdot}3 \times 10^{-4}$	125×10^{-4}

It is possible to infer from Table 1 that the agreement between the predicted and observed scales depends critically on the value of the coefficient of thermal diffusivity which is assumed to be a constant. This assumption is of dubious merit. We therefore point out that if it is assumed that the vertical scale H of the equatorial current is known and if assumption (6) is discarded then all the scales are independent of the coefficient of diffusion. For example,

The downstream velocity $\sim (\alpha g\, \Delta T\, H)^{1/2}$
The width $\sim (\alpha g\, \Delta T\, H R^2/4\Omega^2)^{1/4}$.

From this point of view the depth-scale is determined by extra-equatorial considerations, i.e. by the requirement that there be smooth matching as $Y \to \infty$. Since the equatorial current is maintained by the equatorward flow in the extra-equatorial thermocline, it is reasonable to assume that the depth-scale of the current is simply that of the thermocline just outside the current. A realistic description of the thermocline away from the equator will, therefore, eliminate the need to resort to constant coefficients of eddy diffusivity in the scaling of the equatorial current.

The equations of motion governing flow in the equatorial thermocline may now be written as follows

$$-\nu_v U_{zz} - \Gamma U_{yy} + UU_x + VU_y + WU_z + YV + P_x = 0 \tag{4.5a}$$

$$0(E^{3/5}\varepsilon^{-2/5}) - YU + P_y = 0 \tag{4.5b}$$

$$-T + P_z = 0 \tag{4.5c}$$

$$U_x + V_y + W_z = 0 \tag{4.5d}$$

$$-K_v T_{zz} - \Gamma T_{yy} + UT_x + VT_y + WT_z = 0. \tag{4.5e}$$

The validity of the scaling is ensured by conditions (2.4). The value of the coefficient

$$\Gamma = \frac{\nu_H}{\nu_v}\left(\frac{\Omega^5 K_v^{\,3}}{RL\alpha^4 g^4 \Delta T^4}\right)^{1/5} \tag{4.6}$$

is assumed to be not larger than one. This restricts the value of the coefficients of lateral friction ν_H to

$$\nu_H < \nu_v^{\,2/5}\, 10^8 \text{ cm}^2 \text{ sec}^{-1}. \tag{4.7}$$

Without lateral friction and lateral conductivity in the model it is possible for the fields to have cusps (discontinuous meridional derivatives) at the equator. If $\Gamma \ll 1$, then a thin shear layer centered on the equator and of width $0(\Gamma^{1/2} L)$ serves the purpose of removing these cusps. To resolve such a thin boundary layer in a numerical model, a very dense grid is required. To avoid this complication we choose $\Gamma \lesssim 1$.

The singularity of the extra-equatorial flow could have been removed by invoking lateral friction and diffusion rather than the inertial terms. This would be necessary if $\Gamma \gg 1$ in which case the scaling would be different. This parameter range will not be considered here.

Because of the geostrophic equation (4.5b) the temperature and zonal velocity are not entirely independent and care must be taken to have consistent boundary conditions:

(a) $W = 0$, $T = T_0(x)$, $U_z = 0$ on the surface $Z = 0$. (4.8a)

(b) $T_y = U_y = V = 0$ at the equator $y = 0$, which is assumed to be a line of symmetry. (4.8b)

(c) The influx of heat (T) and mass (V) must be specified in a region where $y \ll 1$ but $Y \gg 1$. (4.8c)

(d) If the ocean is assumed to be infinitely deep, then

$$U,\ V,\ T \to 0 \text{ as } Z \to -\infty. \tag{4.8d}$$

(e) If a rigid but slippery bottom is assumed, then U_z, T and the normal velocity must vanish on the bottom

$$Z = -Z_0;\ Z_0 \gg 1. \tag{4.8e}$$

6. THE SIMILARITY TRANSFORMATION

We attempt to eliminate explicit x dependence from equations (4.5) by means of a similarity transformation. Write

$$\begin{aligned} U &= \bar{u}(x)\mu(\eta, \zeta) & P &= \bar{p}(x)\pi(\eta, \zeta) \\ V &= \bar{v}(x)\nu(\eta, \zeta) & T &= \bar{T}(x)\theta(\eta, \zeta) \\ W &= \bar{w}(x)\omega(\eta, \zeta) & & \end{aligned} \tag{5.1}$$

where $Y = h(x)\eta$ $\qquad Z = g(x)\zeta.$ (5.2)

