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A minimalist probabilistic description of root zone soil water

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Abstract. The probabilistic response of depth-integrated soil water to given climatic forcing can be described readily using an existing supply-demand-storage model. An apparently complex interaction of numerous soil, climate, and plant controls can be reduced to a relatively simple expression for the equilibrium probability density function of soil water as a function of only two dimensionless parameters. These are the index of dryness (ratio of mean potential evaporation to mean precipitation) and a dimensionless storage capacity (active root zone soil water capacity divided by mean storm depth). The first parameter is mainly controlled by climate, with surface albedo playing a subsidiary role in determining net radiation. The second is a composite of soil (through moisture retention characteristics), vegetation (through rooting characteristics), and climate (mean storm depth). This minimalist analysis captures many essential features of a more general probabilistic analysis, but with a considerable reduction in complexity and consequent elucidation of the critical controls on soil water variability. In particular, it is shown that (1) the dependence of mean soil water on the index of dryness approaches a step function in the limit of large soil water capacity; (2) soil water variance is usually maximized when the index of dryness equals 1, and the width of the peak varies inversely with dimensionless storage capacity; (3) soil water has a uniform probability density function when the index of dryness is 1 and the dimensionless storage capacity is large; and (4) the soil water probability density function is bimodal if and only if the index of dryness is <1, but this bimodality is pronounced only for artificially small values of the dimensionless storage capacity.

1. Introduction

Because soil water is replenished by precipitation, the temporal variability of soil water content has a strong random component. Consequently, a probabilistic description of soil water may provide a productive framework for analyses of processes that interact with soil water, complementing the equilibrium-based approach [*Eagleson*, 1978a, 1978b]. *Cordova and Bras* [1981] and *Hosking and Clarke* [1990] presented investigations of the soil water probability density function (pdf) under conditions of random rainfall and soil water–dependent losses. *Milly* [1993] derived an analytic solution to the stochastic soil water problem using a nonlinear (double step, described below) loss function; the loss function is the combined loss to evaporation and runoff as a function of soil water amount. *Rodríguez-Iturbe et al.* [1999] recently solved the problem using a more general form of the loss function, in which *Milly*'s [1993] steps are generalized to ramps, and the approach that they used allows for even more general forms of the loss function. The essay of *Rodríguez-Iturbe* [2000] called attention to the probabilistic nature of soil water and speculated on its potential to yield insights into ecosystem dynamics.

Although the "minimalist" model of *Milly* [1993] is a special case of the model of *Rodríguez-Iturbe et al.* [1999], it nevertheless contains many essential features of the more general problem. In particular, the amount of soil water storage is limited by a definite storage capacity, and both the refilling and the depletion of soil water storage are dependent on the soil water content. The use of step functions by *Milly* [1993] is a minimum-parameter formulation that represents soil water dependence of the loss function in both the dry and wet limits. In light of interpretations by *Rodríguez-Iturbe et al.* [1999] and *Rodríguez-Iturbe* [2000] to the contrary, an explanation may be helpful. *Milly* [1993] assumed that soil water is replenished at the precipitation rate for all attainable levels of soil water except capacity (at which point, all precipitation is excluded). Likewise, evaporation was allowed to proceed at the potential rate for all attainable storage levels except the minimum (at which point, all evaporation ceases). This is equivalent to a loss function whose value is zero for soil water below the minimum, stepping up to some constant value at some minimum soil water level, and stepping up again to an infinite loss rate at the maximum soil water level.

Mathematical models can play a crucial role in the understanding of the dynamic interactions among climate, soil, soil water, and vegetation. The relative simplicity and dimensionless form of *Milly*'s [1993] solution yields simple expressions for the pdf and low-order moments of soil water, presented here, that contribute to the elucidation of physical controls on soil water dynamics. They also offer a potential path toward the analysis of space-time links between climate, soil, and vegetation that *Rodríguez-Iturbe* [2000] has encouraged.

Modeling always presents the analyst with a tradeoff between simplicity and detail, and all points on the tradeoff curve are valid points of departure. It is acknowledged that the solution presented by *Milly* [1993], applied here, does not recognize the generally accepted gradual decrease in evaporation with depletion of soil water, and less importantly, it does not recognize the nonstep function nature of soil drainage. On the

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other hand, the analysis rewards parametric frugality with simplicity and perspicuity of results. Furthermore, as shown here, the resulting probabilistic characteristics of soil water are very similar to those implied by the more general model, implying that the exact shape of the loss function is only of secondary importance.

The objectives of this paper are (1) to compare the formulations and results of the models of *Milly* [1993] and *Rodríguez-Iturbe et al*. [1999] and (2) to use the simpler model to explain the control of the soil water pdf, mean, and variance by soil, climate, and vegetation characteristics.

