A Hierarchy of Turbulence Closure Models for Planetary Boundary Layers

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ABSTRACT

Turbulence models centered on hypotheses by Rotta and Kolmogoroff are complex. In the present paper we consider systematic simplifications based on the observation that parameters governing the degree of anisotropy are small. Hopefully, we shall discern a level of complexity which is intuitively attractive and which optimizes computational speed and convenience without unduly sacrificing accuracy.

Discussion is focused on density stratified flow due to temperature. However, other dependent variables—such as water vapor and droplet density—can be treated in analogous fashion. It is, in fact, the anticipation of additional physical complexity in modeling turbulent flow fields that partially motivates the interest in an organized process of analytical simplification.

For the problem of a planetary boundary layer subject to a diurnally varying surface heat flux or surface temperature, three models of varying complexity have been integrated for 10 days. All of the models incorporate identical empirical constants obtained from neutral flow data alone. The most complex of the three models requires simultaneous solution of 10 partial differential equations for turbulence moments in addition to the equations for the mean velocity components and temperature; the least complex eliminates all of the 10 differential equation whereas a "compromise" model retains two differential equations for total turbulent energy and temperature variance.

We conclude that all of the models give nearly the same results. We find the two-differential-equation model particularly attractive.

1. Introduction

We undertake this study in the belief that it will provide a framework for a hierarchy of turbulent flow models centered on hypotheses by Kolmogoroff (1942), Prandtl and Wieghardt (1945) and Rotta (1951); we hope to determine a model devoid of complications unnecessary to predictive power. However, what we gain by an organized process of model simplification could be lost in a welter of semantics. At this time the *kind* of models we are discussing are variously termed "turbulent field" models (Daly and Harlow, 1970), "mean turbulent field" models or "mean turbulent energy" models (Reynolds, 1970; Mellor and Herring, 1973[2]), "invariant" models (Donaldson and Rosenbaum, 1968), or "second-order" models. The last designator is popular and simple but begs the question—second order in what parameter? Here we supply an answer to that question but at the same time require more extensive nomenclature. Therefore, we introduce semantically neutral model designators; Levels 1, 2, 3 and 4. The specific Level 4 version with which we begin has been presented by one of us [Mellor (1973) henceforth paper A; and Mellor and Herring (1973) henceforth paper B].

As a kind of table of content and referring to equation numbers in the text, each model level is comprised of the following equations in addition to Eqs. (1), (2) and (3):

Level 4 Eqs. (4), (5), (6)

↓ Advection and diffusion $= O(a)$
Neglect $O(a^2)$ terms

Level 3 Eqs. (20), (21), (22), (23)

↓ Advection and diffusion $= O(a^2)$
Neglect $O(a^2)$ terms

Level 2 Eqs. (25), (26), (27), (28)

↓ Neglect $O(a)$ terms

Level 1 Eqs. (29), (30), (31), (32)

In the above, a denotes the degree of anisotropy such that $a \to 0$ is the isotropic limit. Note that for a turbu-

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[2] Here, one will find the basic approach exploited in the present paper. Similar thoughts toward model simplification have been expressed by Deardorff (1973) and Donaldson (1973).

lent layer no physical process exists such that $a \rightarrow 0$ as it does in the kinetic theory of gases where $a \rightarrow 0$ as the mean free path approaches 0. One can only assume that a is small enough for present purposes and then test the consequences of such an assumption.

History may be recognized in this hierarchy. Level 1 and 2 models in the neutral case bear direct resemblence to eddy or mixing length models utilized by many investigators. For example, for density-stratified flow, Level 1 relates somewhat to KEYPS (see, for example, Lumley and Panofsky, 1964) type models.

A Level 2 model has enjoyed considerable success (Mellor, 1973), compared to the surface layer data of Businger *et al.* (1971). The model predicted a critical Richardson number of 0.21 beyond which turbulence was extinguished by stable buoyancy and seemed also to give reasonable results in the unstable free convection limit. Virtually no parametric adjustment was required over and above that required by neutral turbulent flow data [Similar success was reported by Lewellen and Teske (1973); however, specific parametric adjustment to accomodate stability effects seemed to be required by them.]

Models closely resembling Level 3 have been exploited in the case of neutral boundary layers (Glushko, 1965; Mellor and Herring, 1968; Beckwith and Bushnell, 1968; Ng and Spalding, 1972; and others) but apparently have not been utilized in the case of stratified planetary boundary layers, previous to this paper.

Models closely resembling Level 4 have been pursued by Donaldson and Rosenbaum (1968) and Hanjalic and Launder (1972) although the latter authors' subsequent simplifications do not follow the methodology developed here. The model recently proposed by Lumley and Khajeh-Nouri (1974) might be classed as a Level 5 model. (Obviously, still higher order levels are possible.) Although inspired by Lumley's work the calculations of Wyngaard *et al.* (1973) are more nearly a Level 4 model.

All of the aforementioned models are unanimous in including Rotta's energy redistribution hypothesis.[3]

Clarke (1974) has recently surveyed and compared a variety of older models much in the manner of this paper. The models he tested most nearly conform to our Level 1 or 2 models where, however, the stability functions [S_M and S_H in Eqs. (67a, b)] are empirical; here, they can be regarded as theoretical extensions from a base of neutral turbulent flow data.

The kind of turbulent field closure discussed here might also be termed ensemble mean closure wherein turbulent statistics can be defined as the average over many realizations of a spatially and temporally local variable or the time average of a variable in stationary flow or the spatial average of the flow homogeneous in one dimension. It is noteworthy that essentially the same model as that which we label Level 4 has been used by Deardorff (1973) to model "subgrid-scale turbulence" in fully three-dimensional and unsteady flow simulations in which case the length scale l adopted here is made proportional to grid spacing.

In the present case a general prescription for l (which automatically adjusts to each problem) seems illusive.[4] We have, therefore, chosen a rather simple length-scale function which we apply identically to the three model levels 4, 3 and 2 for the problem of a planetary boundary layer subjected to a diurnally varying surface temperature.

Having satisfied ourselves on the relative merits of the different model levels in the present paper, we plan to continue further work in comparing observations and predictions with the Level 3 model; it is sufficiently simple and does not appear to compromise accuracy relative to the Level 4 model.

2. Governing equations

We let $D(\ )/Dt \equiv U_k\partial(\ )/\partial x_k + \partial(\ )/\partial t$ (in this paper the combination of advective and tendency terms will be shortened to *advective term*). Then, the equations of motion for the mean velocity and mean potential temperature Θ are

$$\frac{\partial U_k}{\partial x_k}=0, \tag{1}$$

$$\frac{DU_j}{Dt}+\epsilon_{jkl}f_kU_l=\frac{\partial}{\partial x_k}(-\overline{u_ku_j})-\frac{\partial P}{\partial x_j}-g_j\beta\Theta, \tag{2}$$

$$\frac{D\Theta}{Dt}=\frac{\partial}{\partial x_k}(-\overline{u_k\theta}), \tag{3}$$

where P is the mean kinematic pressure, $g_j=(0,0,-g)$ the gravity vector, $f_j=(0,f_y,f)$ the Coriolis parameter (the vertical component will not be subscripted), and $\beta\equiv-(\partial\rho/\partial T)_p/\rho$ the coefficient of thermal expansion. Molecular diffusive terms have been neglected. The overbars represent ensemble averages; u_k and θ are the fluctuating components of the velocity and temperature.

a. The Level 4 Model

The full mean Reynolds stress model equations (paper A) are as follows:

$$\frac{D\overline{u_iu_j}}{Dt}-\frac{\partial}{\partial x_k}\left[q\lambda_1\left(\frac{\partial\overline{u_iu_j}}{\partial x_k}+\frac{\partial\overline{u_iu_k}}{\partial x_j}+\frac{\partial\overline{u_ju_k}}{\partial x_i}\right)\right]$$
$$=-\overline{u_ku_i}\frac{\partial U_j}{\partial x_k}-\overline{u_ku_j}\frac{\partial U_i}{\partial x_k}-\frac{2}{3}\frac{q^3}{\Lambda_1}\delta_{ij}$$

[3] In the light of the present analysis we assert that this hypothesis is implicit even in eddy viscosity or mixing length models.

[4] For example, we feel that the popular use of a transport equation for dissipation to supply a macro-length scale is fundamentally incorrect.

$$-\frac{q}{3l_1}\left(\overline{u_iu_j}-\frac{\delta_{ij}}{3}q^2\right)+C_1q^2\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)$$
$$-\beta(g_j\overline{u_i\theta}+g_i\overline{u_j\theta})$$
$$+C_2\beta(g_j\overline{u_i\theta}+g_i\overline{u_j\theta}-\tfrac{2}{3}\delta_{ij}g_k\overline{u_k\theta}) \quad (4)$$

$$\frac{D\overline{\theta^2}}{Dt}-\frac{\partial}{\partial x_k}\left[q\lambda_2\frac{\partial\overline{\theta^2}}{\partial x_k}\right]=-2\overline{u_k\theta}\frac{\partial\Theta}{\partial x_k}-2\frac{q}{\Lambda_2}\overline{\theta^2} \quad (5)$$

$$\frac{D\overline{u_j\theta}}{Dt}-\frac{\partial}{\partial x_k}\left[q\lambda_3\left(\frac{\partial\overline{u_j\theta}}{\partial x_k}+\frac{\partial\overline{u_k\theta}}{\partial x_j}\right)\right]$$
$$=-\overline{u_ju_k}\frac{\partial\Theta}{\partial x_k}-\overline{\theta u_k}\frac{\partial U_j}{\partial x_k}-\beta g_j\overline{\theta^2}-\frac{q}{3l_2}\overline{u_j\theta}+C_3g_j\beta\overline{\theta^2}. \quad (6)$$

Terms that have been modeled contain either l_1, l_2, C_1, C_2 or C_3 [and are based on the energy redistribution hypothesis by Rotta (1951)], Λ_1 or Λ_2 [and are based on the local isotropy hypothesis by Kolmogoroff, (1941)], and λ_1, λ_2 or λ_3 which are length parameters in diffusion terms. As discussed in paper A several tensorial forms are available for diffusion terms and we have chosen one rather arbitrarily. (One attractive feature of the Level 3 model is that all forms collapse to identical expressions.)