Substitute the above expressions into equations (4.5) and demand that each term separate into the product of a function of x and a function of η and ζ. For example, from the hydrostatic equation

$$\bar{T}\theta = \frac{\bar{p}}{g}\pi\zeta \tag{5.3a}$$

the equation

$$\bar{T} = a_1 \bar{p}/g \tag{5.3b}$$

where a_1 is an arbitrary constant, is obtained. On solving the various equations for $\bar{u}, \bar{v} \ldots$, it is found that

$$\eta = x^{\alpha} y \qquad \zeta = x^{\beta} Z \tag{5.4a, b}$$

$$U = x^{-2\alpha} \mu \qquad V = x^{-3\alpha-1} v \tag{5.4c, d}$$

$$W = x^{-2\alpha-\beta-1}\omega \qquad T = x^{-4\alpha+\beta}\theta \tag{5.4e, f}$$

$$P = x^{-4\alpha}\pi. \tag{5.4g}$$

The constants α and β are, for the time being, arbitrary. The equations of motion may now be written

$$\mu_t = -\eta v + 4\alpha\pi - \alpha\eta\pi_\eta - \beta\zeta\pi_\zeta + \mu_{\zeta\zeta} + \Gamma\mu_{\eta\eta} + (5\alpha + \beta)\mu^2 - [(v + \alpha\eta\mu)\mu]_\eta - [(\omega + \beta\zeta\mu)\mu]_\zeta \tag{5.5a}$$

$$\bar{\varepsilon}v_t = \eta\mu - \pi_\eta \tag{5.5b}$$

$$0 = -\theta + \pi_\zeta \tag{5.5c}$$

$$0 = -(3\alpha + \beta)\mu + (v + \alpha\eta\mu)_\zeta + (\omega + \beta\zeta\mu)_\zeta \tag{5.5d}$$

$$\theta_t = \theta_{\zeta\zeta} + \Gamma\theta\eta\eta + 7\alpha\mu\theta - [(v + \alpha\eta\mu)\theta]_\eta - [(\omega + \beta\zeta\mu)\theta]_\zeta. \tag{5.5e}$$

(We are interested in the steady-state solution so that the time-dependent terms on the left-hand side may be ignored. The presence of these terms will be explained in Section 5.)

The boundary conditions (4.8c) at $Y \gg 1$, in terms of the new variables $\eta, \zeta, \mu, v \ldots$ etc. are, after the use of (3.10),

$$x^{-4\alpha+\beta}\,\theta(\eta_0, \zeta) = x^{3r+1-\alpha(3s+2)+t}\,\eta_0{}^{3s+2}\,F''(x^{r-\alpha s-\beta}\,\eta_0{}^{s}\,\zeta), \tag{5.6}$$

where $\eta_0 \gg 1$. It follows that

$$t = -1 - 2\alpha - 2\beta \tag{5.7a}$$

$$r = \alpha s + \beta. \tag{5.7b}$$

Once the x dependence of the coefficient of diffusivity is given, the value of t is known. A given surface temperature implies that the values of r and s are known [see (3.10)]. The constants α and β may then be calculated from (5.7). In other words, for a given extra-equatorial flow, the equatorial flow is completely determined (in the sense that there are no free parameters left).

It has already been pointed out that if the sea surface temperature is not to increase with latitude and if it is not to be singular at the equator, then the value of s must be

$$s = -2/3, \tag{5.8}$$

so that the sea surface temperature varies with longitude only. One of the observed features of the density structure of the equatorial oceans which we wish to incorporate into our model is the upward slope of the lines of constant density as one proceeds from west to east. If we choose

$$\beta < 0 \tag{5.9}$$

and if we choose the origin of the reference system to be at the eastern coast, then (5.4 b) and (5.4 f) together imply that lines of constant density in the interior have the desired slope, even in the event of a constant surface temperature. (This explains the choice of the left-handed coordinate system in Section 2.) For the sea surface temperature to increase from east to west (as it is observed to do), it is necessary that

$$4\alpha < \beta < 0. \tag{5.10}$$

These inequalities enable us to infer certain qualitative features of the current:

(1) The zonal pressure gradient at the equator is

$$P_x = -4\alpha\pi + \beta\zeta\theta.$$

If it is assumed that the pressure gradient attenuates with increasing distance from the ocean surface and if θ is assumed to be positive then, on integrating the hydrostatic equation (5.5c) we learn that

$$\pi > 0.$$

Since the domain of ζ is $-\infty < \zeta < 0$ it follows that $P_x > 0$. Hence, the zonal driving force and the equatorial current is eastward.

