A Comparative Study of Curved Flow and Density-Stratified Flow[1]

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ABSTRACT

A semi-empirical theory, used to predict buoyancy effects in a density-stratified and shear-driven flow, is also applied to the case of a boundary layer with curvature. Curved flow data are available and interesting in their own right since it can be seen that the Reynolds stress is reduced to zero at a critical "curvature Richardson" number predicted reasonably well by the theory.

1. Introduction

In 1929 Prandtl wrote a paper citing the fact that two somewhat analogous physical factors—buoyancy and flow curvature—can profoundly influence turbulence [and reinvented the nondimensional buoyancy parameter defined by Richardson (1920, 1925)]. Using his mixing length approach one can arrive at a heuristic idea of why flow with a normal pressure gradient related to gravity or a centrifugal field are stable or unstable. Later, Bradshaw (1969) appealed to this analogy to transfer stratified experimental correlations, as understood then, to the curved, centrifugal flow case.

So and Mellor (1972) presented data and the results of a turbulence model based primarily on hypotheses by Rotta (1951) and Kolmogorov (1941) applied to the curvature effect. The model neglected advection and diffusion terms in the turbulent moment equations; this simplification results in algebraic expressions for the effect of curvature. Donaldson (1973) similarly simplified his model and applied it to density stratified flow as well as curved flow. However, his assumption for dissipation did not adhere to Kolmogorov's hypothesis of small-scale isotropy (Corrsin, 1973) and apparently is the reason his results are in disagreement with existing data; for example, he obtained a critical Richardson number of 1.64 instead of 0.22 in the stratified flow case.

Mellor (1973; henceforth referred to as paper I) presented the results of his turbulence model; then simplified it by neglect of advection and diffusion and compared the result to the (constant flux) near surface, atmospheric boundary layer data of Businger *et al.* (1971). Agreement with the data was quite good. A similar result was also presented by Lewellen and Teske (1973) who included turbulent diffusion but not advection. Their model was an extension of Donaldson's model, but correctly applied Kolmogorov's hypothesis.

Mellor and Yamada (1974; henceforth referred to as paper II) reexamined paper I and attached a "Level 4" label to the original complex of equations which calls for the simultaneous solution of differential equations for six turbulent stress components, three heat flux components and temperature variance, in addition to the mean equations.

Identification of a small parameter related to departures from isotropy suggested a systematic process by which terms in these equations could be neglected. Thus, the Level 3 model eliminated all but two differential equations for turbulent energy and temperature variance; the remaining information was reduced to algebraic form. The Level 2 model reduced all of the turbulent moment equations to algebraic equations and is identical to that which was ultimately applied to the surface atmospheric data in paper I; now, however, from the derivation in paper II the Level 2 model need not be constrained to the near-surface layers, but is generalized to three-dimensional flow and is considered an approximation to the Level 4 model across the entire boundary layer. The result is a simple closed expression for the stabilizing or destabilizing effect of buoyancy as expressed through the Richardson number.

The purpose of this paper is to bring together in a common format pieces of information which heretofore have been disconnected, and we do so on the basis of the simplest (Level 2) model which, it appears, contains the essential ingredients of the other more complicated models.[2]

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[2] In our most recent atmospheric boundary layer numerical simulations (not yet published), we have chosen the Level 3 model. The underlying conceptual simplicity of the Level 2 model has been important in the success of these simulations, however.

In the density-stratified, near-surface layer case, the Level 2 model and data indicated a local flux Richardson number of 0.21, or gradient Richardson number of 0.22 beyond which turbulence could not exist.[3] In the curvature data to be reviewed here a critical "curvature Richardson" number of about 0.23 is obtained whereas the model predicts 0.185 (adding diffusion as in the Level 3 or 4 models would increase this value). What is important about these data is that, unlike the available atmospheric data, one has a clear observation of the rather abrupt extinction of the Reynolds stress in the middle of the layer where the velocity gradient is nonzero but where the Richardson number is 0.23. Presumably, atmospheric data, if they could be obtained under controlled laboratory conditions, would exhibit the same behavior. The data are also important because they demonstrate the power of the turbulence model in predicting the most salient feature of two seemingly disparate flow situations. Note that the model does contain some adjustable constants, two of which are inoperative in the curvature case. However, these have been fixed once and for all from neutral flow data.