In this paper, to simplify discussion we have not included the Coriolis terms $f_k(\epsilon_{jkl}\overline{u_lu_i}+\epsilon_{ikl}\overline{u_lu_j})$ in (4) and $f_k\epsilon_{jkl}\overline{u_l\theta}$ in (6); within the boundary layer they are relatively small.

Level 4 modeling consists of solving Eqs. (1), (2), (3), (4), (5) and (6). For a PBL—after the hydrostatic approximation—this consists of solving 13 simultaneous partial differential equations. All length scales are assumed proportional so that

$$l_1,l_2=A_1l,A_2l, \quad (7a,b)$$

$$\Lambda_1,\Lambda_2=B_1l,B_2l. \quad (7c,d)$$

It should be noted that the values (A_1,A_2,B_1,B_2,C_1) = (0.78, 0.78, 15.0, 8,0. 0.056) have been gleaned from neutral turbulence data as described in papers A & B when, if $l=kz$, the stratified surface layer data of Businger *et al.* (1971) were predicted quite well. The constants, C_2, C_3, are the only coefficients specifically related to bouyancy terms. We take $C_2=C_3=0$ as in paper A. Here we also assume that $\lambda_3=\lambda_2=\lambda_1$ and from paper B we learn that $\lambda_1=0.23l$.

Ordering of terms

To reduce the complexity of the Level 4 model we wish to neglect or simplify terms in a systematic way.

Now Eq. (4) may be separated into an isotropic part and an anisotropic part. The isotropic part is the energy equation obtained by contracting (4).

Thus, if $\overline{u_k^2}\equiv q^2$, we have

$$\frac{Dq^2}{Dt}-\frac{\partial}{\partial x_k}\left[q\lambda_1\left(\frac{\partial q^2}{\partial x_k}+2\frac{\partial\overline{u_iu_k}}{\partial x_i}\right)\right]$$
$$=-2\overline{u_ku_i}\frac{\partial U_i}{\partial x_k}-2\beta g_i\overline{u_i\theta}-2\frac{q^3}{\Lambda_1}, \quad (8)$$

whereas, if we subtract the product of (8) and $\delta_{ij}/3$ from (4), we have

$$\frac{D}{Dt}\left(\overline{u_iu_j}-\frac{\delta_{ij}}{3}\overline{u_k^2}\right)-\frac{\partial}{\partial x_k}\left\{q\lambda_1\left[\frac{\partial\overline{u_iu_j}}{\partial x_k}+\frac{\partial\overline{u_iu_k}}{\partial x_j}+\frac{\partial\overline{u_ju_k}}{\partial x_i}\right.\right.$$
$$\left.\left.-\frac{\delta_{ij}}{3}\left(\frac{\partial\overline{u_l^2}}{\partial x_k}+2\frac{\partial\overline{u_lu_k}}{\partial x_l}\right)\right]\right\}$$
$$=-\overline{u_ku_i}\frac{\partial U_j}{\partial x_k}-\overline{u_ku_j}\frac{\partial U_i}{\partial x_k}+\tfrac{2}{3}\delta_{ij}\overline{u_ku_l}\frac{\partial U_l}{\partial x_k}$$
$$-\beta(g_j\overline{u_i\theta}+g_i\overline{u_j\theta}-\tfrac{2}{3}\delta_{ij}g_l\overline{u_l\theta})$$
$$-\frac{q}{3l_1}\left(\overline{u_iu_j}-\frac{\delta_{ij}}{3}q^2\right)+C_1q^2\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right). \quad (9)$$

We now define nondimensional departures from isotropy, a_{ij} and b_i, so that

$$\overline{u_iu_j}\equiv\left(\frac{\delta_{ij}}{3}+a_{ij}\right)q^2;\quad a_{ii}=0, \quad (10)$$

$$\overline{u_i\theta}\equiv b_iq\varphi, \quad (11)$$

where $\varphi^2\equiv\overline{\theta^2}$. Using (10) and (11) we write Eqs. (8) and (9) as (12) and (13) in Table 1. Eq. (5) has no anisotropic part and is repeated as (14) in Table 1 whereas (6) has no isotropic part and is repeated as (15). We now let $l\equiv O(l_1)=O(l_2)$ and $\Lambda\equiv O(\Lambda_1)=O(\Lambda_2)$, and further define $a^2\equiv O(a_{ij}^2)$, $U_x^2\equiv O[(\partial U_i/\partial x_j)^2]$, $\Theta_x^2\equiv O[(\partial\Theta/\partial x_i)^2]$, $b^2\equiv O(b_i^2)$ and $g^2\equiv g_i^2$.

The ordering of each term in Table 1 is accomplished in two steps corresponding to the two rows of terms below each equation. It is our plan to assume that l/Λ, a and b are small and, in fact, we will show that $a^2=O(b^2)=O(l/\Lambda)$. The procedure is inspired by the kinetic theory of gases where l plays a role like the mean free path. Unlike the kinetic theory of gases, there is no real limiting process $l/\Lambda\rightarrow 0$ for a specific turbulence problem.[5] Note, however, that we have already determined that $l/\Lambda\approx 0.05$–0.10 from neutral experimental data and that this ratio apparently prevails for density stratified flow (see paper A); similarly, we find that

[5] As, for example in the limiting large Reynolds number process (Yajnik, 1970; Mellor, 1972). On this point it should be mentioned that all equations and quantities discussed here are considered to be lowest order in inverse Reynolds number, *ab initio.*

TABLE 1. Ordering of terms. The second rows of ordering parameters make use of Eqs. (16a,b), (17a,b) and (18).

$$\frac{Dq^2}{Dt}-\frac{\partial}{\partial x_k}\left[q\lambda_1\frac{5}{3}\frac{\partial q^2}{\partial x_k}\{1+O(a)\}\right]=-2a_{ki}q^2\frac{\partial U_i}{\partial x_k}-2b_kg_k\beta q\varphi-2\frac{q^3}{\Lambda}\tag{12}$$

Uq^2/L Uq^2/L aq^2U_x $\beta bgq\varphi$ q^3/Λ

q^3/Λ q^3/Λ q^3/Λ

$$\frac{D}{Dt}(a_{ij}q^2)-\frac{\partial}{\partial x_k}\left[\frac{q\lambda_1}{3}\left\{\delta_{ik}\frac{\partial q^2}{\partial x_j}+\delta_{jk}\frac{\partial q^2}{\partial x_i}-\frac{2}{3}\delta_{ij}\frac{\partial q^2}{\partial x_k}\right\}\{1+O(a)\}\right]=-q^2\left[\left\{\frac{\delta_{ki}}{3}+a_{ki}\right\}\frac{\partial U_j}{\partial x_k}+\left\{\frac{\delta_{kj}}{3}+a_{kj}\right\}\frac{\partial U_i}{\partial x_k}\right.$$

aUq^2/L Uq^2/L $q^2U_x[1+O(a)]$

$a^{-1}q^3/\Lambda[1+O(a)]$

$$\left.-\frac{2}{3}\delta_{ij}a_{kl}\frac{\partial U_l}{\partial x_k}-C_1\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)\right]-\beta q\varphi(g_jb_i+g_ib_j-\tfrac{2}{3}\delta_{ij}g_lb_l)-\frac{q^3}{3l_1}a_{ij}\tag{13}$$

$b\beta q\varphi g$ aq^3/l

q^3/Λ $a^{-1}q^3/\Lambda$

$$\frac{D\overline{\theta^2}}{Dt}-\frac{\partial}{\partial x_k}\left[q\lambda_2\frac{\partial\overline{\theta^2}}{\partial x_k}\right]=-2q\varphi b_k\frac{\partial\Theta}{\partial x_k}-2\frac{q}{\Lambda_2}\overline{\theta^2}\tag{14}$$

$U\varphi^2/L$ $U\varphi^2/L$ $q\varphi b\Theta_x$ $q\varphi^2/\Lambda$

$q\varphi^2/\Lambda$ $q\varphi^2/\Lambda$

$$\frac{D}{Dt}(b_jq\varphi)-\frac{\partial}{\partial x_k}\left[q\lambda_3\left\{\frac{\partial}{\partial x_k}(b_jq\varphi)+\frac{\partial}{\partial x_j}(b_kq\varphi)\right\}\right]=-q^2\left(\frac{\delta_{jk}}{3}+a_{jk}\right)\frac{\partial\Theta}{\partial x_k}-q\varphi b_k\frac{\partial U_j}{\partial x_k}-g_j\beta\overline{\theta^2}-q^2\varphi b_j/3l_2\tag{15}$$

$bUq\varphi/L$ $bUq\varphi/L$ $q^2\Theta_x[1+O(a)]$ $q\varphi bU_x$ $g\beta\varphi^2$ $q^2\varphi b/l$

$b^{-1}q^2\varphi/\Lambda[1+O(a)]$ $q^2\varphi/\Lambda$ $b^{-1}q^2\varphi/\Lambda$ $b^{-1}q^2\varphi/\Lambda$

$a_{ij}^2\approx 0.15$, a figure representing a sum of the squares of all a_{ij} terms whereas in each term of a given equation only one component is generally operative. Thus, although the parameters of this analysis are not as small as one would wish (if they were, turbulence would have been deemed simpler long ago), we assume that meaningful approximations based on their smallness can nevertheless be found. Furthermore, the procedure lends consistency to model simplification which historically has already been imposed by model builders without benefit of guidelines—save intuition—and generally without regard to the *relative* ordering of all terms.

We now consider the first row of ordering terms and temporarily let the flow be neutral ($g=0$). We first assume that the first and third terms on the right side of (12), i.e., production and dissipation, are dominant[6] so that $aq^2U_x=q^3/\Lambda$ and that the first and third terms on the right side of (13) are also dominant so that $q^2U_x=aq^3/l$. Therefore,

$$a^2=l/\Lambda,\quad U_x=a^{-1}q/\Lambda.\tag{16a,b}$$

From (14) we have $q\varphi b\Theta_x=q\varphi^2/\Lambda$ and assuming the first and last terms on the right of (15) are dominant we obtain $q^2\Theta_x=q^2\varphi b/l$. Therefore,

$$b^2=l/\Lambda,\quad \Theta_x=b^{-1}\varphi/\Lambda.\tag{17a,b}$$

Obviously, $a=b$.

[6] For near-neutral and stable flow, shear production and dissipation are of equal order. However, if we consider the free convection limit ($\partial U_i/\partial x_j=0$) and repeat the analysis, no change in the final equations for Levels 4, 3, 2 is affected; a small correction to the final equations for Level 1 is required.