(2) The zonal transport of the current ($\int\int U dY dZ$) behaves like $x^{-3\alpha-\beta}$ and therefore decreases downstream.

(3) Since the width of the jet is proportional to $x^{-\alpha}$ it gets narrower as it moves eastward.

(4) The jet also becomes shallower downstream since its depth has a $x^{-\beta}$ dependence.

(5) The zonal velocity (proportional to $x^{-2\alpha}$) decreases downstream.

(6) Given that there is an influx of mass at the sides of the jet, the current can decrease its transport as it moves downstream only if there is downwelling in the equatorial thermocline. In the introduction we explained how this implies a westward current at depth.

The picture which emerges is that of an eastward current which has its maximum transport at its western edge and which, because of a downward loss of fluid, decreases in intensity downstream. Since neither equations (4.5) nor the similarity transformation is valid in the vicinity of either the eastern or western coasts the question of where the water comes from and goes cannot be answered here. In view of the poleward flux below the equatorward flow in the extra-equatorial thermocline [this follows if (1.1) is integrated from the ocean floor to the surface], it is probable that the deep westward flow at the equator also has a poleward velocity component. We may speculate further and arrive at a circulation pattern in which the eastward equatorial current on the surface sinks at the eastern coast where it feeds fluid into the deep westward current which wells up at the western edge of the basin thus completing the circuit. Whether there is any truth in this description can only be learnt from a study of the three-dimensional, thermally driven circulation in a closed ocean basin.

7. METHOD OF SOLUTION

For the purpose of solving the equations numerically, it is expedient to convert the problem into a time-dependent one. Non-dimensionalize time with respect to the diffusive time scale $\bar{H}^2/\nu_v$ where $\bar{H}$ is the depth of the equatorial thermocline then the relevant equations are (5.5). (The coefficient $\bar{\varepsilon} = E^{3/5}\,\varepsilon^{-2/5}$). Given initial values of the variables, θ, μ_ζ and ν_ζ may be predicted from (5.5e, a and b), respectively. The vertically integrated zonal and meridional transports are predicted from an integrated vorticity equation which may be derived from (5.5a, b and c). The values of μ, ν and ω follow readily. The iterations are continued until a steady state is reached. Initially, the temperature is assumed not to vary with latitude and to have the vertical structure of the given extra-equatorial temperature. The velocity field consistent with this temperature is calculated and iterations are then continued for fifteen non-dimensional time units. Changes after ten units are very small if the iterations converge at all.

The appropriate boundary conditions are those given in equations (4.8). The 'open boundary' conditions (4.8d) were found to lead to numerical difficulties—the iteration scheme would not converge. Conditions (4.8e) corresponding to a rigid, slippery bottom were used.

The non-linear terms in equations (5.5a and e) could lead to numerical instabilities unless a difference scheme which conserves μ^2, θ^2 and the square of the vorticity is used. Because of the presence of undifferentiated non-linear terms in (5.5a and e), the situation here differs from the usual one and it is not clear that any of the conserving schemes which have been devised will, in fact, be stable unless $\alpha = \beta = 0$. In practice, the scheme given in (6.3) was found to be stable for small values of α and β.

The finite difference scheme used was the following. Let

$$\eta = j\Delta\eta \qquad \zeta = k\Delta\zeta \qquad t = m\Delta t \tag{6.1}$$

where j, k and m are integers so that

$$\mu(\eta, \zeta, t) = u^m_{jk}; \quad \nu(\eta, \zeta, t) = v^m_{jk}, \text{ etc.} \tag{6.2}$$

$\Delta\eta$ and $\Delta\zeta$ were given the value 0·25 (corresponding to sixty points in the vertical of $\zeta_0 = 15$ [see (4.8e)]. Numerical stability then necessitated $\Delta t = 0{\cdot}004$. Decreasing the values of $\Delta\eta$ and $\Delta\zeta$ to 0·15 (and that of Δt correspondingly) did not affect the results when the calculations which led to Figs. 2 and 3 (in Section 8) were repeated.