2. Review of the model equations and their application to density-stratified layers

We define the mean velocity $U_j=(U,V,W)$; Θ is the potential temperature; $\overline{u_k u_i}$ and $\overline{u_k\theta}$ the Reynolds stress and heat flux; $q^2\equiv\overline{u_k^2}$; and $\overline{\theta^2}$ is the temperature variance; $g_k=(0,0,-g)$ is the gravity vector; and β is the coefficient of thermal expansion. From the Level 2 model of paper II, the equations are[4]

$$\frac{q^3}{\Lambda_1}=-\overline{u_k u_i}\frac{\partial U_i}{\partial x_k}-\beta g_k\overline{u_k\theta}, \tag{1}$$

$$\frac{q\overline{\theta^2}}{\Lambda_2}=-\overline{u_k\theta}\frac{\partial\Theta}{\partial x_k}, \tag{2}$$

$$\overline{u_i u_j}=\frac{\delta_{ij}}{3}q^2-3\frac{l_1}{q}\Big[(\overline{u_k u_i}-Cq^2\delta_{ik})\frac{\partial U_j}{\partial x_k}+(\overline{u_k u_j}-Cq^2\delta_{kj})\frac{\partial U_i}{\partial x_k}-\tfrac{2}{3}\delta_{ij}\overline{u_k u_l}\frac{\partial U_l}{\partial x_k}\Big]-3\frac{l_1}{q}\beta[g_i\overline{u_j\theta}+g_j\overline{u_i\theta}-\tfrac{2}{3}\delta_{ij}g_l\overline{u_l\theta}], \tag{3}$$

$$\overline{u_i\theta}=-3\frac{l_2}{q}\Big[\overline{u_j u_k}\frac{\partial\Theta}{\partial x_k}+\overline{\theta u_k}\frac{\partial U_j}{\partial x_k}+\beta g_j\overline{\theta^2}\Big], \tag{4}$$

where $l_1=Al$, $l_2=Al$, $\Lambda_1=B_1 l$, $\Lambda_2=B_2 l$ and $(A,B_1,B_2,C)=(0.78, 15.0, 8.0, 0.056)$. As noted in paper I, only three of these constants are independent and have been determined from neutral turbulent flow data. The constraining relation is $B_1^{1/3}=A(B_1-6A-3B_1C)$.

In paper II Eqs. (1)–(4) were first simplified by the boundary layer approximation. The result[5] can then be presented in a number of forms. If we choose a conventional K-theory format, we obtain

$$-\begin{bmatrix}\overline{wu}\\ \overline{wv}\\ \overline{w\theta}\end{bmatrix}=l^2\Big[\Big(\frac{\partial U}{\partial z}\Big)^2+\Big(\frac{\partial V}{\partial z}\Big)^2\Big]^{\frac{1}{2}}\begin{bmatrix}S_M\partial U/\partial z\\ S_M\partial V/\partial z\\ S_H\partial\Theta/\partial z\end{bmatrix}, \tag{5a, b, c}$$

where

$$S_M=B_1^{\frac{1}{2}}[3A(\gamma_1-\gamma_2\Gamma)]^{\frac{3}{2}}(1-R_f)^{\frac{1}{2}}\times\Big[\frac{\gamma_1-C-9A\Gamma/B_1}{\gamma_1-\gamma_2\Gamma+3A\Gamma/B_1}\Big]^{\frac{3}{2}}, \tag{6a}$$

$$S_H=B_1^{\frac{1}{2}}[3A(\gamma_1-\gamma_2\Gamma)]^{\frac{3}{2}}(1-R_f)^{\frac{1}{2}}\times\Big[\frac{\gamma_1-C-9A\Gamma/B}{\gamma_1-\gamma_2\Gamma+3A\Gamma/B_1}\Big]^{\frac{1}{2}}, \tag{6b}$$

and where $\gamma_1\equiv(\frac{1}{3})-2(A/B_1)$, $\gamma_2\equiv(B_2/B_1)+6(A/B_1)$, and the flux Richardson number is conventionally defined as the ratio of negative buoyant production to the shear production:

$$R_f\equiv\frac{\beta g\,\overline{w\theta}}{\overline{uw}\frac{\partial U}{\partial z}+\overline{vw}\frac{\partial V}{\partial z}}. \tag{7a}$$

Also

$$\Gamma\equiv\frac{R_f}{1-R_f} \tag{7b}$$

is the ratio of negative buoyancy production to total production. Expressions can also be obtained for the other components of the Reynold stress and heat flux tensors. Note that the gradient Richardson number is given by

$$\mathrm{Ri}\equiv\frac{\beta g\partial\Theta/\partial z}{(\partial U/\partial z)^2+(\partial V/\partial z)^2}=\frac{S_M}{S_H}R_f.$$

The functions S_M and S_H are plotted as functions of R_f in Fig. 1. In paper I, these functions, reinterpreted somewhat,[6] have been compared with the constant flux

[3] Miles (1961), following the work of Taylor (1931) and Goldstein (1931), showed that $\mathrm{Ri}>\frac{1}{4}$ *everywhere* in the flow field is a sufficient condition for the stability of a flow subject to *small* perturbations.