2. Relation Between the Two Models

2.1. Climatic Forcing

Milly [1993] and *Rodríguez-Iturbe et al*. [1999] made identical assumptions about the nature of climatic forcing. The depth of rain h for any storm is drawn from an exponential distribution. In the notation of Rodríguez-Iturbe et al. the expected value of this storm depth is expressed (by scaling against soil characteristics) as nz/γ, in which n is soil porosity, z is "depth of soil" (i.e., effective depth of root zone), and γ is a dimensionless constant. Storm arrival is a Poisson process with mean arrival rate λ (events per unit time). Storms are assumed to have zero duration (and infinite intensity). Evaporative demand is expressed through a constant "potential" or maximum (i.e., not water limited) evaporation rate E. Seasonality of forcing is ignored.

2.2. Loss Functions

In both analyses, it is assumed that precipitation will enter the soil store as long as the soil is not saturated. Thus the water balance can be readily discussed in terms of differences in loss functions. *Rodríguez-Iturbe et al*. [1999] assume that the rate of water loss, under unsaturated conditions, can be expressed by a piecewise linear function of saturation s, which is the total amount of soil water divided by the available soil void volume nz. For s less than some value s^* the loss (depth per unit time) is given by Es/s^*. For s between s^* and some greater value s_1, the loss is given simply by E. Over both of these regions the loss is considered to be due only to evaporation. For s greater than s_1 the evaporative loss E is supplemented by a "leakage" (or soil drainage) loss that varies linearly from zero at s_1 to K_s at soil saturation.

The loss function used by *Rodríguez-Iturbe et al*. [1999] reduces to that used by *Milly* [1993] in the limit as s^* approaches zero and K_s approaches infinity. This means that (1) evaporation proceeds at the maximum rate until the soil is completely dry and (2) any excursion of soil water above s_1 is damped instantaneously due to the strong nonlinearity of the soil hydraulic conductivity function.

2.3. Dimensionless Groups of *Milly* [1993]

Let us pause to take stock of the parameters used in the two models. The models share the parameters nz, γ, λ, E, and s_1. *Rodríguez-Iturbe et al*. [1999] use the additional parameters s^* and K_s. In *Milly*'s [1993] analysis, it is shown that the control of soil water and water balance dynamics can be explained in terms of the relative magnitudes of only three water depth scales, corresponding to soil water storage capacity, mean storm depth, and mean interstorm evaporative depth, which collectively can form only two independent ratios. In the present notation, these groups are

$$\alpha = \gamma s_1, \tag{1}$$

$$\beta = (s_1 nz\lambda)/E. \tag{2}$$

In alternate notation, we write [*Milly*, 1993]

$$\alpha = w_o/\langle h \rangle, \tag{3}$$

$$\beta = w_o/(E\langle t_a \rangle), \tag{4}$$

in which w_o is the effective water capacity of the soil (nzs_1), $\langle h \rangle$ is the expected value of storm depth (nz/γ), and $\langle t_a \rangle$ is the expected value of storm interarrival time (λ^{-1}), so that $E\langle t_a \rangle$ is the expected value of cumulative evaporative demand between storms.

Given the symmetry embodied in the parameters α and β, we shall use them in further derivations here. In plotting and analyzing the results from a dimensionless viewpoint (Figures 4–6), however, we shall generally use the dimensionless numbers α and α/β instead. The ratio α/β (also equal to $E\langle t_a \rangle/\langle h \rangle$) is known as the index of dryness, and is the climatic ratio of potential evaporation to precipitation [*Budyko*, 1974].

3. Derivation of Soil Water Probability Density Function

Milly [1993] formulated the problem in terms of the pdfs of soil water immediately before and after any storm. Let us define three random variables: $S = s/s_1$, a rescaled soil water saturation at a randomly chosen point in time; S^-, the value of S conditional on the time being immediately before a randomly chosen storm; and S^+, the value of S conditional on the time being immediately after a randomly chosen storm. We denote the pdfs of these three variables by $f_S(x)$, $f_S^-(x)$, and $f_S^+(x)$, respectively. *Milly* [1993] presented the solutions for the last two of these but not for the first; its derivation from the others is straightforward and is presented here.