If we now allow buoyant energy production [the second term on the right side of (12)] to be equal in order to the other terms we finally obtain

$$g\beta\varphi=b^{-1}q^2/\Lambda.\tag{18}$$

Eqs. (16a,b), (17a,b) and (18) may now be used to established the second row of ordering terms in Table 1 which differ only by factors of a or b or, equivalently, factors of $(l/\Lambda)^{\frac{1}{2}}$.

b. The Level 3 Model

We have postponed consideration of the advection and diffusion terms. The advection terms are problem-dependent. The diffusion terms are also uncertain although from boundary layer experience[7] we believe that $\lambda/\Lambda=O(a^2)$ and $\partial(\ )/\partial z=O(\Lambda^{-1})$ so that one would guess that diffusion is of order a^2q^3/Λ. Our experience is that the advection terms are also generally small. Therefore, in Table 1 we have assumed the terms to be equal and $O(Uq^2/L)$ where L is temporarily undefined. In Levels 3 and 2 we neglect $O(a^2)$ terms. However, to cover a range of possibilities we distinguish between the two levels according to two assumptions for advection and diffusion.

Here, for Level 3, we assume that

$$\frac{Uq^2}{L}=a\frac{q^3}{\Lambda}.\tag{19}$$

[7] Accumulated from experimental laboratory boundary layer measurements, and boundary layer models. See Mellor and Herring (1968, 1973).

Then after multiplying every ordering term in (13) and (15) by a and b, respectively, and after neglecting all $O(a^2)$ and $O(b^2)$ terms we obtain

$$\frac{Dq^2}{Dt}-\frac{\partial}{\partial x_k}\left[\frac{5}{3}q\lambda_1\frac{\partial q^2}{\partial x_k}\right]=-2\overline{u_k u_i}\frac{\partial U_i}{\partial x_k}-2\beta g_k\overline{u_k\theta}-2\frac{q^3}{\Lambda_1} \tag{20}$$

$$\overline{u_iu_j}=\frac{\delta_{ij}}{3}q^2-\frac{3l_1}{q}\left[(\overline{u_ku_i}-C_1q^2\delta_{ki})\frac{\partial U_j}{\partial x_k}+(\overline{u_ku_j}-C_1q^2\delta_{kj})\frac{\partial U_i}{\partial x_k}-\tfrac{2}{3}\delta_{ij}\overline{u_ku_l}\frac{\partial U_l}{\partial x_k}\right]$$
$$-3\frac{l_1}{q}\beta(g_j\overline{u_i\theta}+g_i\overline{u_j\theta}-\tfrac{2}{3}\delta_{ij}g_l\overline{u_l\theta})+3\frac{l_1}{q}\frac{\partial}{\partial x_k}\left[\frac{q\lambda_1}{3}\left(\delta_{ik}\frac{\partial q^2}{\partial x_j}+\delta_{jk}\frac{\partial q^2}{\partial x_i}-\tfrac{2}{3}\delta_{ij}\frac{\partial q^2}{\partial x_k}\right)\right] \tag{21}$$

$$\frac{D\overline{\theta^2}}{Dt}-\frac{\partial}{\partial x_k}\left[q\lambda_2\frac{\partial\overline{\theta^2}}{\partial x_k}\right]=-2\overline{u_k\theta}\frac{\partial\Theta}{\partial x_k}-2\frac{q}{\Lambda_2}\overline{\theta^2} \tag{22}$$

$$\overline{u_j\theta}=-3\frac{l_2}{q}\left[\overline{u_ju_k}\frac{\partial\Theta}{\partial x_k}+\overline{\theta u_k}\frac{\partial U_j}{\partial x_k}+\beta g_j\overline{\theta^2}\right]. \tag{23}$$

Thus, Eqs. (4), (5) and (6), representing ten differential equations for all of the components, are reduced to two differential equations and eight algebraic equations. Strangely enough, diffusion terms are retained in (21), while not in (23); the diffusion terms in (21) are known once (20) is solved. It should be noted that (21) represents only five independent equations since $\overline{u_ju_j}=q^2$ to which (21) reduces upon contraction.

c. *The Level 2 Model*

The assumption here is that advection and diffusive terms are higher order; that is, instead of (19), we assume that

$$\frac{Uq^2}{L}=\frac{a^2q^3}{\Lambda}. \tag{24}$$

Now the net effect of neglecting terms of $O(a^2)$ is to neglect the left side of (20) and (22) and the last term on the right of (21).

In order to easily identify this model we will repeat the equations with the aforementioned simplifications. Thus we have

$$\frac{q^3}{\Lambda}=-\overline{u_ku_i}\frac{\partial U_i}{\partial x_k}-\beta g_k\overline{u_k\theta}, \tag{25}$$

$$\overline{u_iu_j}=\frac{\delta_{ij}}{3}q^2-3\frac{l_1}{q}\left[(\overline{u_ku_i}-C_1q^2\delta_{ki})\frac{\partial U_j}{\partial x_k}+(\overline{u_ku_j}-C_1q^2\delta_{kj})\frac{\partial U_i}{\partial x_k}-\tfrac{2}{3}\delta_{ij}\overline{u_ku_l}\frac{\partial U_l}{\partial x_k}\right]$$
$$-3\frac{l_1}{q}\beta[g_j\overline{u_i\theta}+g_i\overline{u_j\theta}-\tfrac{2}{3}\delta_{ij}g_l\overline{u_l\theta}], \tag{26}$$

$$\frac{q\overline{\theta_2}}{\Lambda_2}=-\overline{u_k\theta}\frac{\partial\Theta}{\partial x_k}, \tag{27}$$

$$\overline{u_j\theta}=-3\frac{l_2}{q}\left[\overline{u_ju_k}\frac{\partial\Theta}{\partial x_k}+\overline{\theta u_k}\frac{\partial U_j}{\partial x_k}+\beta g_j\overline{\theta^2}\right]. \tag{28}$$

When applied to boundary layers further simplifications are possible in solving the now wholly algebraic equations (25), (26), (27) and (28).

d. *The Level 1 Model*

Discussion of this level is included mainly for historical reasons. Referring back to (12), (13), (14) and (15) we now neglect all terms of $O(a)$, yielding

$$\frac{q^3}{\Lambda}=-\overline{u_ku_i}\frac{\partial U_i}{\partial x_k}-\beta g_k\overline{u_k\theta}, \tag{29}$$

$$\overline{u_iu_j}=\frac{\delta_{ij}}{3}q^2-ql_1\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right), \tag{30}$$

$$\overline{\theta^2}=-\frac{\Lambda_2}{q}\overline{u_k\theta}\frac{\partial\Theta}{\partial x_k}, \tag{31}$$

$$\overline{u_j\theta}=-ql_2\frac{\partial\Theta}{\partial x_j}-\frac{3\beta l_2}{q}g_i\overline{\theta^2}. \tag{32}$$

3. The boundary layer approximation

a. *Level 4*

The boundary layer or hydrostatic approximations to (1), (2) and (3) are

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}=0, \tag{33}$$

$$\frac{D}{Dt}\begin{pmatrix}U\\V\\0\end{pmatrix}+\frac{\partial}{\partial z}\begin{pmatrix}\overline{uw}\\\overline{vw}\\0\end{pmatrix}=-\begin{pmatrix}\partial P/\partial x\\\partial P/\partial y\\\partial P/\partial z\end{pmatrix}+\begin{pmatrix}fV\\-fU\\g\beta\Theta\end{pmatrix}, \tag{34a,b,c}$$

$$\frac{D\Theta}{Dt}-\frac{\partial\overline{w\theta}}{\partial z}=0. \tag{35}$$

If we define the diffusion operators

$$\mathfrak{D}_\alpha(\ )\equiv\frac{\partial}{\partial z}\left[q\lambda_\alpha\frac{\partial}{\partial z}(\ )\right], \tag{36a}$$

and the production components

$$P_{ij}\equiv-\overline{wu_i}\frac{\partial U_j}{\partial z}, \tag{36b}$$

the diagonal components of (4) may be written

$$\frac{D}{Dt}\begin{bmatrix}\overline{u^2}\\ \overline{v^2}\\ \overline{w^2}\end{bmatrix}-\mathfrak{D}_1\begin{bmatrix}\overline{u^2}\\ \overline{v^2}\\ 3\overline{w^2}\end{bmatrix}=2\begin{bmatrix}P_{xx}\\ P_{yy}\\ \beta g\overline{w\theta}\end{bmatrix}-\frac{q}{3l_1}\begin{bmatrix}\overline{u^2}-q^2/3\\ \overline{v^2}-q^2/3\\ \overline{w^2}-q^2/3\end{bmatrix}-\frac{2}{3}\frac{q^2}{\Lambda_1}\begin{bmatrix}1\\1\\1\end{bmatrix}. \tag{37a,b,c}$$

Note that here it is easy to see that, if Λ_1 is fixed and $l_1\rightarrow 0$, the flow becomes isotropic. The off-diagonal components of (4) are

$$\frac{D}{Dt}\begin{bmatrix}\overline{uv}\\ \overline{wu}\\ \overline{wv}\end{bmatrix}-\mathfrak{D}_1\begin{bmatrix}\overline{uv}\\ 2\overline{wu}\\ 2\overline{wv}\end{bmatrix}=-\begin{bmatrix}P_{xy}+P_{yx}\\ (\overline{w^2}-C_1q^2)\partial U/\partial z-\beta g\overline{u\theta}\\ (\overline{w^2}-C_1q^2)\partial V/\partial z-\beta g\overline{v\theta}\end{bmatrix}-\frac{q}{3l_1}\begin{bmatrix}\overline{uv}\\ \overline{wu}\\ \overline{wv}\end{bmatrix}. \tag{38a,b,c}$$

Eq. (5) may be written

$$\frac{D\overline{\theta^2}}{Dt}-\mathfrak{D}_3\overline{\theta^2}=-2\overline{w\theta}\frac{\partial\Theta}{\partial z}-2\frac{q\overline{\theta^2}}{\Lambda_2}, \tag{39}$$

whereas (6) becomes

$$\frac{D}{Dt}\begin{bmatrix}\overline{u\theta}\\ \overline{v\theta}\\ \overline{w\theta}\end{bmatrix}-\mathfrak{D}_2\begin{bmatrix}\overline{u\theta}\\ \overline{v\theta}\\ \overline{w\theta}\end{bmatrix}=-\begin{bmatrix}\overline{uw}\\ \overline{vw}\\ \overline{w^2}\end{bmatrix}\frac{\partial\Theta}{\partial z}-\begin{bmatrix}\overline{w\theta}\partial U/\partial z\\ \overline{w\theta}\partial V/\partial z\\ -\beta g\overline{\theta^2}\end{bmatrix}-\frac{q}{3l_2}\begin{bmatrix}\overline{u\theta}\\ \overline{v\theta}\\ \overline{w\theta}\end{bmatrix}. \tag{40a,b,c}$$

Boundary conditions

In this paper we restrict attention to horizontally homogeneous flow so that $D(\ )/Dt=\partial(\ )/\partial t$ and the pressure gradients

$$\begin{bmatrix}\partial P/\partial x\\ \partial P/\partial y\end{bmatrix}=f\begin{bmatrix}V_g\\ -U_g\end{bmatrix}-\frac{\partial}{\partial t}\begin{bmatrix}U_g\\ V_g\end{bmatrix}. \tag{41a,b}$$

Therefore, the outer boundary conditions are $(U,V)\sim(U_g,V_g)$ as $z\rightarrow\infty$ and coincides with the condition $(\overline{wu},\overline{wv})\sim 0$. It will furthermore be stipulated that all other turbulent moments vanish as $z\rightarrow\infty$.