[4] In paper II, Eqs. (25), (26), (27) and (28).

[5] In paper II, Eqs. (52a, b, c) with the diffusion terms eliminated and Eqs. (53a, b, c) and (54a, b, c).

[6] In the constant flux layer, we let $l=\kappa z$, $-\overline{wu}=u_\tau^2$, $-\overline{wv}=0$ and $-\overline{w\theta}=H$. We define $\varphi_M\equiv\kappa z u_\tau^{-1}(\partial U/\partial z)$, $\varphi_H\equiv\kappa z u_\tau H^{-1}\times(\partial\Theta/\partial z)$ and $\zeta\equiv z/L$, where $L\equiv u_\tau^3(\kappa g\beta H)^{-1}$ is the Monin-Obukhov length scale. It is then easy to show that $\varphi_M=S_M^{-\frac{1}{2}}$, $\varphi_H=S_M^{\frac{1}{2}}S_H^{-1}$ and $\zeta=\varphi_M R_f$. The data were presented in the form $\varphi_M(\zeta)$ and $\varphi_H(\zeta)$.

data of Businger *et al.* (1971) where we had set $l=\kappa z$. The comparison was quite favorable. A flux Richardson number of 0.21 or gradient Richardson number of 0.22 beyond which turbulence is extinguished is the salient result of both the model and the data.

3. The curvature data and theory

Fig. 2 illustrates the flow bounded by a curved surface and potential flow.

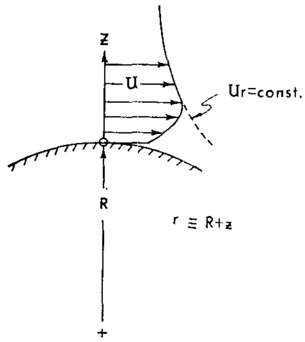

FIG. 2. The two-dimensional boundary layer flow over a curved wall.

If we now return to Eqs. (1) and (2), write these equations in curvilinear form, and then make the boundary layer approximation, we obtain

$$\frac{q^3}{\Lambda_1} = -\overline{wu}\frac{\partial U}{\partial z} + \overline{uw}\frac{U}{r}, \tag{8}$$

$$\begin{bmatrix} \overline{u^2} \\ \overline{w^2} \\ \overline{v^2} \end{bmatrix} = \frac{q^2}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{l_1}{q}\overline{uw}\begin{bmatrix} -4\partial U/\partial z - 8U/r \\ 2\partial U/\partial z + 10U/r \\ 2\partial U/\partial z - 2U/r \end{bmatrix}, \tag{9a, b, c}$$

$$-\overline{uw} = \frac{3l_1}{q}\left[\overline{w^2}\left(\frac{\partial U}{\partial z}+\frac{U}{r}\right) - 2\overline{u^2}\frac{U}{r} - Cq^2\left(\frac{\partial U}{\partial z}-\frac{U}{r}\right)\right]. \tag{9d}$$

We now note that the flux Richardson number, defined in (7a), can be interpreted as the ratio of the negative buoyant production term in the vertical component of turbulent energy equation to the sum of the shear production terms in the horizontal components.[7] In the case of curvature, this suggests that we also define a curvature Richardson in analogous fashion so that

$$R_c \equiv \frac{2U/r}{\dfrac{\partial U}{\partial z}+\dfrac{U}{r}} = \frac{2U}{\dfrac{\partial}{\partial z}(Ur)}. \tag{10}$$

Then Eqs. (8) and (9a–d) can be solved for any component of the Reynolds stress. In particular, $-\overline{uw}$ can be obtained in the form

$$-\overline{uw} = l^2\left|\frac{\partial U}{\partial z}-\frac{U}{r}\right|\left(\frac{\partial U}{\partial z}-\frac{U}{r}\right)S_c(R_c), \tag{11}$$

(note that $\partial U/\partial z - U/r$ is the shear strain whereas $\partial U/\partial z + U/r$ is the vorticity) where

$$S_c = \left[1 - \frac{36A_1^2}{B_1^{4/3}}\frac{R_c}{(1-R_c)^2}\right]^{3/2}. \tag{12}$$