The variable S^+ is a simple function of the previous S^- and the normalized storm depth, which is exponentially distributed; S^- is a simple function of the previous S^+ and the normalized cumulative interstorm potential evaporation, which is also exponentially distributed (by virtue of the constant rate E and the Poisson storm arrival process). Because of these similarities and because of the similarity of the step functions used to describe fluxes at the two limits of soil water storage, *Milly*'s [1993] analysis led to a symmetric set of relations between $f_S^-(x)$ and $f_S^+(x)$. These were solved to find

$$f_S^+(x) = \alpha q e^{(\beta-\alpha)x} + p\delta(x-1) \qquad 0 \le x \le 1, \tag{5}$$

$$f_S^-(x) = \beta p e^{(\alpha-\beta)(1-x)} + q\delta(x) \qquad 0 \le x \le 1, \tag{6}$$

in which $\delta(z)$ is the Dirac delta function, p is the probability that soil water is at capacity following any given storm, and q is the probability that soil is dry at the end of any given interstorm period. These latter two probabilities were shown to be

$$p = (\alpha - \beta)/(\alpha e^{\alpha-\beta} - \beta), \tag{7}$$

$$q = (\beta - \alpha)/(\beta e^{\beta-\alpha} - \alpha). \tag{8}$$

One step in the derivation of the solution presented above was the expression of $f_S^-(x)$ as a derived distribution depending on $f_S^+(x)$ and the pdf of normalized storm interarrival time. The distributions are linked through the relation

$$S^- \max = (S^+ - u, 0), \tag{9}$$

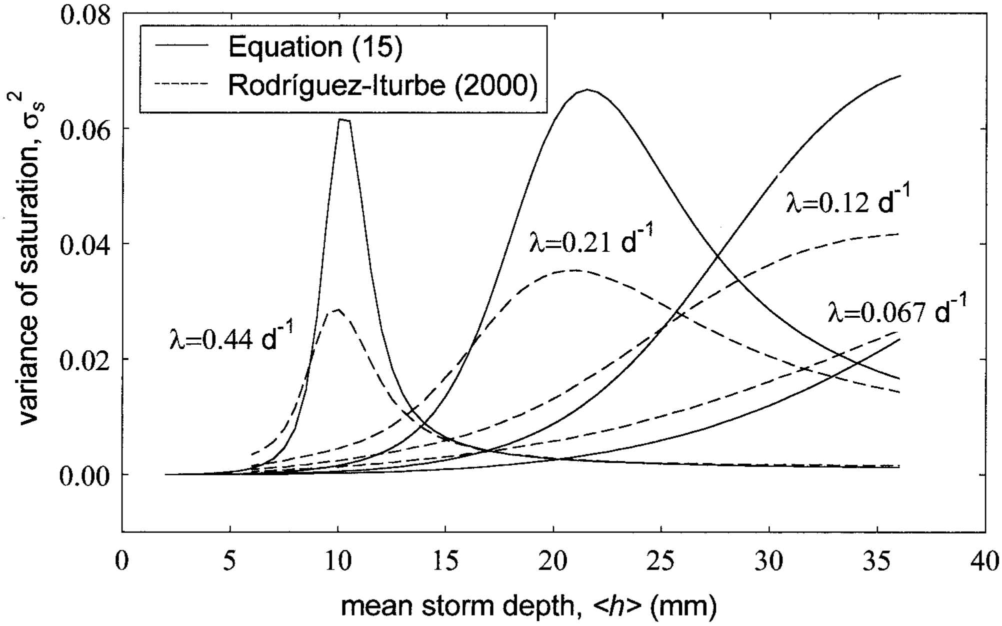

Figure 1. Variance of soil water saturation s as a function of storm arrival rate λ (events per day) and mean storm depth $\langle h \rangle$ (mm). Here, $nz = 400$ mm, $s^* = 0.35$, $s_1 = 0.85$, $K_s = 800$ mm d^{-1}, and $E = 4.5$ mm d^{-1}. Solid curves are based on (15), and dashed curves are from *Rodríguez-Iturbe* [2000].

in which u is a normalized storm interarrival time. (The normalized storm interarrival time is defined as the product of storm interarrival time and the potential evaporation rate E, divided by the available soil void volume nz. Thus it is the ratio of storm interarrival time to the time that would be needed to deplete the soil water store from capacity to dryness by evaporation.)

Having summarized the pertinent points from *Milly* [1993], we are now ready for the brief derivation of the pdf of S. The value of S at any randomly chosen time can be viewed as a function of two independent random variables,

$$S = \max(S^+ - U, 0), \tag{10}$$

in which U is the normalized time since the most recent storm. For a randomly chosen point in time and given the Poisson storm arrival model, however, the pdf of U is identical to the pdf of normalized storm interarrival time. It follows that the pdf of S, $f(S)$, is identical to the pdf of S^-, so

$$f_S(x) = \beta p e^{(\alpha-\beta)(1-x)} + q\delta(x) \qquad 0 \le x \le 1 \tag{11}$$

or, equivalently,

$$f_S(x) = \beta q e^{(\beta-\alpha)x} + q\delta(x) \qquad 0 \le x \le 1. \tag{12}$$