In setting surface boundary conditions we wish to be certain that they are compatible with the differential equations.

We now assert that any viable model for the length scale l must have $l\sim kz$ as $z\rightarrow 0$, where k is a constant (and where we have, in fact, contrived our definitions of A_1, B_1, etc., so that k will emerge as von Kármán's constant). We view this as governing the inner asymptotic behavior of our model which itself is an outer functional representation of the complete layer.[8]

In the absence of a full asymptotic analysis (a fully rough inner layer is hard to represent) we augment the assumption, $l\sim kz$ as $z\rightarrow 0$ by assuming temporarily that all turbulent moment functions are regular as $z\rightarrow 0$ so that, after multiplying all terms by z, the left sides of (37a,b,c), (38a,b,c), (39) and (40a,b,c) vanish as $z\rightarrow 0$. If we *define* u_τ^2 and α such that

$$\begin{bmatrix}-\overline{wu}\\ -\overline{wv}\end{bmatrix}_{z=0}\equiv u_\tau^2\begin{bmatrix}\cos\alpha\\ \sin\alpha\end{bmatrix}, \tag{42a,b}$$

$$(-\overline{w\theta})_{z=0}\equiv H, \tag{43c}$$

these statements seem sufficient to establish that

$$\begin{bmatrix}U\\ V\end{bmatrix}\sim\frac{u_\tau}{k}\ln\left(\frac{z}{z_0}\right)\begin{bmatrix}\cos\alpha\\ \sin\alpha\end{bmatrix}\quad\text{as}\quad z\rightarrow 0, \tag{44a,b}$$

$$\Theta-\Theta(0)\sim\frac{H}{ku_\tau}P_{rt}\ln\frac{z}{z_{0t}}\quad\text{as}\quad z\rightarrow 0, \tag{45}$$

where k is the von Kármán constant; $P_{rt}=A_1(\gamma_1-c_1)\div(A_2\gamma_1)$ is the turbulent Prandtl number where $\gamma_1=\frac{1}{3}-2A_1/B_1$; and z_0, z_{0t} are empirically determined surface roughness parameters. Furthermore, we obtain

$$q^3(0)=B_1u_\tau^3, \tag{46}$$

$$\begin{bmatrix}\overline{u^2}\\ \overline{v^2}\\ \overline{w^2}\\ \overline{uv}\end{bmatrix}_{z=0}=\frac{q^2(0)}{3}\begin{bmatrix}1\\1\\1\\0\end{bmatrix}+q^2(0)\frac{A_1}{B_1}\begin{bmatrix}4\cos^2\alpha-2\sin^2\alpha\\ 4\sin^2\alpha-2\cos^2\alpha\\ -2\\ -6\sin\alpha\cos\alpha\end{bmatrix}, \tag{47a,b,c,d}$$

[8] Asymptotically in the limit as $z_0/\delta\rightarrow 0$ where z_0 is the characteristic length scale of the inner (surface) flow [z_0 will later be specifically designated as the roughness height. For a smooth surface $z_0=\nu/u_\tau$; see Mellor (1972)] and δ is the outer boundary length scale.

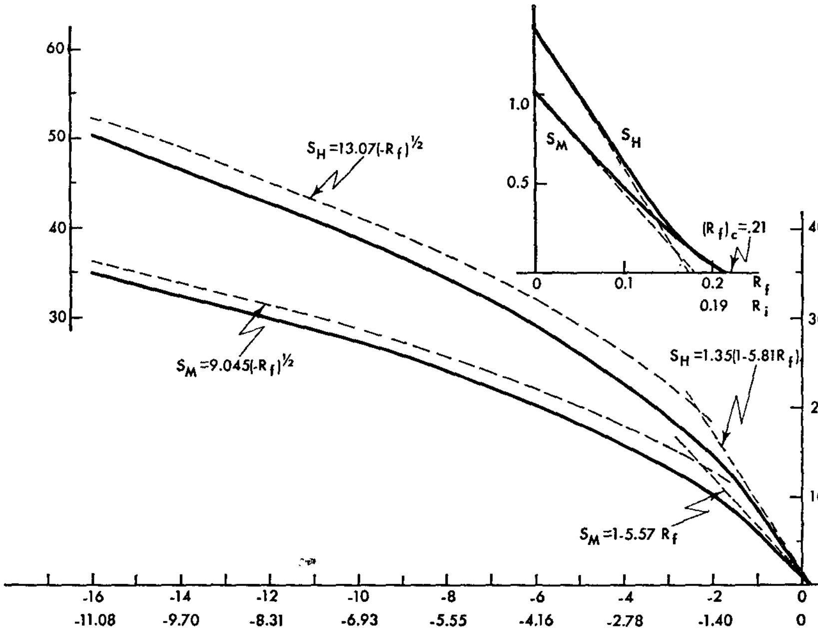

FIG. 1. The stability functions S_M and S_H as functions of R_f or Ri as defined (67a, b). The inset is a detail near $\mathrm{R}_f=\mathrm{Ri}=0$.

$$-\begin{bmatrix}\overline{u\theta}\\ \overline{v\theta}\end{bmatrix}_{z=0}=-H\frac{3A_2}{B_1^{1/3}}(\mathrm{P}_{rt}+1)\begin{bmatrix}\cos\alpha\\ \sin\alpha\end{bmatrix}, \quad (48a,b)$$

$$\overline{\theta^2}(0)=\frac{H^2}{u_\tau^2}\frac{B_2}{B_1^{1/3}}\mathrm{P}_{rt}. \quad (49)$$

Another way of stating what has happened here is that the differential equation for the turbulent moments is a second-order equation in z with a regular singular point at $z=0$. One solution is regular; the other is not. We have rejected the singular solution in prescribing (46)–(49). Having understood the situation on this simple basis we now note, using (44a,b) in (34a,b) for example, that all of the moments are not regular but behave like $a_1+a_2z\ \ln z+a_3z\ldots$ as $z\rightarrow 0$. Nevertheless, further analysis shows that the above results prevail.

In the sample computations discussed in Section 5 the frictional velocity u_τ and α were obtained from (44) utilizing U, and V at $z=z_2(=2.5\text{ m})$. Then the heat flux at the surface H was computed from (45) utilizing Θ at $z=z_2$ and u_τ obtained above. The boundary values for the turbulence moments are obtained from (46), (47), (48) and (49). Boundary values for U, V, Θ are then obtained at $z=z_1(=0.32\text{ m})$ from (44) and (45).

b. Level 3

Eqs. (33), (34a,b,c) and (35) apply at this level. However, the boundary layer approximations to (20)–(22) are

$$\frac{D(q^2/2)}{Dt}-\frac{5}{3}\mathfrak{D}_1\frac{q^2}{2}=P_{xx}+P_{yy}+\beta g\overline{w\theta}-\frac{q^3}{\Lambda_1}, \quad (50)$$

$$\frac{D(\overline{\theta^2}/2)}{Dt}-\mathfrak{D}_3\frac{\overline{\theta^2}}{2}=-\overline{w\theta}\frac{\partial\Theta}{\partial z}-\frac{q\overline{\theta^2}}{\Lambda_2}. \quad (51)$$

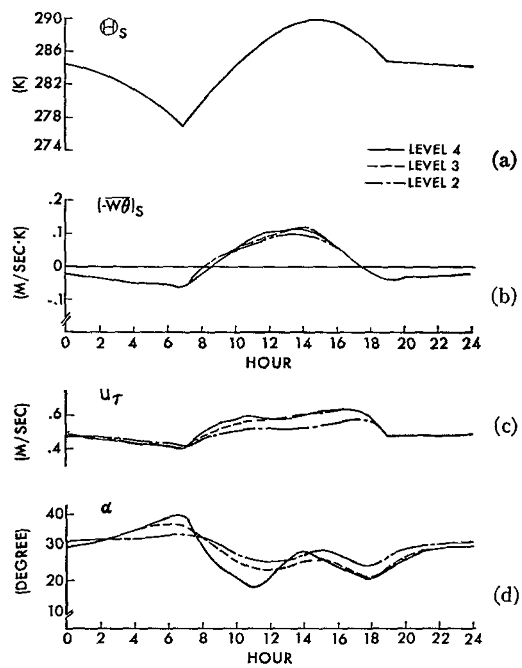

FIG. 2. Surface variables: (a) surface potential temperature, (b) heat flux, (c) friction velocity, and (d) angle included between geostrophic velocity vector and surface stress vector.