In finite difference form (5.5a) reads

$$\begin{aligned}
(u^{m+1} - u^{m-1})/(2\Delta t) = {} & (u^{m-1}_{jk+1} - 2u^{m-1}_{j,k} + u^{m-1}_{j,k-1})/\Delta\zeta^2 + \frac{\Gamma}{\Delta\eta^2}(u^{m-1}_{j+1,k} - 2u^{m-1}_{j,k} + u^{m-1}_{j-1,k}) \\
& - j\Delta\eta\, v^{m+1}_{j,k} + 4\alpha\pi^{m+1}_{j\,k} - \alpha j\Delta\eta \quad \pi^{m+1}_{j+1,k} - \pi^{m+1}_{j-1,k})/(2\Delta\eta) - \beta k\Delta\zeta\theta^{m+1}_{j,k} \\
& - \frac{1}{4\Delta\eta}(v^m_{j+1,k} + \alpha(j+1)\Delta\eta u^m_{j+1,k} + v^m_{jk} + \alpha_j\Delta\eta u^m_{jk})(u^m_{j+1,k} + u^m_{j,k}) \\
& + \frac{1}{4\Delta\eta}(v^m_{j,k} + \alpha j\Delta\eta u^m_{j,k} + v^m_{j-1,k} + \alpha(j-1)\,\Delta\eta u^m_{j-1,k})(u^m_{j,k} + u^m_{j-1,k}) \\
& - \frac{1}{4\Delta\zeta}(w^m_{jk+1} + \beta\Delta\zeta(k+1)u^m_{jk+1} + w^m_{j,k} + \beta\Delta\zeta k u^m_{j,k})(u^m_{j,k} + u^m_{j,k+1})
\end{aligned}$$

$$+\frac{1}{4\Delta\zeta}(w_{j,k}^m+\beta\Delta\zeta k u_{j,k}^m+w_{j,k-1}^m+\beta\Delta\zeta(k-1)u_{j,k-1}^m)(u_{j,k}^m+u_{j,k-1}^m)$$
$$+(5\alpha+\beta)\,u_{j,k}^{m-1}\,u_{j,k}^{m-1}. \tag{6.3}$$

Note that the Coriolis term is treated implicitly and that the undifferentiated non-linear term is lagged. The other equations may be written similarly. The pressure is obtained by integrating the hydrostatic equation after the temperature had been predicted. Only the depth-dependent part of the pressure can be obtained in this manner so that we are free to add to the velocity obtained from (6.3), a function which does not depend on ζ. This depth-independent part of the flow is obtained in the following manner. Write

$$\int_{-\zeta_0}^{0} u\,d\zeta = \phi_\eta\,;\quad \int_{-\zeta_0}^{0} v\,d\zeta = -\alpha\eta\phi_\eta + (3\alpha+\beta)\phi \tag{6.4}$$

then the integrated continuity equation is satisfied. Integrate (5.5a and b) with respect to ζ over the total depth and eliminate the pressure by cross differentiation to obtain

$$\frac{1}{\Gamma}\Phi_t = \lambda\Phi_{\eta\eta} - \lambda^2(a+4\bar{\varepsilon}\alpha^2)\eta\Phi_\eta + \lambda^3[(a+4\bar{\varepsilon}\alpha^2)(a+2\bar{\varepsilon}\alpha^2)\eta^2 - \lambda(2a+11\bar{\varepsilon}\alpha^2+3\bar{\varepsilon}\alpha\beta)]\Phi$$
$$+[(\alpha-\beta)/\Gamma + 3a^2\lambda^2 + (9\bar{\varepsilon}\alpha^2+3\bar{\varepsilon}\alpha\beta)(2a+4\bar{\varepsilon}\alpha^2)\lambda^2 + 6\bar{\varepsilon}\alpha^2 a\lambda^2 - \lambda^3 a(a+4\bar{\varepsilon}\alpha^2)\times$$
$$(a+2\bar{\varepsilon}\alpha^2)\eta^2]\eta\phi_\eta$$
$$-\left[\frac{3\alpha+\beta}{\Gamma}+\lambda^2(9\bar{\varepsilon}\alpha^2+3\bar{\varepsilon}\alpha\beta)\{-(2a+11\bar{\varepsilon}\alpha^2+3\bar{\varepsilon}\alpha\beta)+\eta^2\lambda(a+4\bar{\varepsilon}\alpha^2)(a+2\bar{\varepsilon}\alpha^2)\}\right]\phi$$
$$-\frac{\beta}{\Gamma}\int_{-\zeta_0}^{0}\zeta\theta_\eta\,d\zeta + N \tag{6.5}$$