In the limit of the special situation where $\alpha = \beta$, this is simply

$$f_S(x) = \frac{\beta}{(1+\beta)} + \frac{\delta(x)}{(1+\beta)} \qquad \alpha = \beta \qquad 0 \le x \le 1. \tag{13}$$

4. Mean and Variance of Soil Water

Expressions for the mean and variance of soil water follow by direct computation from (12). The mean is

$$\langle s/s_1 \rangle = \langle S \rangle = -1/(\beta - \alpha) + (1 + \beta e^{\beta-\alpha})/(\beta e^{\beta-\alpha} - \alpha), \tag{14}$$

and the variance is

$$\mathrm{Var}\,[s/s_1] = \mathrm{Var}\,[S] = 1/(\beta-\alpha)^2 - [1 + (\alpha+2)\beta e^{\beta-\alpha}]/(\beta e^{\beta-\alpha} - \alpha)^2. \tag{15}$$

Results for the special case where $\alpha = \beta$ cannot be evaluated directly by (14) or (15). In the limit as $\beta - \alpha \to 0$ we find

$$\langle s/s_1 \rangle = \alpha/[2(1+\alpha)] \qquad \alpha = \beta, \tag{16}$$

$$\mathrm{Var}\,[s/s_1] = (\alpha^2 + 4\alpha)/[12(\alpha+1)^2] \qquad \alpha = \beta. \tag{17}$$

5. Illustration and Interpretation of Solution

5.1. Comparison with Examples of More General Solution

Rodríguez-Iturbe [2000] showed the dependence of variance of saturation s on mean storm arrival rate and mean storm depth for all other parameters fixed. Here we evaluate the variance of s by means of (15), using the parameter values of *Rodríguez-Iturbe* [2000]. The simpler model produces a maximum variance that is about twice that in the detailed model (Figure 1). This is not surprising because the simpler model allows soil water to vary freely to low values, without reduction of losses when s drops below s^*, but only when s reaches 0. Given the ambiguity in definition of the datum for zero soil water in both models, this discrepancy in scale should possibly not be given great emphasis. Also, despite the overall scale difference both solutions share essential qualitative characteristics, such as the shape and relative positions of curves. Indeed, the simpler model even reproduces the locations of variance maxima quantitatively.

Figure 2 compares probability distributions from this analysis and from the more general analysis, using the examples of *Rodríguez-Iturbe et al.* [1999]. To facilitate the comparison in the presence of the delta function pdf, the plots are in terms of the integral of the pdf, or the cumulative density function (cdf). The differences in solutions are readily understood in terms of the approximations of the simpler model. For cases a and d, in