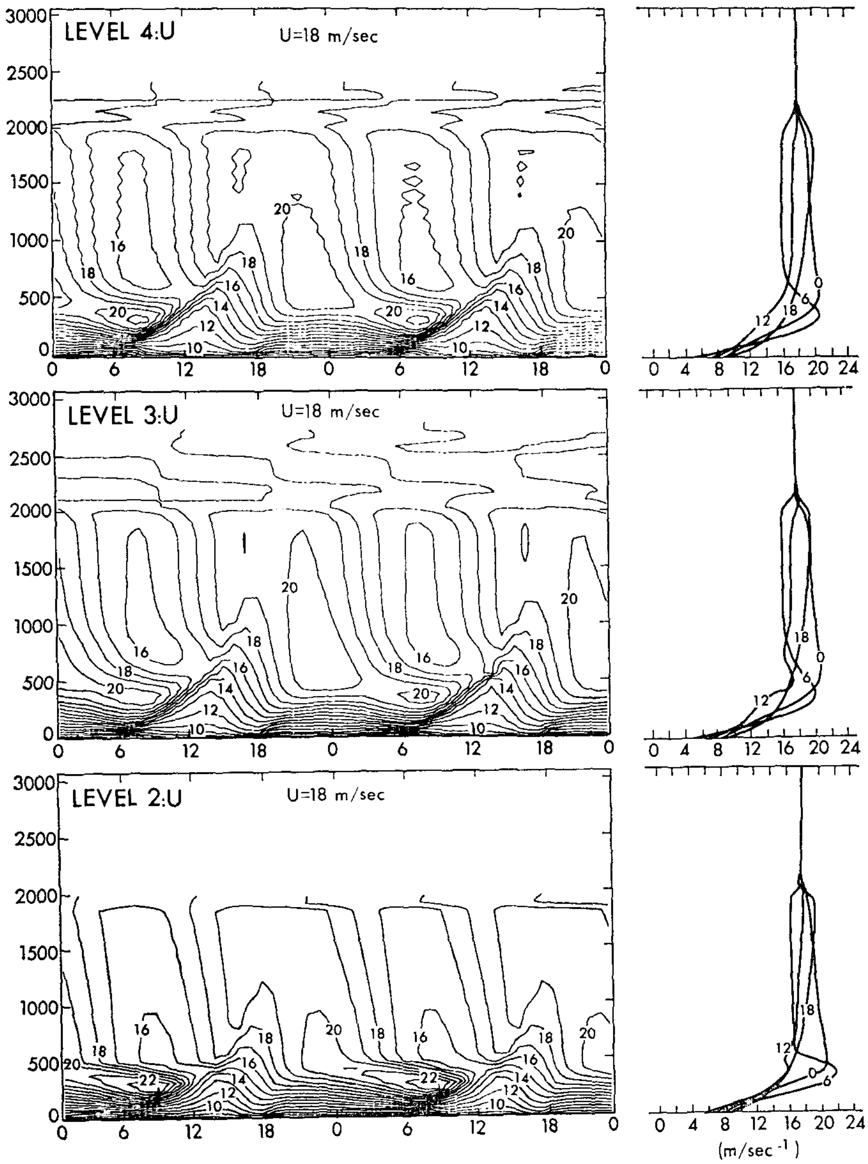

FIG. 3. Mean wind component parallel to the geostrophic velocity vector.

The diagonal terms of (21) may be written

$$\begin{pmatrix}\overline{u^2}\\ \overline{v^2}\\ \overline{w^2}\end{pmatrix} = \frac{q^2}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} + \frac{l_1}{q}\begin{pmatrix}4P_{xx}-2P_{yy}-2\beta g\overline{w\theta}\\ -2P_{xx}+4P_{yy}-2\beta g\overline{w\theta}\\ -2P_{xx}-2P_{yy}+4\beta g\overline{w\theta}\end{pmatrix} - \frac{l_1}{q}\mathfrak{D}_1 q^2\begin{pmatrix}\frac{2}{3}\\ \frac{2}{3}\\ -\frac{4}{3}\end{pmatrix}, \quad \text{(52a,b,c)}$$

where it is noteworthy that diffusion terms survive, but may be determined diagnostically once (50) is solved.

The off-diagonal terms are

$$-\begin{pmatrix}\overline{uv}\\ \overline{wu}\\ \overline{wv}\end{pmatrix} = 3\frac{l_1}{q}\begin{pmatrix}P_{yx}+P_{xy}\\ (\overline{w^2}-C_1q^2)\partial U/\partial z-\beta g\overline{u\theta}\\ (\overline{w^2}-C_1q^2)\partial V/\partial z-\beta g\overline{v\theta}\end{pmatrix}. \quad \text{(53a,b,c)}$$

Finally (23) may be written

$$-\begin{pmatrix}\overline{u\theta}\\ \overline{v\theta}\\ \overline{w\theta}\end{pmatrix} = 3\frac{l_2}{q}\begin{pmatrix}\overline{uw}\partial\Theta/\partial z+\overline{w\theta}\partial U/\partial z\\ \overline{vw}\partial\Theta/\partial z+\overline{w\theta}\partial V/\partial z\\ \overline{w^2}\partial\Theta/\partial z-\beta g\overline{\theta^2}\end{pmatrix}. \quad \text{(54a,b,c)}$$

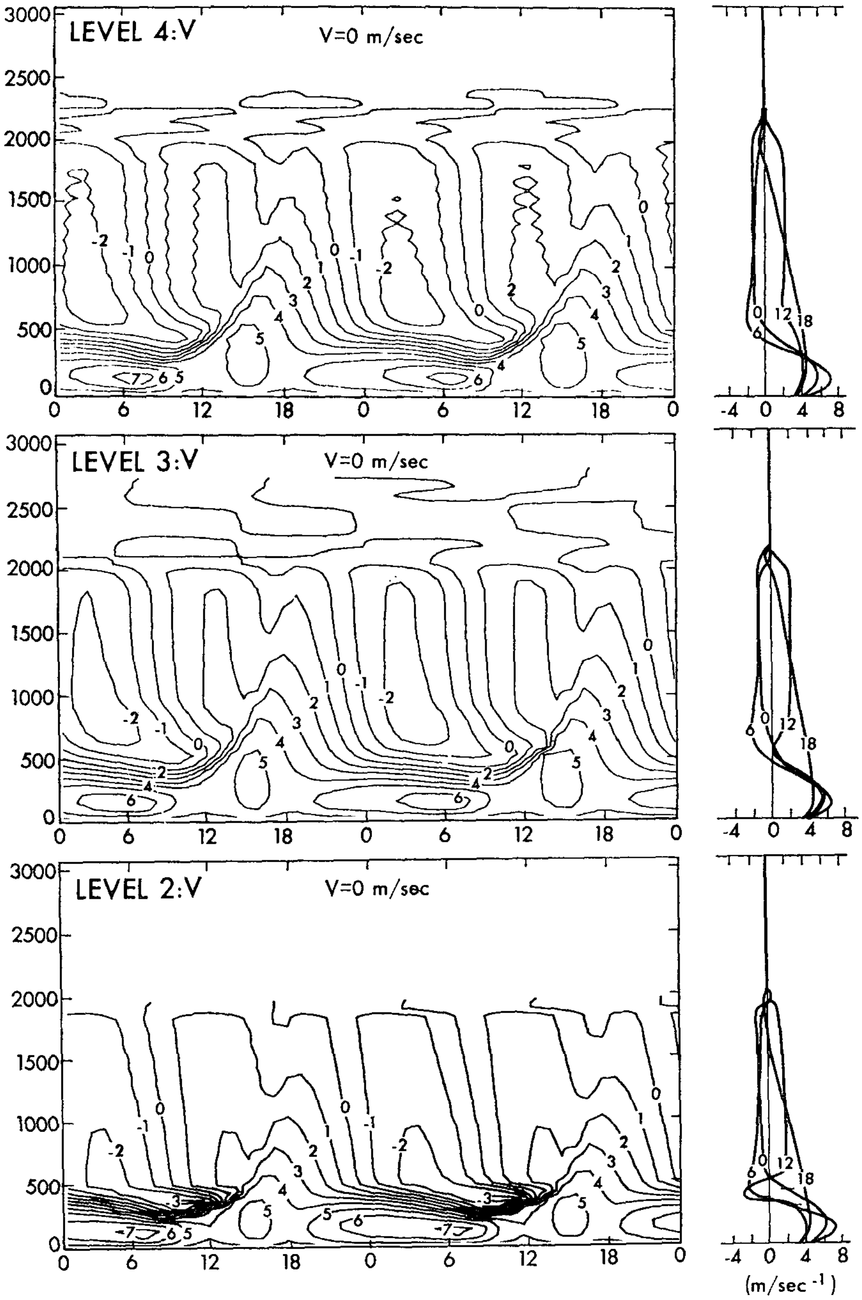

FIG. 4. Mean wind component normal to the geostrophic velocity vector.

All but $\overline{uw}$, $\overline{vw}$, $\overline{w\theta}$, $\overline{q^2}$ and $\overline{\theta^2}$ may be eliminated (but obtained later diagnostically); the following complicated but prognostically useful expressions are obtained:

$$-(\overline{wu},\overline{wv}) = A_1 l[(1-3C_1)q^5 + 3q^2\mathfrak{D}_f - 9\beta g A_2 l^2\{4A_1C_1q^3 + A_2(q^3+3\mathfrak{D}_f)\}\partial\Theta/\partial z + 9(\beta g)^2 q A_2 l^2(4A_1+3A_2)\overline{\theta^2}] \div [q^4 + 6A_1^2l^2q^2|\partial \mathbf{V}/\partial z|^2 + 3A_1A_2l^2\{7q^2 - 18A_1A_2l^2|\partial\mathbf{V}/\partial z|^2 + 36A_1A_2l^2(\beta g)\partial\Theta/\partial z\}(\beta g)\partial\Theta/\partial z]\left(\frac{\partial U}{\partial z},\frac{\partial V}{\partial z}\right) \quad (55a,b)$$

$$-\overline{w\theta} = A_2 l[\{(q^3+3\mathfrak{D}_f) - 6A_1 l(P_{xx}+P_{yy})\}\times(\partial\Theta/\partial z) - 3\beta g q\overline{\theta^2}] \div (q^2 + 12A_1A_2l^2\beta g\,\partial\Theta/\partial z) \quad (56)$$

where

$$\mathfrak{D}_f \equiv \tfrac{4}{3}A_1 l \frac{\partial}{\partial z}\left[\lambda_1 q \frac{\partial q^2}{\partial z}\right], \quad (57)$$

$$|\partial\mathbf{V}/\partial z|^2 \equiv (\partial U/\partial z)^2 + (\partial V/\partial z)^2. \quad (58)$$

Surface boundary conditions given by (44a,b), (45), (46) and (49) are now all that are required.