where
$$a = -\bar{\varepsilon}\alpha(\beta+5\alpha) \tag{6.6a}$$
$$\lambda = 1/(1+\bar{\varepsilon}\alpha^2\eta^2) \tag{6.6b}$$
$$\Phi = \frac{1}{\lambda}\phi_{\eta\eta} - a\eta\phi_\eta + 3\alpha\bar{\varepsilon}(3\alpha+\beta)\phi. \tag{6.6c}$$
$$N = \left[(5\alpha+\beta)\frac{\partial}{\partial\eta} - \alpha\eta\frac{\partial^2}{\partial\eta^2}\right]\int_{-\zeta_0}^{0}\mu^2\,d\zeta - \frac{\partial^2}{\partial\eta^2}\int_{-\zeta_0}^{0} v\,d\zeta \tag{6.6d}$$

The appropriate boundary for (6.5) and (6.6) are

$$\Phi = \phi = 0 \text{ at } \eta = 0. \tag{6.7a}$$
$$\phi \text{ and } \phi_\eta \text{ are given at } \eta = \eta_0 \gg 1. \tag{6.7b}$$

Once ϕ is known (6.6c) and (6.7b) may be used to calculate a value for Φ at $\eta = \eta_0$ at every time step.

8. RESULTS

In this model, the region under consideration gains heat because of equatorward advection of heat from higher latitudes, eastward advection of heat from warmer regions and downward conduction of heat from the surface. The main sink is the deep westward current but it only partially offsets the gain in heat. Consequently, severe restrictions have to be placed on the amount of heat advected equatorward. Choosing small values for α and β is sufficient to ensure a stable temperature gradient everywhere since a small amount of heat can be lost at the bottom through conduction. Since both the temperature and temperature gradient decay away from the surface, an increase in the depth (ζ_0) of the floor of the model results in a smaller loss of heat at the bottom. To compensate, the positive temperature gradient at the surface becomes smaller. The effect is shown in Fig. 1 where temperature profiles for the case $\Gamma = 0{\cdot}75$, $\eta_0 = 4$, $\alpha = -0{\cdot}025$, $\beta = 0$ and for two different values of ζ_0 are depicted. For these values of α and β, $\zeta_0 = 15$ is the largest value which ζ_0 may assume without the temperature gradient becoming unstable at the surface. (Note that in the subsequent figures the values for α, β and η_0 are those given above, the value for ζ_0 is 12.)

Figures 2 and 3 depict isotherms and isotachs of the zonal flow in a meridional

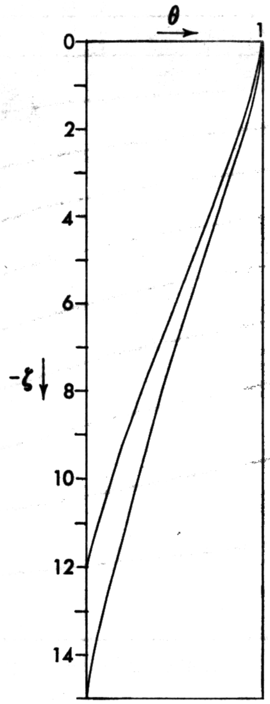

Fig. 1. Temperature as a function of depth at $\eta = 1$ for two values of ζ_0.

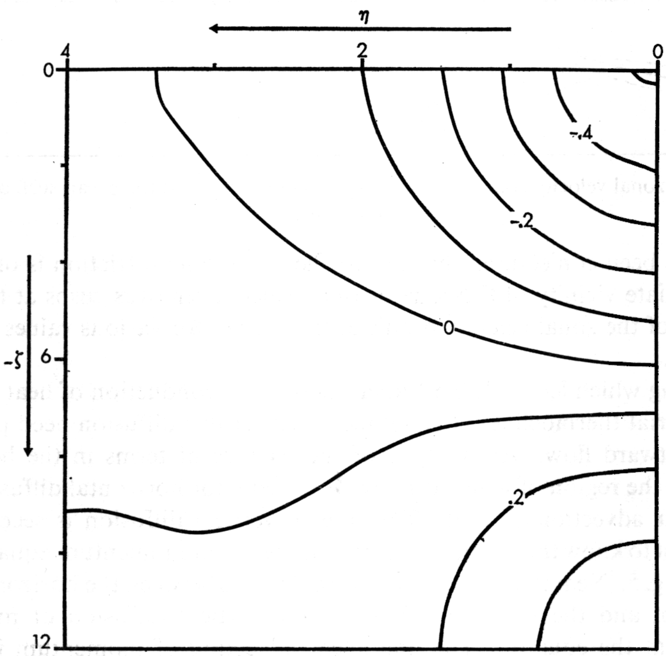

Fig. 2. Lines of constant μ in the η, ζ plane.