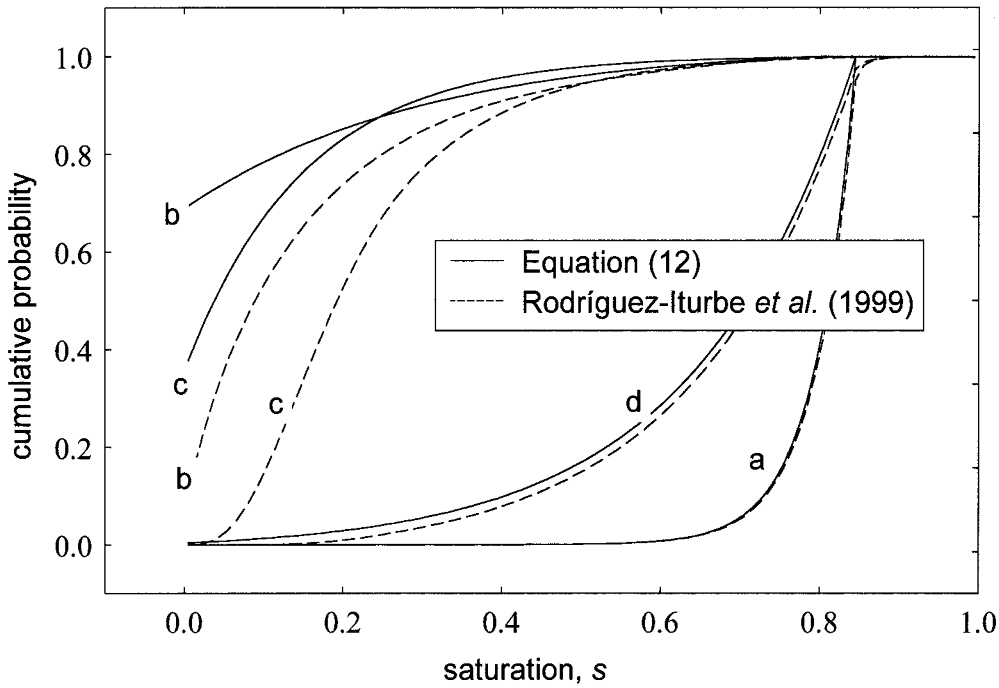

Figure 2. Cumulative density functions (cdf) of soil water saturation s for various combinations of parameters. Solid curves are based on (12), and dashed curves are from *Rodríguez-Iturbe* [2000]. For case a, nz = 450 mm, s^* = 0.3, s_1 = 0.85, K_s = 900 mm d^{-1}, $\langle h \rangle$ = 15 mm, λ^{-1} = 1.5 days, and E = 6 mm d^{-1}. For case b, nz = 100 mm, s^* = 0.45, s_1 = 0.8, K_s = 5000 mm d^{-1}, $\langle h \rangle$ = 20 mm, λ^{-1} = 10 days, and E = 6 mm d^{-1}. For case c, nz = 200 mm, s^* = 0.3, s_1 = 0.9, K_s = 200 mm d^{-1}, $\langle h \rangle$ = 10 mm, λ^{-1} = 6 days, and E = 2.5 mm d^{-1}. For case d, nz = 300 mm, s^* = 0.3, s_1 = 0.85, K_s = 300 mm d^{-1}, $\langle h \rangle$ = 15 mm, λ^{-1} = 3 days, and E = 4 mm d^{-1}.

which soil tends to be wet, very close agreement is found over a wide range of soil water. However, the more general solution allows s to exceed s_1 temporarily after a storm, whereas the simpler solution simply takes s_1 as an upper bound on storage. Because soil generally drains rapidly down to s_1, this incurs only a small error overall. A more significant difference arises in connection with treatment of evaporation when s is below s^*. The simple model attaches no significance to s^* and allows

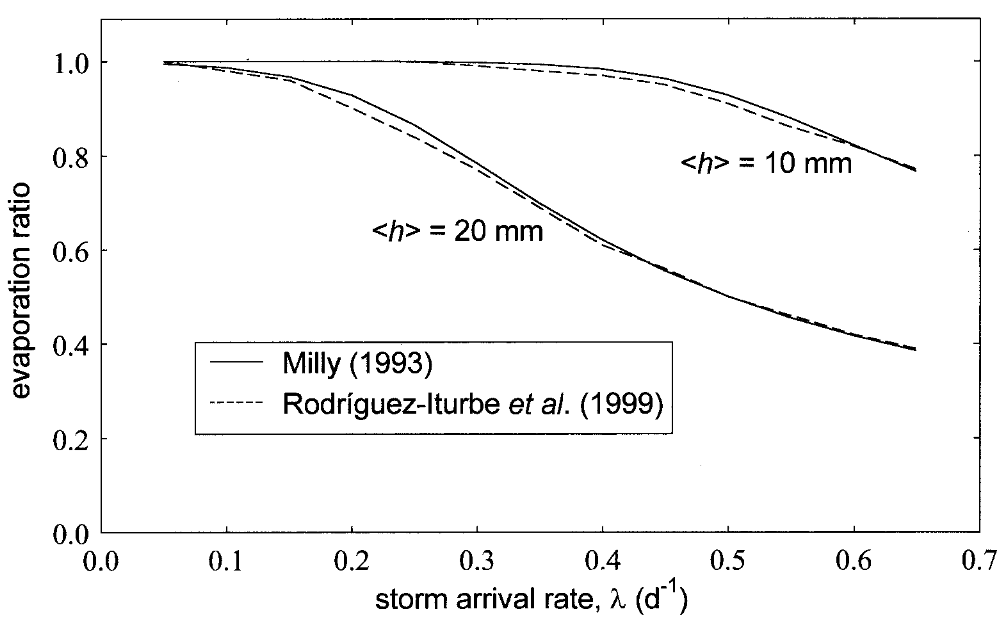

Figure 3. Evaporation ratio (mean evaporation as a fraction of mean precipitation) as a function of storm arrival rate λ. Here nz = 150 mm, s^* = 0.35, s_1 = 0.85, K_s = 1000 mm d^{-1}, and E = 5 mm d^{-1}. Solid curves are based on (18), and dashed curves are from *Rodríguez-Iturbe* [2000].