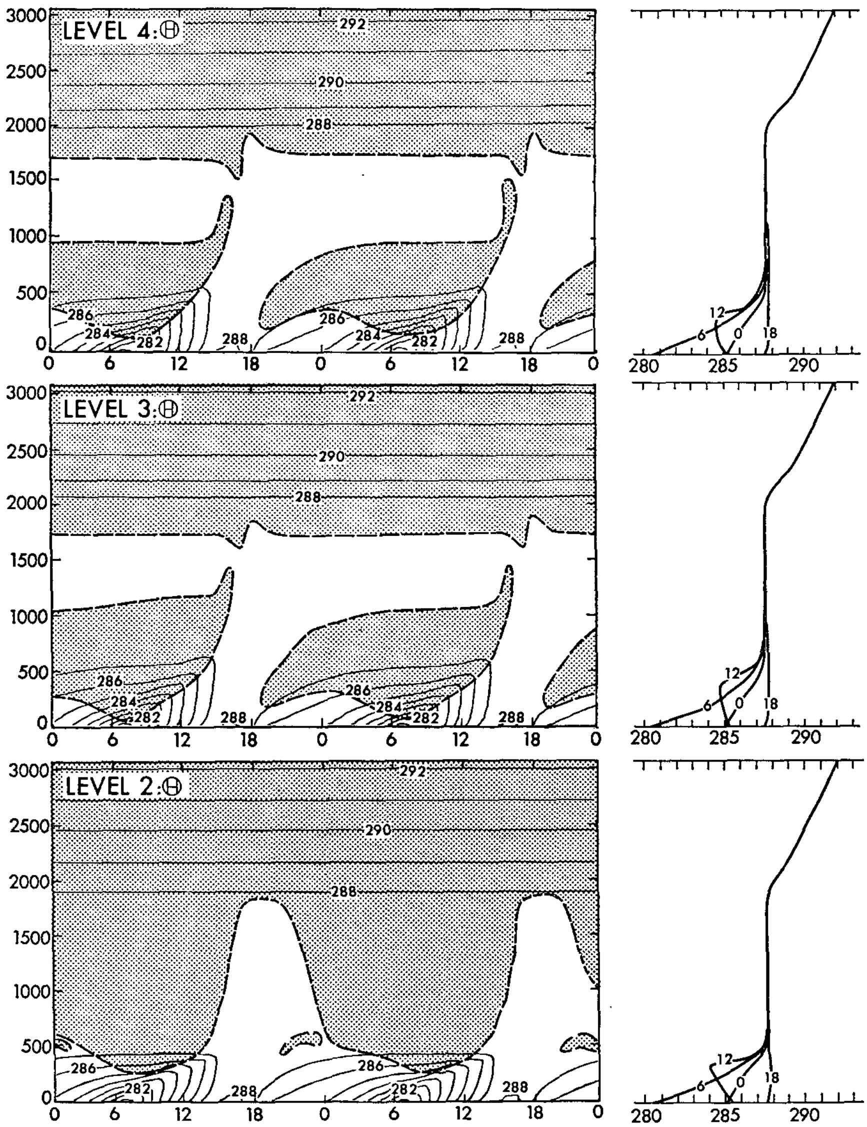

FIG. 5. Potential temperature: the shaded region corresponds to Ri⩾0.21. For Level 2, this is identically a region of zero turbulence intensity.

c. Level 2

Level 2 equations correspond to (52a,b,c) with the diffusion terms neglected and (53a,b,c), (54a,b,c); instead of (50), (51), however, we have more simply

$$\frac{q^3}{\Lambda_1} = -\overline{uw}\frac{\partial U}{\partial z} - \overline{uw}\frac{\partial V}{\partial z} + \beta g\overline{w\theta}, \tag{59}$$

$$\frac{q\overline{\theta^2}}{\Lambda_2} = -\overline{w\theta}\frac{\partial \Theta}{\partial z}. \tag{60}$$

It is possible to solve (52), (53), (54), (59) and (60) and present results in a number of forms. We define the flux Richardson number

$$R_f \equiv -\beta g\overline{w\theta}/(P_{xx}+P_{yy}); \tag{61a}$$

it is also convenient to define

$$\Gamma \equiv R_f/(1-R_f), \tag{61b}$$

which is the ratio of negative buoyant production to total energy production. Then we obtain

$$-(\overline{uw},\overline{vw}) = lq\tilde{S}_M\left(\frac{\partial U}{\partial z},\frac{\partial V}{\partial z}\right), \tag{62a,b}$$

$$-\overline{w\theta} = lq\tilde{S}_H\frac{\partial \Theta}{\partial z}, \tag{63}$$

where

$$\tilde{S}_M = 3A_1\frac{\gamma_1 - C_1 - (6A_1+3A_2)\Gamma/B_1}{\gamma_1-\gamma_2\Gamma+3A_1\Gamma/B_1}(\gamma_1-\gamma_2\Gamma), \tag{64a}$$

$$\tilde{S}_H = 3A_2(\gamma_1-\gamma_2\Gamma), \tag{64b}$$

$$\gamma_1 \equiv \tfrac{1}{3} - (2A_1/B_1),\ \gamma_2 \equiv (B_2/B_1)+(6A_1/B_1).$$

In this formulation q is obtained from (59) and lq then has the form of an eddy viscosity modified by the stability functions $\tilde{S}_M$ and $\tilde{S}_H$ which are functions of R_f.

The critical flux Richardson number is determined by the condition

$$\gamma_1-\gamma_2\Gamma = 0, \tag{65a}$$

or using (61a,b,c) we obtain

$$R_{fc} = \frac{\gamma_1}{\gamma_1+\gamma_2} = \frac{B_1-6A_1}{B_1+3B_2+12A_1}. \tag{65b}$$

For $(A_1,B_1,B_2) = (0.78,\ 15.0,\ 8.0)$ we have $R_{fc}=0.21$ as the Richardson number above which turbulence and mixing cease to exist.

Alternatively, in a more traditional format, we may write

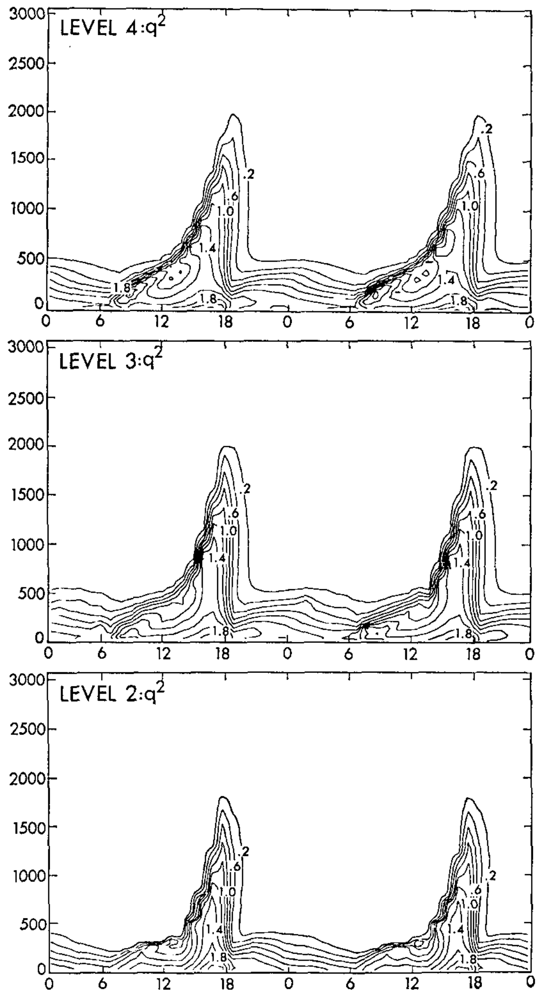

FIG. 6. Turbulent kinetic energy.

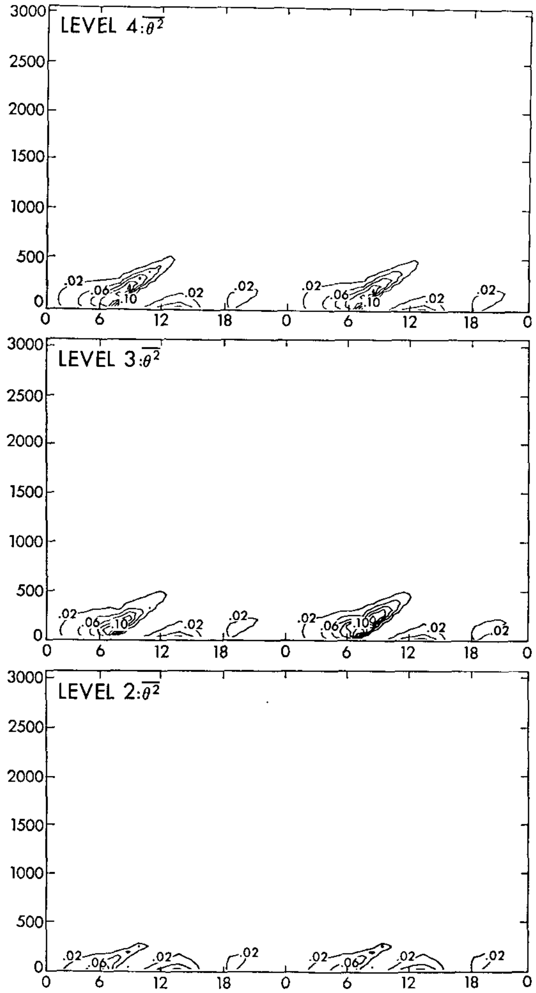

FIG. 7. Temperature variance.

$$-(\overline{wu},\overline{wv}) = l^2\left[\left(\frac{\partial U}{\partial z}\right)^2+\left(\frac{\partial V}{\partial z}\right)^2\right]^{\frac{1}{2}} S_M\left(\frac{\partial U}{\partial z},\frac{\partial V}{\partial z}\right), \tag{66a}$$

$$-\overline{w\theta} = l^2\left[\left(\frac{\partial U}{\partial z}\right)^2+\left(\frac{\partial V}{\partial z}\right)^2\right]^{\frac{1}{2}} S_H\frac{\partial \Theta}{\partial z}, \tag{66b}$$

where now

$$S_M = B_1^{\frac{1}{2}}(1-R_f)^{\frac{1}{2}}\tilde{S}_M^{\frac{3}{2}} \tag{67a}$$

$$S_H = B_1^{\frac{1}{2}}(1-R_f)^{\frac{1}{2}}\tilde{S}_M^{\frac{1}{2}}\tilde{S}_H. \tag{67b}$$

It should be noted that $\tilde{S}_M$, $\tilde{S}_H$ or S_M, S_H may also be determined as functions of the gradient Richardson number, $\mathrm{Ri} \equiv [\beta g\partial\Theta/\partial z]\times[(\partial U/\partial z)^2+(\partial V/\partial z)^2]^{-1}$, since $\mathrm{Ri} = (S_M/S_H)R_f$.[9] Since the turbulent Prandtl number, $S_M/S_H \approx 1.04$ as $R_f \rightarrow 0.21$, we also have $\mathrm{Ri} \rightarrow 0.22$.