The function $S_c(R_c)$, is plotted in Fig. 3. From (12) one obtains $S_c=0$ when $R_c=0.185$, and again at $R_c=5.416$. Within the region, $0.185<R_c<5.416$, we assume $S_c=0$; otherwise the combination of (8), (11) and (12) would yield negative turbulent kinetic energies. It will be noted that flat plate flow corresponds to $R_c=0$, potential curved flow to $R_c\sim\pm\infty$, and solid-body rotation (as in the eye of a tornado or hurricane?) to $R_c=1.0$.

As described by So and Mellor (1973), a two-dimensional boundary layer was established on a flat surface and was then made to flow over a curved wall whose radius of curvature was around 30 cm (and

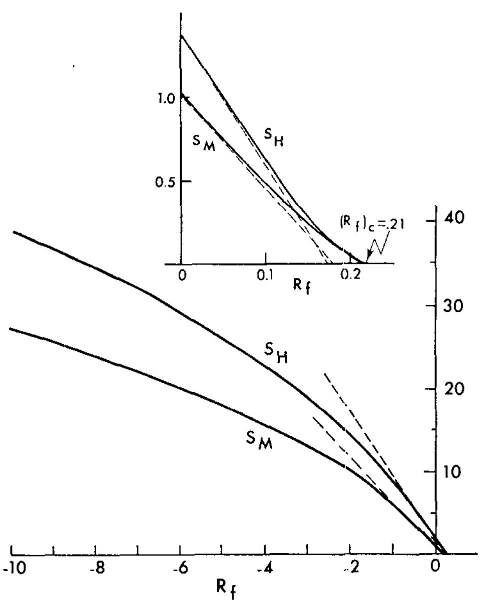

FIG. 1. The buoyancy stability functions S_M and S_H vs R_f. The inset is a detail near $R_f=0$. The dashed lines are given by $S_M=1.0-5.57R_f$ and $S_H=1.35(1.0-5.81R_f)$.

[7] To see this, (9a, b, c) might better have been written in the form

$$(q/l_1)[\overline{u^2},\overline{w^2},\overline{v^2}] - (q^3/3l_1)[1,1,1] = -2(q^3/\Lambda_1)[1,1,1] - 6\overline{uw}[\partial U/\partial z + U/r, -2U/r, 0].$$

Elimination of the second term on the right, using (8), yields (9a, b, c). The form of (9a, b, c), however, has the advantage that the full differential equation may replace (8) in the alternate Level 3 scheme of paper II.

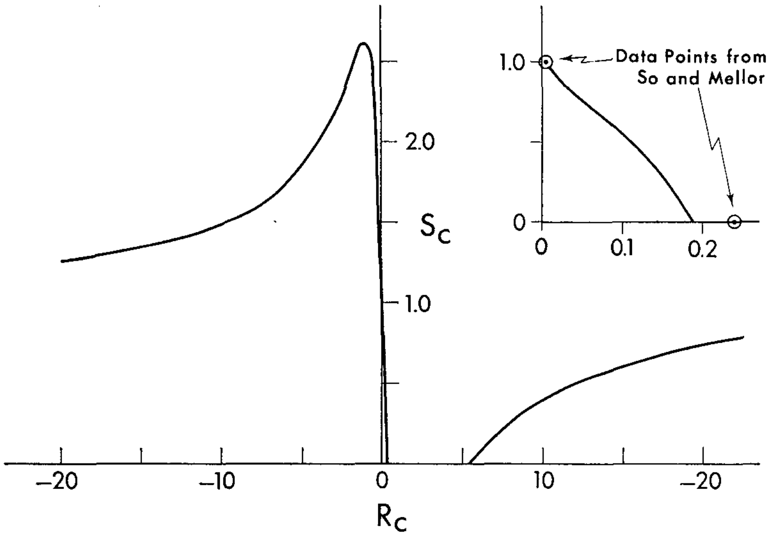

FIG. 3. The curvature stability function S_c vs R_c.

varied slightly along the flow path). Some of the mean profiles, first on the flat surface and then on the curved section, are shown in Fig. 4. As $z\rightarrow\infty$, $U\sim U_{pw}(R/r)=U_{pw}(1-zR^{-1})$ which is the potential flow profile; U_{pw} is the potential flow wall velocity and R is the radius of curvature. The transition from flat to curved flow occurred at $x=122$ cm, and at this point the profile looks very much like the profile at $x=62$ cm. The Reynolds stress $\overline{uw}$ has been measured and is also shown in Fig. 4. Other data are available in the original paper.