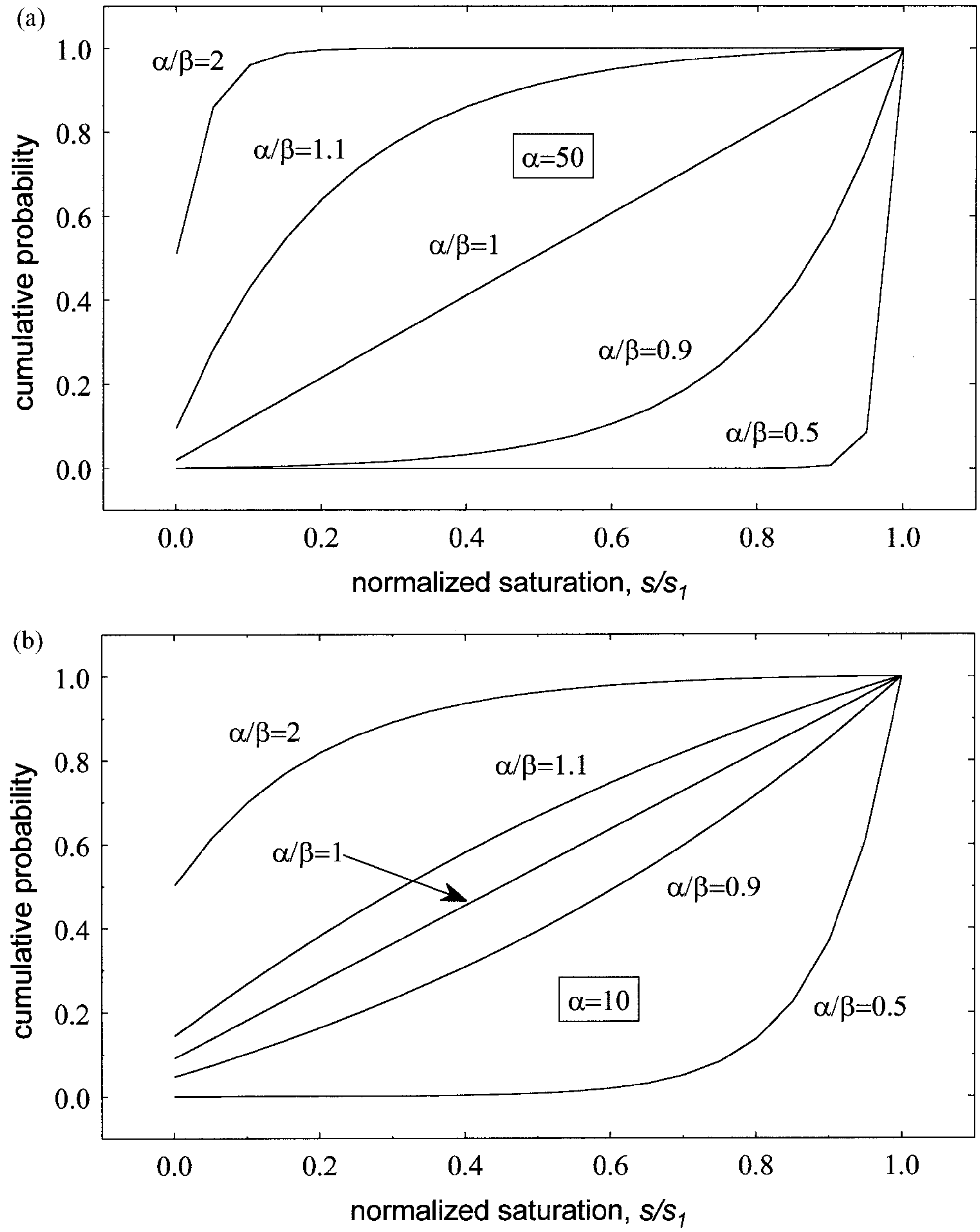

Figure 4. The cdf of normalized soil water saturation s/s_1 for various values of dimensionless storage capacity α and index of dryness α/β; (top) $\alpha = 50$; (bottom) $\alpha = 10$.

evaporation to continue unabated until the soil is dry. The more general model suppresses evaporation in this range, tending to keep soil relatively wet when water is scarce. In situations where the detailed shape of the pdf or cdf is a concern, especially under arid conditions, one might prefer to employ the more general model.

Milly [1993] has shown that the fraction of precipitation lost to evaporation is given by

$$(e^{\alpha-\beta} - 1)/(e^{\alpha-\beta} - \beta/\alpha) \; . \qquad (18)$$

The partitioning of precipitation into runoff and evapotranspiration by the two models is compared in Figure 3 using the examples of *Rodríguez-Iturbe et al.* [1999]. Differences are very small.