[9] From which one can determine that

$$R_f = 0.725[\mathrm{Ri}+0.186-(\mathrm{Ri}^2-0.316\,\mathrm{Ri}+0.0346)^{\frac{1}{2}}],$$

using the previously cited values of A_1, B_1, A_2, B_2, C_1.

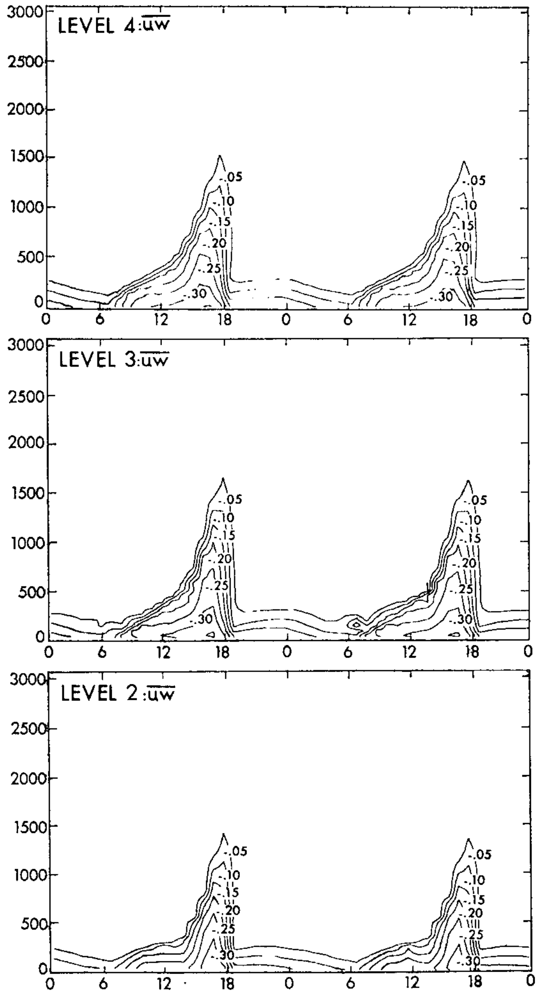

FIG. 8. Reynold stress component parallel to geostrophic velocity vector.

The functions S_M and S_H are plotted in Fig. 1 and are compared with simple linear relations obtained asymptotically from (67a,b) for $R_f \rightarrow 0$ and $R_f \rightarrow -\infty$.

The result, neatly summarized in (67a,b) when specialized to the constant flux layer, has been shown by Mellor (1973) to predict the Kansas surface layer data of Businger *et al.* (1971) with considerable accuracy.[10]

d. Level 1

Effectively, Level 1 does not yield further substantive simplification but is included here for completeness.

If (29), (30), (31) and (32) are reduced to the boundary layer approximation, we obtain (59) and (60) again, whereas (30) and (32) yield

$$(-\overline{wu}, -\overline{wv}) = ql_1\left(\frac{\partial U}{\partial z}, \frac{\partial V}{\partial z}\right), \tag{68a,b}$$

$$-\overline{w\theta} = ql_2\frac{\partial \Theta}{\partial z} - 3l_2\frac{\beta g}{q}\overline{\theta^2}. \tag{69}$$

Eqs. (59), (60), (68a,b) and (69) also reduce to the form (66a,b), except that we now have

$$S_M = A_1(A_1B_1)^{\frac{1}{2}}(1-R_f)^{\frac{1}{2}}, \tag{70a}$$

$$S_H = A_2(A_1B_1)^{\frac{1}{2}}(1-R_f)^{\frac{1}{2}}\left[1-3\frac{B_2}{B_1}\frac{R_f}{1-R_f}\right]. \tag{70b}$$

Using the aforementioned values of A_1, A_2, B_1, B_2 we find the coefficient $A_1(A_1B_1)^{\frac{1}{2}} = A_2(A_1B_1)^{\frac{1}{2}} = 2.66$, whereas $B_2/B_1 = 0.53$. S_M limits to zero at $R_f = 1.0$

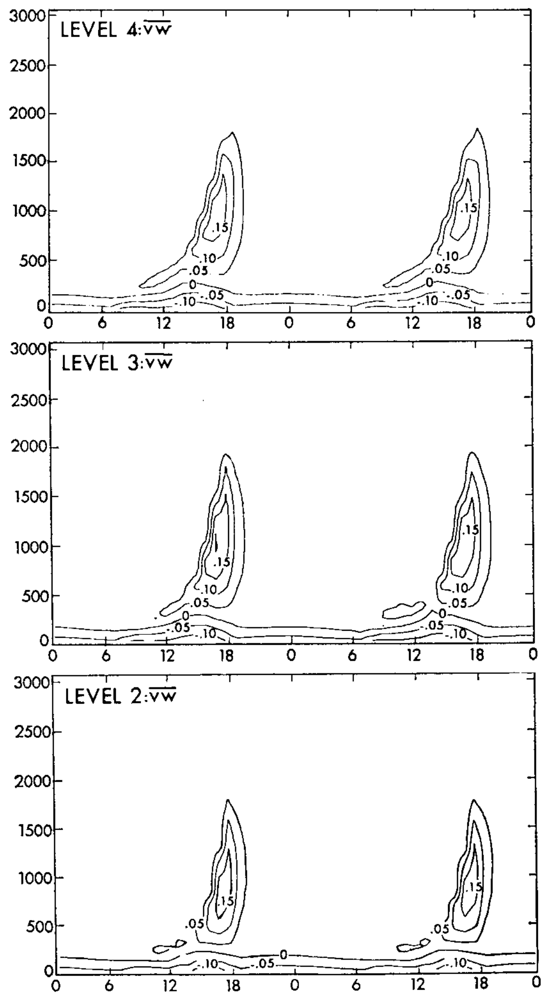

FIG. 9. Reynold stress component normal to geostrophic velocity vector.

[10] In the constant flux layer, we let $l = kz$, $-\overline{wu} = u_\tau^2$, $-\overline{wv} = 0$ and $-\overline{w\theta} = H$. We define $\varphi_M \equiv kzu_\tau^{-1}(\partial U/\partial z)$, $\varphi_H \equiv kzu_\tau H^{-1}(\partial \Theta/\partial z)$ and $\zeta \equiv z/L$ where $L \equiv u_\tau^3(kg\beta H)^{-1}$ is the Monin-Obukhov length scale. It is then easy to show that $\varphi_M = S_M^{-\frac{1}{4}}$, $\varphi_H = S_M^{\frac{1}{4}}S_H^{-1}$ and $\zeta = \varphi_M R_f$. The data were presented in the form $\varphi_M(\zeta)$ and $\varphi_H(\zeta)$.

whereas S_H limits to zero (due to the term in brackets) at $R_f=0.38$. Thus, the result expressed by (70a,b) differs significantly from that given by (67a,b) as plotted in Fig. 1. We expect the latter to be correct as detailed by Mellor (1973).

4. Length scale stipulation

All of our results thus far are independent of a prescription for $l(z)$ except for the fact that $l\sim kz$ as $z\rightarrow 0$ which was used in conjunction with our discussion of boundary conditions. Note also that the critical Richardson number is not dependent on l, but only on the constants of proportionality of the various length scales or, more precisely, on the ratios $\Lambda_1/l_1=B_1/A_1$ and $\Lambda_2/l_1=B_2/A_1$, as seen in (65b).

However, to proceed toward concrete calculations we must now stipulate the entire function $l(z)$.

There have been a number of proposals for equations to provide l, as reviewed in paper B, all of which are on considerably shakier ground (in our opinion) than is

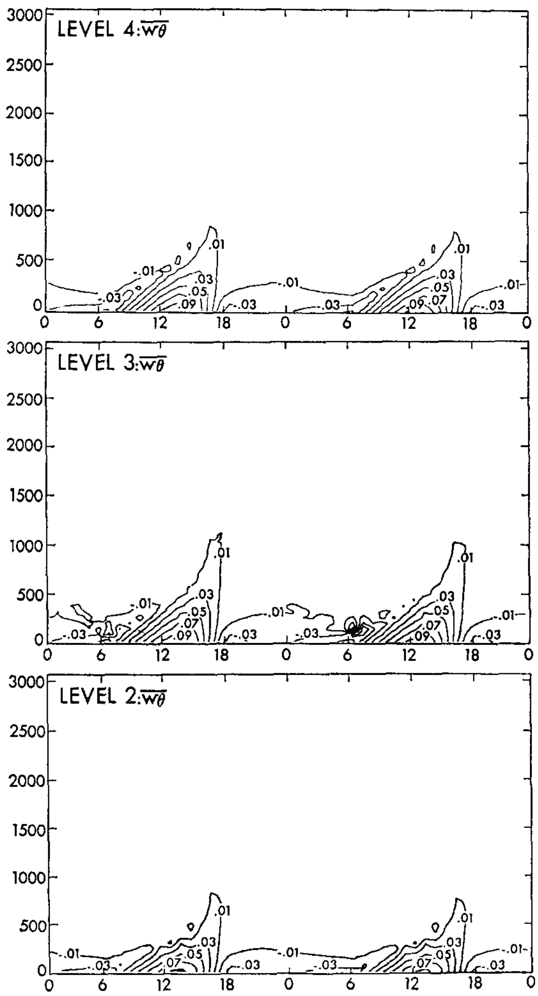

Fig. 10. Heat flux.

