A SCHEME FOR NUMERICAL INTEGRATION OF THE EQUATIONS OF MOTION ON AN IRREGULAR GRID FREE OF NONLINEAR INSTABILITY

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In the long-term numerical integration of the equations of motion required for medium-range weather forecasting, the study of the atmospheric general circulation, and other applications, the finite difference formulation of the nonlinear terms may give rise to a special type of instability. This difficulty was first noted in the meteorological literature by Phillips [4]. Phillips pointed out that a unique feature of this instability is that it cannot be suppressed by using shorter time steps. Arakawa [1] has made a very valuable contribution in showing that a numerical scheme which retains certain integral properties of the continuous equations eliminates nonlinear instability.* It must be pointed out that the formulation proposed by Arakawa does not guarantee accuracy. This is assured only if all the significant energy of the flow is in scales of motion that are large enough to be adequately resolved by the numerical grid. A system free of nonlinear instability has the merit, however, that relatively minor truncation errors in grid-scale motions do not lead to large spurious increases in energy.

The work of Arakawa [1] has been extended by Lilly [2]. Lilly has devised a method of numerically integrating the primitive equations which, except for time truncation, exactly conserves finite difference expressions for the kinetic energy of both the divergent and nondivergent components of the flow. This method is currently being used in an extension of investigations of the atmospheric general circulation initiated by Smagorinsky [5].

The present note is concerned with a generalization of the ideas of Arakawa [1] and Lilly [2]. In many applications of the techniques of numerical weather forecasting to other geophysical problems it may be necessary to use grids with irregularly spaced points. For example, it may be important to join two different types of nets together, or the peculiar geometry of the region under consideration requires an irregular arrangement of points.

Consider the following equation

$$\frac{\partial\alpha}{\partial t}+\mathbf{V}\cdot\nabla\alpha=0 \tag{1}$$

$$\nabla\cdot\mathbf{V}=0 \tag{2}$$

*That "nonlinear instability" is somewhat of a misnomer has been pointed out by Miyakoda [3] who showed that a similar instability occurs for linear equations with nonconstant coefficients.

α is a scalar quantity and $\mathbf{V}=u\mathbf{i}+v\mathbf{j}+w\mathbf{k}$.

Let the total region of interest be denoted as R. The normal velocity at the outer surface of R is taken to be zero. The region R is subdivided into J subvolumes, each of volume r_j. If α_j is the average value of the scalar α in the jth subvolume, the total amount of α is given by

$$\sum_{j=1}^{J}\alpha_j r_j=I_1. \tag{3}$$

A lower bound on the total variance of α is given by

$$\sum_{j=1}^{J}\alpha_j^2 r_j=I_2. \tag{4}$$

Combining (1) and (2) and integrating over the subvolume, r_j, results in

$$r_j\frac{\partial\alpha_j}{\partial t}=-\iint_S V_n\alpha_s ds \tag{5}$$

where use has been made of the divergence theorem. V_n and α_s are the normal velocity and the local value of α on the surface of the subvolume, respectively. In the finite difference approximation of the right-hand side of (5), the surface is considered to consist of K_j plane interfaces of area $A_{k,j}$. The average normal velocity for each interface is given by $V_{k,j}$. The value of α on the interface is approximated by $(\alpha_j+\alpha_k)/2$ where α_k is the average value of α in the subvolume adjacent to the kth surface. With these substitutions (5) becomes

$$r_j\frac{\partial\alpha_j}{\partial t}=-\sum_{k=1}^{K_j}V_{k,j}(\alpha_j+\alpha_k)A_{k,j}/2. \tag{6}$$

The corresponding continuity equation is

$$\sum_{k=1}^{K_j}V_{k,j}A_{k,j}=0. \tag{7}$$

We wish to show that (6) and (7) lead to an exact conservation of I_1 and I_2 except for truncation caused by differencing with respect to the time coordinate.

Summing over all the subvolumes, we note that the time change of I_1 is

$$\frac{\partial}{\partial t} I_1 = -\sum_{j=1}^{J} \sum_{k=1}^{K} V_{k,j}(\alpha_j+\alpha_k)A_{k,j}/2. \tag{8}$$

It can be seen that the terms on the right side of (8) fall into two groups. Those terms which involve interfaces between subvolumes occur in couples which are equal and opposite. These terms cancel. The remaining terms are due to surfaces which lie on the outside of R. These terms are zero since the normal velocity along the outer boundaries of R is zero.

On the other hand, the change of I_2 is given as

$$\frac{\partial}{\partial t} I_2 = -\sum_{j=1}^{J} \sum_{k=1}^{K_j} V_{k,j}(\alpha_j^2+\alpha_k\alpha_j)A_{k,j}. \tag{9}$$

This may be rewritten as

$$\frac{\partial}{\partial t} I_2 = -\sum_{j=1}^{J} \alpha_j^2 \sum_{k=1}^{K_j} V_{k,j}A_{k,j} - \sum_{j=1}^{J} \sum_{k=1}^{K_j} \alpha_k\alpha_j A_{k,j}V_{k,j}. \tag{10}$$

The first term on the right vanishes through the continuity relation (7). The same argument applied to (8) is also true for the second term of (10). All contributions for interfaces will occur as couples, which are equal and opposite. The remaining terms lie on the outer boundaries of the region R.

To see how the general formula (5) and (6) might actually be applied, consider the following equations of motion for an incompressible, homogeneous fluid under rotation,

$$\frac{\partial}{\partial t} u + \nabla\cdot(\mathbf{V}u) = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + F^x + fv \tag{11}$$

$$\frac{\partial}{\partial t} v + \nabla\cdot(\mathbf{V}v) = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + F^y - fu \tag{12}$$

$$\rho_0 g = -\frac{\partial p}{\partial z} \tag{13}$$

$$\nabla\cdot\mathbf{V} = 0. \tag{14}$$

Application of (6) to the left side of (11) and (12) will guarantee that the finite difference expressions for the advective terms will not alter the finite difference equivalent of the kinetic energy integral

$$\text{K.E.} = \int_0^L \int_0^L \rho_0 \frac{(u^2+v^2)}{2}\, dx dy$$

provided the continuity relation (6) is used as the diagnostic relation to determine the vertical component. This will eliminate the possibility of spurious changes in the energy level associated with the nonlinear numerical instability described by Phillips [4].

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CORRECTION

Vol. 93, October 1965, p. 582: The first of the expressions appearing three lines below equation (25) should be:

$$\frac{u_g}{fy}$$