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ADJUSTMENT OF GEODETIC FIELD DATA USING A SEQUENTIAL METHOD

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ABSTRACT. *Using remote terminals, National Geodetic Survey (NGS) field parties are now able to carry out limited adjustments for the purpose of evaluating their observations. Such an adjustment must be able to handle incomplete networks. The method adopted was developed by Creusen* [1965]. *It is a sequential adjustment using a modified arithmetic which automatically handles all problems of possible singularities, giving for indeterminate parameters their pseudo-inverse solution accompanied by appropriate flags. The modified arithmetic is based on the Laurent series* $\sum = A_1\varepsilon^{-1}+A_2+A_3\varepsilon+A_4\varepsilon^2+ \ldots$ *with* ε *a dummy variable.* A_1 *thus has infinite variance relative to* A_2 *which has infinite variance relative to* A_3. *For a sequential adjustment only two terms are necessary.* A_1 *becomes a projection matrix and* A_2 *a covariance matrix.* A_1 *is initially set to* I *and at each step of the sequential adjustment becomes* $I-N^+N$, *while* $A_2=N^+$. *At determinacy* N^+ *becomes* N^{-1} *and* A_1 *becomes zero, thus providing a flag on the determination of the unknowns. The main disadvantage of this method is large core requirements in a computer. With the residuals from the adjustment and their standard errors, the NGS field observers can immediately determine if their work satisfies first-order requirements.*

I. INTRODUCTION

Our problem is to carry out a least-squares adjustment of observations of unknown quality, such that best estimates of unknowns are obtained and poor observations are automatically detected. Further, we do not know at any time if sufficient observations have been made to guarantee a unique solution for the parameters.

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The problem arises as National Geodetic Survey crews observe a triangulation network. They need to have real time estimates, as well as control, of the quality of their work. Some of their observations may close triangles, but others would be part of still indeterminate triangles. Further, we wish to automate this whole evaluation process, leaving ourselves the options to remove the effects of poor observations from the network and to replace them, where necessary, with reobservations.

One method is to have the geodetic surveyor adjust the network by gathering together observations until there are enough determinations of the unknowns in the starting area, and then do a conventional batch least-squares adjustment. As more observations become available, another complete adjustment is made adding the new observations in with the old. This requires continual evaluation of the data to avoid entering observations that would create a singular normal matrix. Also a tremendous amount of duplicated computation would occur.

Another method available is a sequential adjustment. This has the advantage that the system does not have to be readjusted for each new batch of observations. However, in the conventional sequential adjustment an a priori covariance matrix of the parameters is needed which is difficult to automate. Singularities in part of the network can still exist which are not automatically exposed in the results.

II. DESCRIPTION OF THE METHOD

The method we have adopted is a sequential adjustment based on that developed in 1965 by Creusen. The Creusen sequential includes both the unknowns and the observations in the covariance matrix, and uses a modified arithmetic to handle parameters as "observations" with infinite variance relative to the actual observed quantities. The terms for the observables are initialized with the estimates of the variance of the observations and the terms for the parameters with the identity matrix. The final result gives the variance of the adjusted unknowns and observations as well as all correlations between them. Thus a complete analysis of the system is possible.

The advantage of this method lies in its full generality. The use of modified arithmetic permits the solution of any system of equations, whatever its rank. The pseudoinverse is produced if the rank is less than the number of unknowns; a unique solution results if they are equal, and a least-squares solution if the observations are redundant. Further, flags on all incompletely determined unknowns are provided.

Let us begin by describing the concept of modified arithmetic as formulated in *Pope and Hanson* [1972]. It was pointed out that the semi-infinite formal Laurent series

$$\sum = A_1 \frac{1}{\varepsilon} + A_2 + A_3 \varepsilon + A_4 \varepsilon^2 + \ldots$$

(ϵ a dummy variable) can be appropriately initialized and manipulated according to standard algorithms to give a series of matrices expressible in terms of generalized inverses of A_2. If A_i is appropriately chosen, all information relevant to the solution is carried along by the solution process. The pseudoinverse becomes the inverse on determinancy.

In a sequential adjustment Pope determined that only two terms of the series need be carried along. This corresponds with the concept of Creusen, although Creusen does not give an explicit formulation of the rules for manipulating these two terms (the derivation of which requires a third term nowhere mentioned in Creusen). At each arithmetic operation, modified arithmetic is used to find the sum or product or quotient of the components of $\sum$. This process is particularly simple in the case of sequential adjustments.