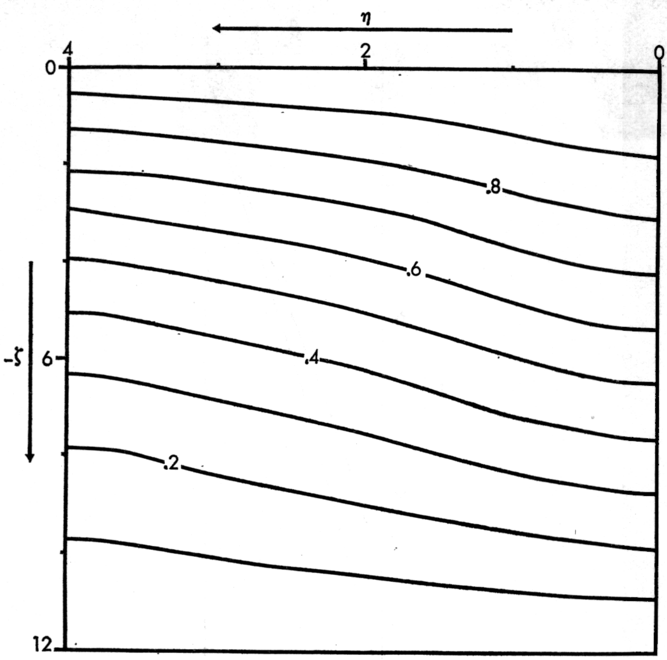

Fig. 3. Isotherms in the η, ζ plane.

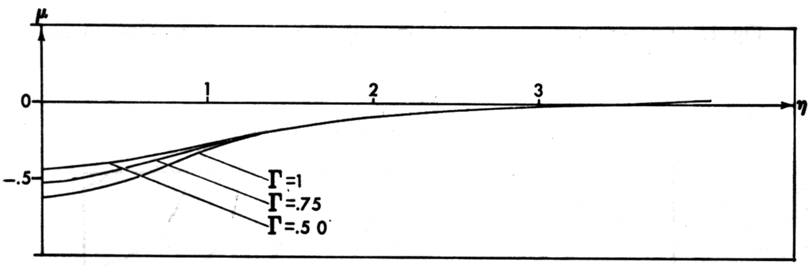

Fig. 4. The zonal velocity component as a function of latitude at the ocean surface for various values of Γ.

plane. It has been conjectured that the coefficient of lateral friction is of significance in the immediate vicinity of the equator only (since it removes cusps at the equator). Comparison of the zonal velocity profile at the surface for various values of Γ (Fig. 4) confirms this.

The scaling which led to Table 1 assumes that the conduction of heat is important in the equatorial thermocline. It is not clear that vertical diffusion need play a role in the deep westward flow. An analysis of the balance of terms in the heat equation shows that in the region of westward flow at the equator horizontal diffusion balances the non-linear advection of heat. The role of vertical diffusion is secondary. It is also of interest to know the balance of terms in the zonal momentum equation. This is depicted in Fig. 5. Near the surface the main balance is between the horizontal diffusion of momentum and the pressure gradient. The vertical diffusion of momentum is unimportant at the equator. The non-linear advection of momentum is significant in the westward flow at depth.

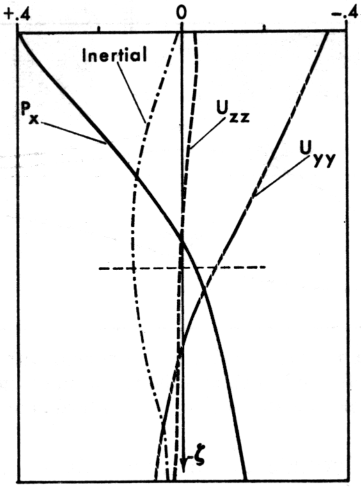

Fig. 5. The balance of terms at the equator in the zonal momentum equation. The zonal velocity is zero at the dotted, horizontal line.