Rodríguez-Iturbe et al. [1999], having introduced the parameter s^* as a level of saturation below which water stress reduces evapotranspiration, are able to quantify also the partitioning of total evapotranspiration into that which occurs while the plants are under stress $s < s^*$ and that which does not. In contrast, *Milly* [1993] makes the minimalist assumption that water stress is absent until all the water is gone, so none of the evaporation occurs under stress. The inability to caculate meaningful values of the measure of stress used by Rodríguez-Iturbe et al. might appear to be a decided disadvantage for "ecohydrological" studies. However, a more physically relevant measure of water stress, which can be obtained from either model, is the temporal mean of difference between stressed and unstressed evaporation rates (that is, between actual and potential evaporation rates). As already shown, actual evaporation rates differ little between the two models.

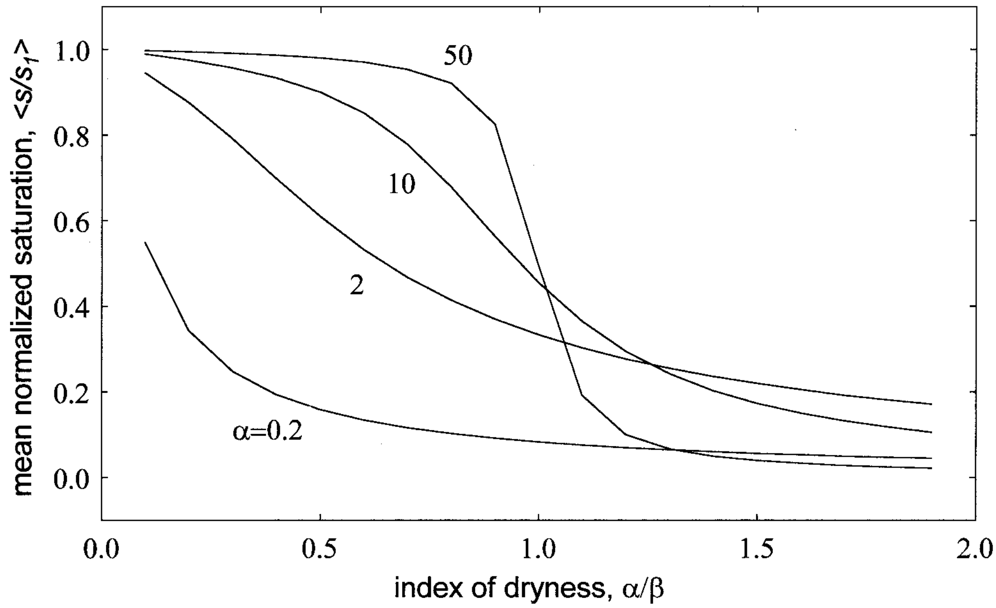

Figure 5. Mean normalized soil water $\langle s/s_1 \rangle$ as a function of dimensionless storage capacity α and index of dryness α/β.

5.2. Solutions for the Full Parameter Space

An attractive characteristic of (14) and (15) is that they depend only on the parameters α and β, facilitating illustration of the full parameter space of the solution. Figure 4 shows the dependence of the cdf on these dimensionless parameters. For a large dimensionless water capacity ($\alpha = 50$), soil water is almost always at or near its maximum when the index of dryness α/β is <1 and almost always at or near its minimum when the index of dryness is >1. It is only when the index of dryness is very close to 1 that soil water is likely to be found at any level. Indeed, for the interesting case where the index of dryness is 1 and α is large, the soil water pdf reduces to a uniform distribution, as is clear also directly from the form of (13).

As the dimensionless soil water capacity is reduced from a very large value, the strong sensitivity of the cdf to the index of dryness α/β in the area of $\alpha/\beta = 1$ is successively reduced. In the case where R is slightly <1 (e.g., $\alpha/\beta = 0.9$, a slightly humid climate), the cdf plotted for $\alpha = 10$ in Figure 4 indicates that the pdf is bimodal. One mode is associated with the finite probability at total dryness. The second can be inferred from the concave upward cdf, indicating pdf increasing with saturation. In fact, the conditions for bimodal pdf of soil water are