A sequential least-squares adjustment is given by [*Bjerhammer*, 1973]

$$X_k = X_{k-1} - Q_{xw} Q_{ww}^{-1} W = X_{k-1} - \Delta X \qquad (1)$$

$$Q_k = Q_{k-1} - Q_{xw} Q_{ww}^{-1} Q_{wx} = Q_{k-1} - \Delta Q \; . \qquad (2)$$

Here Q_k is the estimate of the covariance matrix after processing the kth group of observations. W is the closure term and, when adding one new observation, is a scalar as is also Q_{ww}. Further, using the model,

$$BX = 0 \tag{3}$$

$$Q_{xw} = Q_{wx}^{t} = B_k \, Q_{k-1} \tag{4}$$

and

$$Q_{ww} = B_k \, Q_{k-1} B_k^{t} . \tag{5}$$

Then the modified arithmetic sequential is initialized by the following choice of Q_o and X_o

$$Q_o = \Sigma = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \frac{1}{\varepsilon} + \begin{bmatrix} 0 & 0 \\ 0 & Q_v \end{bmatrix}, \quad X_o = \begin{bmatrix} X \\ V \end{bmatrix} . \tag{6}$$

Here, I corresponds to that part of the solution that contains only unknowns X. Q_v is the a priori covariance matrix of the observables. This, in effect, gives parameters infinite variance relative to the observables, and no a priori covariance matrix of the parameters need be introduced. V is the vector of residuals on the observations.

We do not need to get involved in a full development of the modified arithmetic as would be necessary in a batch solution, but rather can restrict ourselves to the particular operations which we know will occur in our sequential adjustment. The following equations constitute the modified arithmetic and are easily verified.

$$\frac{(a\frac{1}{\varepsilon} + b)(c\frac{1}{\varepsilon} + d)}{(f\frac{1}{\varepsilon} + g)} = \begin{cases} \dfrac{ac}{f}\dfrac{1}{\varepsilon} + \dfrac{afd + fbc - acg}{f^2} & \text{if } f \neq 0 \qquad (7) \\[2ex] \dfrac{cb + ad}{g}\dfrac{1}{\varepsilon} + \dfrac{bd}{g} & \text{if } f = 0 \qquad (8) \end{cases}$$

This corresponds to the operation $\Delta X = \frac{Q_{xw} W}{Q_{ww}}$ and $\Delta Q = \frac{Q_{xw} Q_{wx}}{Q_{ww}}$ of the sequential algorithm. In each of these cases there are terms in $(\varepsilon, \varepsilon^2, \varepsilon^3)$ which must be considered in developing (7) and (8), but have no effect in the final expression and can be ignored.

After the application of (7) and (8), and the recognition of terms never receiving corrections if initially zero, the final expressions for ΔQ and ΔX are:

with

$$Q_{xw} = [R_{x1} \; R_{v1}] \frac{1}{\varepsilon} + [R_{x2} \; R_{v2}] \tag{9}$$

$$Q_{ww} = q_{w1} \frac{1}{\varepsilon} + q_{w2} \tag{10}$$

for $q_{w1} \neq 0$

$$\Delta Q = \begin{bmatrix} \frac{R_{x1}^t R_{v1}}{q_w} & 0 \\ 0 & 0 \end{bmatrix} \frac{1}{\varepsilon} + \begin{bmatrix} \frac{q_{w1}(R_{x1}^t R_{x2} + R_{x2}^t R_{x1}) + q_{w2} R_{x1}^t R_{x1}}{q_{w1}^2} & \frac{R_{x1}^t R_{v2}}{q_{w1}} \\ \frac{R_{v2}^t R_{x1}}{q_{w1}} & 0 \end{bmatrix} \tag{11}$$

$$\Delta X = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \frac{1}{\varepsilon} + \begin{bmatrix} \frac{R_{x1}^t W_2}{q_{w1}} \\ 0 \end{bmatrix} \tag{12}$$

for $q_{w1} = 0$

$$\Delta Q = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \frac{1}{\varepsilon} + \begin{bmatrix} R_{x2}^t \\ R_{v2}^t \end{bmatrix} \frac{1}{q_{w2}} [R_{x2} R_{v2}] = Q_{xw} Q_{ww}^{-1} Q_{wx} \tag{13}$$

$$\Delta X = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \frac{1}{\varepsilon} + \begin{bmatrix} R_{x_2}^t \\ R_{v_2}^t \end{bmatrix} W_2 = Q_{xw} Q_{xx}^{-1} W . \qquad (14)$$

Here R_{x1} and R_{v_1} are part of the projector term (corresponding to A_1) due to the unknowns and observables respectively, and R_{x2} and R_{v_2} are the part of the real term (corresponding to A_2) due to unknowns and observables respectively. (13) and (14) will be recognized as the standard formulas for a sequential adjustment. In a later paper we will show that in (11) the A_1 matrix or projector matrix becomes zero upon determinancy, or rather the term in this matrix which corresponds to each determinant unknown becomes zero. This provides our flag on the status of each unknown.