From the integration of (1.1) across the thermocline it was learnt that the flow in the thermocline is equatorward. This is the source for the eastward equatorial current. If we integrate (1.1) from the ocean bottom (assumed flat) to the ocean surface (assumed stress free) then it may be shown that the integrated meridional transport is zero. Consequently, there is a poleward flux at depth to compensate for the equatorward flow in the thermocline. From Fig. 6 which is a plot of the vectors representing the meridional circulation we learn that this poleward flow is maintained by a loss of fluid from the deep westward current. Figure 7 shows the variation of the meridional velocity component with latitude both at the surface where the flow is equatorward and the bottom where it is poleward. Also on Fig. 7 is the variation of the vertical

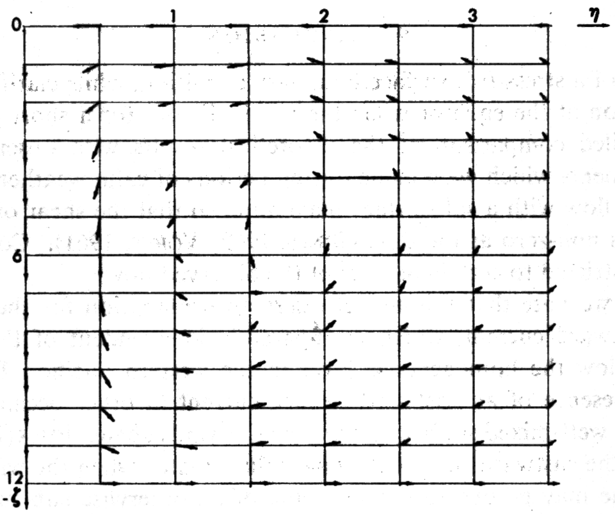

Fig. 6. The meridional circulation presented as a plot of (ν, ω) vectors in the η, ζ plane.

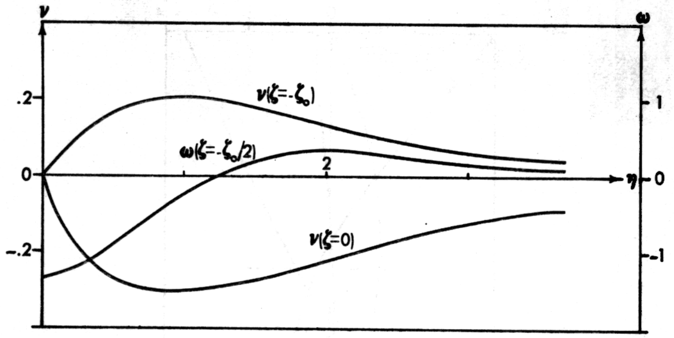

Fig. 7. The variation of the meridional velocity component with latitude η at the surface and bottom (ordinate along the left-hand axis), and the variation of the vertical velocity component ω with latitude at an intermediate depth ($\zeta = -\zeta_0/2$) (ordinate along the right-hand axis).

velocity component with latitude at an intermediate depth. Note the upwelling extra-equatorially, the downwelling at the equator.

It has been pointed out that numerical considerations necessitate that this model have a rigid (slippery) bottom. This bottom does not correspond to that of the ocean so that care must be taken in interpreting the results discussed in the previous paragraph. In particular, one cannot make inferences about the circulation of the deep equatorial waters on the basis of these results. Not only is the bottom of the model much shallower than that of the ocean, but it is not clear that the horizontal Coriolis terms are negligible at depth or that the similarity transformation of Section 6 is appropriate for the deep flow. Ideally, this model would have 'open' boundary conditions at depth. This would make possible the calculation of the deep (non-zero) vertical velocity which drives the abyssal flow. Attempts at such a calculation have unfortunately failed.

Calculations were also performed for other (small) values of α and β. An increase in α leads to more intense flow, but otherwise the results are qualitatively the same.

9. DISCUSSION

The choice of a stress-free surface boundary condition, while clarifying the nature of the circulation at the equator when the winds die out for a short period of time, precludes detailed comparison of the theoretical results with observations. Even those measurements which were made during periods of calm weather at the equator show eastward flow with a sub-surface maximum, so that the shear of the current at the surface was non-zero at the time (JONES, 1969; VOIGT, 1961). Comparisons will therefore be restricted to certain aspects of the observed flow.

First of all we note that this model offers an explanation for the presence of a permanent eastward current, which is apparently independent of the wind, in the thermocline below the homogeneous layer in the western Pacific. The model also explains the presence of an eastward surface current in other ocean basins, where there is no deep well-mixed surface layer, when the winds abate. PULS (1895) originally suggested that the eastward surface flow near the equator when the winds die out for a period of time may be due to the surfacing of an otherwise sub-surface eastward current.