Focussing attention on the $x=180$ cm profiles, the dramatic decrease in Reynold stress compared to the flat plate case is easily seen; at $z/\delta\approx0.45$ the Reynolds stress is essentially zero.

In the application of this closure model it has thus far been assumed—successfully it appears—that the length scale l needed in (5a, b, c) or (11) may be modeled in identical fashion to neutral layers. Near a surface it is then possible to let $l\sim\kappa z$ as $z\rightarrow0$. For large z, a more elaborate empirical function, $l(z)$, is required. In the case of the atmospheric boundary layer theory, data comparison of paper I, all of the data were indeed collected close to the surface where $l=\kappa z$ was appropriate.

For the data of Fig. 4 the points close to the surface have a well-defined, classical logarithmic behavior (see So and Mellor, 1973), where $R_c<0.01$. Together with $l=\kappa z$ Eq. (11) implies an experimental determination, $S_c\approx1.0$, for very small R_c and this datum has been entered into Fig. 3.

From the $x=180$ cm profile, at $z/\delta\approx0.45$, we calculate $R_c\approx0.23$. At this point we observe $\overline{uw}\approx0$ so that, independent of l, we obtain the experimental value, $S_c\approx0$.

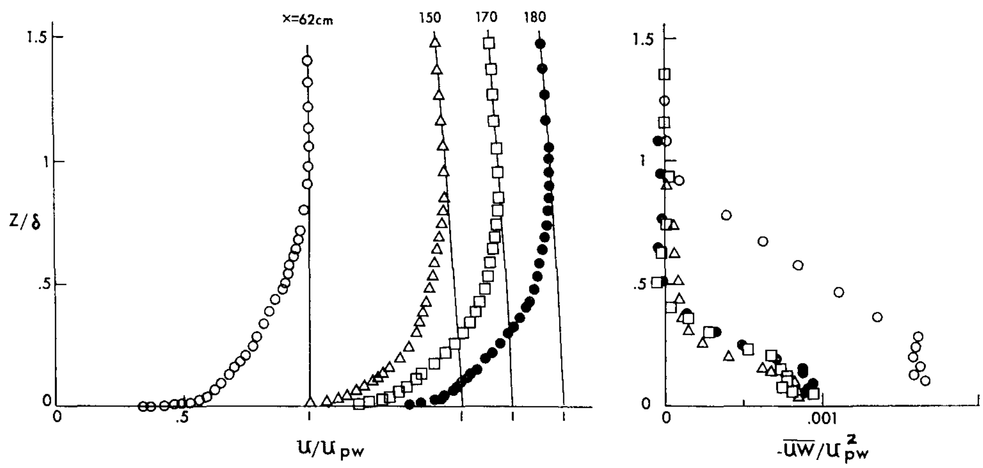

FIG. 4. Non-dimensional velocity and Reynold stress profiles from So and Mellor (1973). When $x<122$ cm, the flow is over a flat surface whereas the surface is curved when $x>122$ cm. At $x=62$ cm, $\delta=1.40$ cm; when $x=122$–180 cm, $\delta=\text{constant}=2.41$ cm.

For $z/\delta > 0.45$ and $R_c > 0.23$ more data could be inserted in Fig. 3 but would also yield $S_c \approx 0$. Somewhere around $z/\delta \approx 1$, R_c increases rapidly so that $S_c > 0$ according to our prediction. However, then the square of the shear strain and the Reynolds stress in (11) are very small.

For the region $0 < z/\delta < 0.45$, the data yields $1.0 > S_c > 0$; however, to evaluate experimental values of S_c, one would have to specify a more elaborate $l(z)$ other than that appropriate to near-surface flow. In this paper, we avoid this elaboration.

For the $x = 170$ cm profile at $z/\delta \approx 0.42$ we estimate $R_c \approx 0.32$ whereas for $x = 150$ cm the Reynolds stress is not zero until $z/\delta \approx 1.0$ although it is greatly surpressed when $z/\delta \approx 0.40$. Our interpretation is that these systematic upstream departures from the predicted critical value, $R_c = 0.185$, might be attributed to the neglect of turbulent advection and diffusion which would be included in the Level 3 or 4 models of paper II. We hope to show this in the future. However, it is believed that this simple discussion, based on the more simple Level 2 model, is persuasive.

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