Using a better known notation, the projector matrix is $Q_x - Q_{xw}^t Q_{ww}^{-1} Q_{xw}$, where Q_{xw} is BQ_x. Q_x is initially I, then $I - B^t(BB^t)^{-1}B$, and so on. This can be shown to be the recussion for $I - N^+N$, where N is the matrix of the normals formed from the observations. This is a projector onto the solution space. The A_2 matrix in (11) is N^+, or the pseudoinverse of the singular normal equations. On achieving determinancy (i.e., rank equal to the number of unknowns) $I - N^+N$ becomes zero and N^+ becomes N^{-1}, a covariance matrix. After determinancy, all subsequent equations that do not introduce new parameters, thus creating new indeterminancies, find $q_{w_1} = 0$ and use (13) and (14).

One disadvantage of this approach is the necessity of computing the covariances between observations and unknowns as well as their variances. Further, as the corrections to the unknowns and observations are obtained simultaneously, all these corrections, with their full covariance matrix, must be in the computer at the same time. Also, the variances and covariances of the unknowns require double storage. In short, a large computer is necessary to solve a small network.

We have chosen this method because, for the application of determining the accuracy of the observations, only a small part of the network needs to be considered, and with the new field terminal system being introduced at NGS a large computer becomes readily available.

There are methods by which the above disadvantages can be greatly reduced. The unknowns and corrections to observations do not have to be solved for simultaneously. If the unknowns are solved for alone, the method reduces to a Kalman filter-type sequential adjustment but still with modified arithmetic to handle singularities.

Another method is to introduce the concept of a local adjustment [*Halmos et al.*, 1974]. As a single equation is introduced into the adjustment, it is possible to determine from the size of the covariance term what other unknowns and observations will be affected, and then construct a Q matrix that only contains terms for these unknowns and observations. The resulting adjustment will be rigorous only to some preselected degree of

accuracy, but very large networks can be handled one small section at a time.

III. IMPLEMENTATION OF THE ADJUSTMENT PROCEDURE

For field use, two modes of adjustment will be made available. In one, the geodetic surveyor will designate which stations will be included in the adjustment. This mode has been essentially completed and will be available for field use in the near future. In the second mode, a localization will be carried out to decide which observations and unknowns will be included in the adjustment. The geodetic surveyor will indicate only the general region in which he is interested. In this mode the total of all the local adjustments will constitute a final adjustment of the entire network at the completion of a project.

In order to eliminate the orientation unknowns, and thereby save space, we chose an alternative form of the basic equation for horizontal directions. If α designates azimuths, and ℓ an observed direction, then it follows that

$$\ell_{12} - \ell_{10} = \alpha_{12} - \alpha_{10} = \text{the angle at 1 between 0 and 2}$$

$$0 = \ell_{10} - \ell_{12} + \alpha_{12} - \alpha_{10}$$

$$0 = w + v_{10} - v_{12} + \delta\alpha_{12} - \delta\alpha_{10}$$

$$0 = w + v_{10} - v_{12} - \frac{M_1}{s}\sin\alpha_{12}\delta\phi_1'' + \frac{N_2}{s}\cos\phi_2\cos\alpha_{21}\delta\lambda_1'' - \frac{M_3}{s}\sin\alpha_{12}\delta\phi_2$$

$$+ \frac{M_1}{s}\sin\alpha_{10}\delta\phi_1'' - \frac{N_2}{s}\cos\phi_2\cos\alpha_{21}\delta\lambda_2'' + \frac{M_0}{s}\sin\alpha_{10}\delta\phi_0$$

$$+ \frac{N_0}{s}\cos\phi_0\cos\alpha_{01}\delta\lambda_0'' - \frac{N_0}{s}\cos\phi_0\cos\alpha_{01}\delta\lambda_0'' .$$

This is the constraint equation used in the adjustment. It introduces two residuals in each equation. The Creusen sequential has the unique advantage of handling this general case without difficulty. Here w is the difference between the angle computed from the initial unknowns and the observed angle. The subscripts designate directions. v_{10}, thus, is the correction to the direction from point one to point zero. The $\delta\alpha$ have been expanded in terms of $\delta\phi_i$, $\delta\phi_j$, $\delta\lambda_i$, and $\delta\lambda_j$ [*Rapp*, 1969].

The present adjustment contains an extended feature comparable to the extended Kalman filter [*Jazwinski*, 1970] where the coefficients and closure terms are evaluated at the most recent updated value of the unknowns. The entire adjustment is then iterated, removing the extended feature. The hope is to reduce the accuracy requirements for the initial approximations of the unknowns. Not enough trial adjustments have yet been carried out to determine if this hope has been realized.

It would be impractical to use this adjustment technique to solve large networks. However, if small sections of a network must be adjusted to provide statistical control of the quality of the work, then a Creusen sequential adjustment with a localization feature provides an effective means of doing this.

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