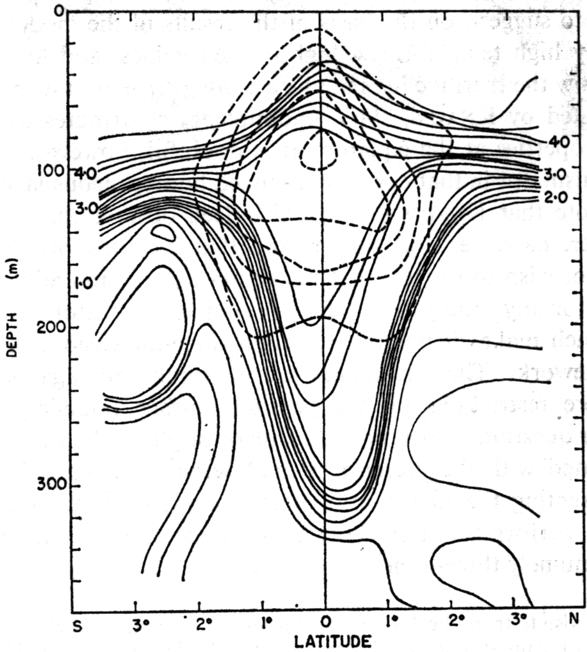

Fig. 8. An oxygen cross-section (in ml/l.) superimposed on a velocity cross-section at 140°W. The interval of the velocity contours (in dotted lines) is 25 cm/sec. The maximum velocity (at the core) is 125 cm/sec. [From KNAUSS (1960)].

The deep westward current predicted by this model has been observed in the western Pacific by KORT, BURKOV and CHEKOTILLO (1966); HISARD, MERLE and VOITURIEZ (1970); and by RUAL (1969). It apparently has its core in the vicinity of 1200 m and attains speeds of up to 40 cm/sec. This is what Taft (private communication) found at 150°W in the spring of 1971. KNAUSS (1960, 1966) observed a deep westward current at longitudes east of 140°W, STALCUP and METCALF (1966) recorded a westward speed of 23 cm/sec at a depth of 405 m at 0° 32°W in the Atlantic. For a summary of the observations of this deep westward current the reader is referred to HISARD and RUAL (1970) who refer to it as the Intermediate Equatorial Current.

It has not been established that there is, on the average, equatorward flow at the level of the core of the undercurrent. The meridional flow close to the equator is difficult to measure since it is small and since small errors in the measurement of the direction of the intense undercurrent imply large errors in the estimate of the meridional flow. It is nonetheless of interest to point out that KNAUSS (1966) measured equatorward flow at the level of the core of the current at 140°W and 118°W in the autumn of 1961.

The vertical motion in the ocean can only be inferred from indirect evidence. One of the most characteristic features of the equatorial undercurrent is the spreading of the equatorial thermocline and the deep penetration of water of high oxygen and low phosphate concentration (see Fig. 8). Note the ridging of the upper isopleths in Fig. 8. This implies that at the equator the cool surface waters have a low oxygen and high phosphate concentration. Several authors have pointed out that this is suggestive of upwelling, caused by the westward surface winds. [CROMWELL (1953) was the first to do so.] We wish to suggest, on the basis of the results of the model presented here, that the unusually high temperatures, high oxygen values, and low phosphate and silicate values below the thermocline may be a consequence of downwelling. Vertical mixing (as suggested by KNAUSS, 1966) undoubtedly contributes to the anomalous distribution of properties at the equator. It is doubtful, however, whether it is the dominant mechanism, for in that case a well-mixed homogeneous layer should result.

Finally, we note that KNAUSS (1966) did indeed observe that the undercurrent becomes shallower, narrower and slower, and that its transport decreases east of 140°W. We do not wish to suggest that the similarity solution adequately describes the flow of those (or any other) longitudes. A similarity transformation is simply an expedient tool which makes it possible to retain three-dimensional effects in a two-dimensional framework. The advantages are enormous but against them must be weighed the severe restrictions on the possible surface boundary conditions, an inability to answer questions concerning the fate or origin of the water, and numerical difficulties associated with the presence of undifferentiated terms in the non-linear expressions representing the advection of heat and momentum when the similarity transformation is performed. For these reasons the present calculation is but a precursor for a genuinely three-dimensional model.

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