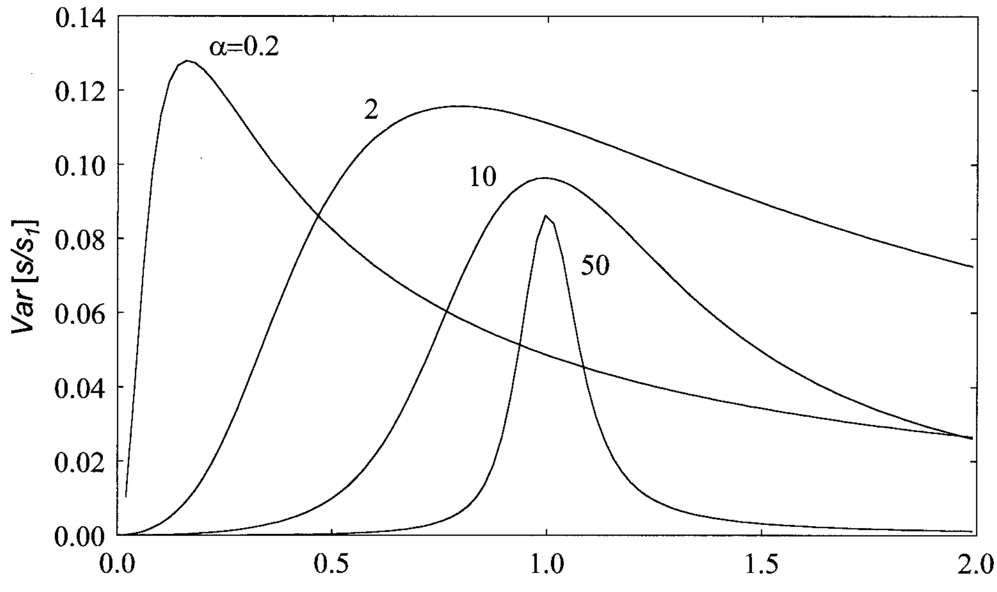

Figure 6. Variance of normalized saturation as function of dimensionless storage capacity and index of dryness.

immediately obvious from (12). The pdf will have two maxima if and only if the index of dryness is <1, and the modal values of normalized soil water are 0 and 1 in all cases.

Figure 5 shows the dependence of the mean soil water on the dimensionless water capacity and the index of dryness. As we would expect, the larger the value of the index of dryness, the drier the mean state of soil water. For the typical case where water capacity is large, the soil is near its maximum possible wetness for most humid climates and near its minimum wetness for most arid climates. Only in a narrow range of climates characterized by an index of dryness near 1, representing a near balance between water supply and evaporative demand, are intermediate values of mean soil water found.

This result is easily understood in terms of the model assumptions. When α and β are large, it is very unlikely that soil water will change much during an individual storm or interstorm event. If, for example, the climatic tendency is for an excess of precipitation over evaporative demand, then each pair of events is more likely to raise than to lower soil water, given some intermediate state of wetness. Over time, the wetness will approach its maximum, and because β is large, soil water will very rarely drop far below the maximum. A similar argument applies to the arid case. In the limit of infinite α and β the curves in Figure 5 approach a step function, with zero mean soil water for all arid climates (index of dryness >1) and maximum possible soil water for all humid climates (index of dryness <1).

If, however, water capacity is not small in comparison to the characteristic water depths of climatic forcing (mean storm depth and mean interstorm evaporative depth), then regardless of the level of soil water at a given time, it is quite possible that soil water may take any value in the near future. Thus, when α and β are not large, a wider range of states is experienced, and as a result, the dependence of the mean state on index of dryness is spread over a wider range of climates, as seen in Figure 5.

Figure 6 illustrates the dependence of variance on water capacity and index of dryness. The large-capacity case exhibits a sharp peak in variance around the boundary between arid and humid climates. The peak broadens and shifts toward lower index of dryness as the water capacity is reduced. The general pattern of the variance curves is readily understood in terms of the means already shown and discussed in Figure 5. For a given value of α, the greatest variance occurs under those conditions that support an intermediate value for the mean soil water. It is only when the mean state of soil water is not "pinned" against one of its two limiting values that it is free to vary up and down most widely.

The peaks in variance grow broader with decreasing values of α and β. Small values of these parameters allow larger changes in normalized wetness during a given storm or interstorm event. This, in turn, allows a wider range of soil water values to be experienced under a given set of climatic conditions.

The variance of a uniform distribution on the interval [0,1] is 0.0833. This is the maximum variance in the case of large α. As α decreases, the maximum variance rises above this value, indicating the presence of the bimodal distribution mentioned earlier. It can be seen that the increase in maximum variance is significant only when α is much less than 10. Natural ecosystems tend to have root zones deep enough to make values of α much larger than 1, probably as a result of the tendency to maximize water use [*Milly and Dunne*, 1994; *Milly*, 1994]. Given this observation and the results of Figure 6, it can be concluded, for most practical purposes, that soil water variance is maximized when the index of dryness is in the neighborhood of 1, that is, when the atmospheric supply of water is approximately balanced by the evaporative demand for water.

6. Summary

The implications of a previously published analysis of the probabilistic response of soil water to random precipitation have been explored. Expressions for the cumulative distribution function, the mean, and the variance of soil water have been derived, presented, and interpreted. Despite the simplicity of the problem considered the solution shows a richness of behavior. Because of the simplicity of the analysis, that richness can be interpreted easily.

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