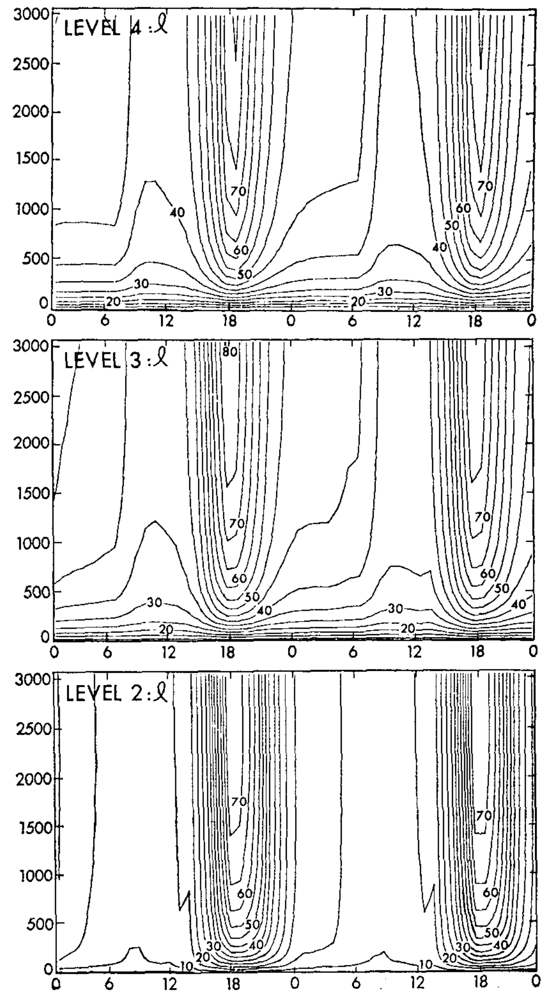

Fig. 11. The length scale l to which all other scales are proportional.

the basic model given by (4), (5) and (6). Hopefully a more persuasive prescription for l will be determined in the future.

In the following calculations we have adopted Blackadar's (1962) interpolation formula

$$l=\frac{kz}{1+kz/l_0}, \tag{71}$$

which interpolates between two limits $l\sim kz$ as $z\rightarrow 0$ and $l\sim l_0$ as $z\rightarrow\infty$. Various propositions for l_0 have appeared in the literature; they range from stipulating a fixed number characteristic of an atmospheric surface layer to stipulating proportionality to U_g/f where U_g is the geostrophic velocity. The latter would not yield proper turbulent Ekman similarity, a fact which could be corrected by using u_τ/f instead. However, in either case, the scaling would be unique to the fact of the problem being a low-Rossby-number, stationary, neutral boundary layer problem.

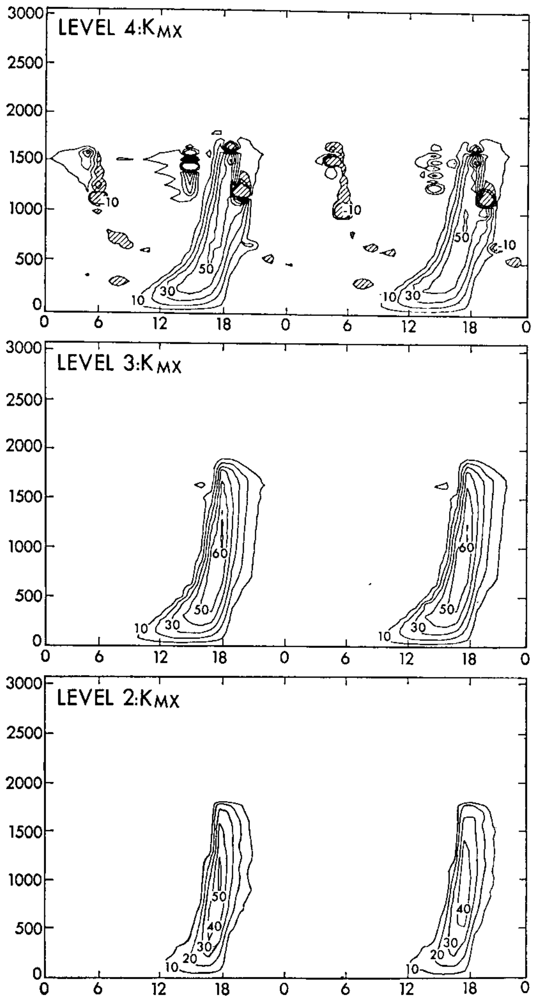

FIG. 12. Momentum exchange coefficients: cross-hatched areas indicate negative values.

At this stage, our thinking is to adopt the simplest—but hopefully general—length scale, characteristic of the extent of the turbulence field. We therefore form the ratio of the first to the zeroth moment of the profile $q(z)$. Thus

$$l_0=\alpha\frac{\int_0^\infty zqdz}{\int_0^\infty qdz}, \qquad (72)$$

where α is an empirical constant (and is the only important constant which we feel is not well known at this stage). We have set $\alpha=0.10$. This value yields a steady, neutral Ekman layer whose height is slightly greater than $0.30\ u_\tau/f$ (see Fig. 14), a value which is uncertain from observations since the conditions necessary for a steady, neutral layer are rare in nature. A goal of future comparisons of prediction and data will be to use one of the model levels to evaluate α which, presuming the distribution (71) is valid [and it is known that calculated profiles are fairly insensitive to variants of (71) so long as its $z\rightarrow 0$ behavior is maintained], governs the velocity profile distribution of a neutral layer.

5. A comparative calculation

We will not show comparisons between data and experiments in this paper; rather, we will compare the calculated results of the various model level using simple boundary conditions. One can then enter into a data comparison program with some feeling of model sensitivity to internal sophistication and complexity.

However, the case we have chosen to investigate is drawn from Clarke *et al.* (1971) but has been simplified. Here the geostrophic wind velocity aligned in the x direction is set at a constant value of 18 m $\sec^{-1}$; the Coriolis parameter is $f=0.88\times10^{-4}$ $\sec^{-1}$, and the roughness parameter $z_0=5$ cm. Initially, the potential temperature is given by $\Theta(z,0)=285\text{K}$ for $0<z\leqslant 1000$ m and $\Theta(z,0)=285\text{K}+(0.0035\text{K m}^{-1})(z-1000\text{ m})$ for $1000\text{ m}<z$. The initial velocity field is computed according to the steady-state equations for the neutral conditions. Thereafter the gound temperature is allowed to vary according to Fig. 2a. Note that the ground temperature is the only unsteady boundary condition.

We allow the calculations[11] to proceed for 10 days with the ground temperature repeating cyclically every 24 hr. The computed field properties require about 3 days before one could say that they are approximately cyclical. For example, during this time the imposed inversion layer has eroded from a base of 1000 to 2000 m (approximately equal to the neutral Ekman layer height) after which no significant change is observed.

In Figs. 2b–d we have plotted the resultant surface heat flux, shear stress magnitude, and the angle formed by the (constant) geostrophic velocity vector and the surface stress vector for the tenth day.

Generally speaking model Levels 4, 3 and 2 (we did not think it worthwhile to make use of Level 1) yield very much the same result with, as inferred by the foregoing ordering analyses, the difference between the Level 4 and 3 results being less than the difference between the Level 3 and 2 results. This fact is also apparent in the detailed comparison of Figs. 3–13.

Figs. 3, 4 and 5 show the computed mean horizontal velocity components and potential temperature; all of these quantities approach the surface values logarithmically. Also shown plotted in Fig. 5 is a shaded region where $\text{Ri}\geqslant 0.21$. For the simpler Level 2 model this is a region where all turbulence moments are iden-

[11] The calculation scheme was basically implicit; 60 vertical grid points were in the region 5200 m $>z>$ 1000 m whereas 20 points were spaced logarithmically in the first 1000 m. Generally the time increment was 1 min, although 10 min was tried successfully for Levels 2 and 3.

tically zero; for the higher level models this is not the case.

In Fig. 3 the velocity component in the direction of the geostrophic velocity indicates minima and maxima. The maxima correspond to the so-called "nocturnal jet." The height of the jet is seen to decrease toward morning. It is about 800 m at 2200 and decreases to about 300 m at 0700. According to Bonner (1968) our result may be classified as a Criterion 1 jet, whose maximum wind must be greater than 12 m sec^{-1} and maximum-minimum difference should exceed 6 m sec^{-1}. All three results in Fig. 3 resemble each other, but the one from Level 2 gives a slightly more intense jet than do the other two. The difference can be explained by considering the diurnal variation of the friction term in the equation of motion as discussed by Blackadar (1957). According to Blackadar's argument, a nocturnal jet is produced by the inertial oscillation of the wind vector decoupled from the surface layer. Decoupling of the layer is the result of the disappearance of turbulent Reynolds stress. Indeed as we will see later the turbulence diminishes above 500 m at 2100 and the boundary of zero stress decreases with time until noon. The sharp cutoff of turbulence in Level 2 permits a completely free inertial oscillation of wind wherever $\mathrm{Ri}>0.21$ and this is very nearly true with the Level 3 and 4 models.

Fig. 5 reveals the large mid-altitude homogeneous regions (constant potential temperature) above which is the stationary inversion above 2000 m. A surface inversion is created between the surface and 600 m by cooling between 1800 and 0800; this inversion is subsequently destroyed from the surface upward by heating between 0800 and 1800. Between 1500 and 1800 the entire layer is essentially neutral allowing turbulent production up to 2000 m as seen in Fig. 6. This may also be seen in Figs. 7–10 where some of the turbulent moments are presented (*not* shown are the individual $\overline{u^2}$, $\overline{v^2}$ and $\overline{w^2}$, the off-diagonal components, $\overline{uv}$, and the heat flux components, $\overline{u\theta}$ and $\overline{v\theta}$).

Fig. 11 shows the behavior of $l(z,t)$. The value l_0 from (72) may be readily discerned as the large z asymptote.

Values of $K_{Mx}\equiv -\overline{uw}/(\partial U/\partial z)$ and $K_H\equiv -\overline{w\theta}/(\partial\Theta/\partial z)$ were computed diagnostically and the results are shown in Figs. 12 and 13. In the case of Levels 2 and 3, $K_{Mx}=K_{My}$ is, *a priori*, a scaler. For Level 4 K_{Mx} and K_{My} were not equal but the difference was insignificant.

Discussion of one feature of the model has been postponed to this point. In Levels 2 and 3, we have insisted that K_M be positive definite; for Level 2, this is simply accomplished by setting $S_M=S_H=0$ when $R_f\approx\mathrm{Ri}>0.21$; for Level 3, whenever a negative value of K_M was calculated, it was reset to zero. In the Level 4 calculations, no special conditions were imposed and diagnostically computed K_M values were negative in local regions as shown in Fig. 12a. Note that negative

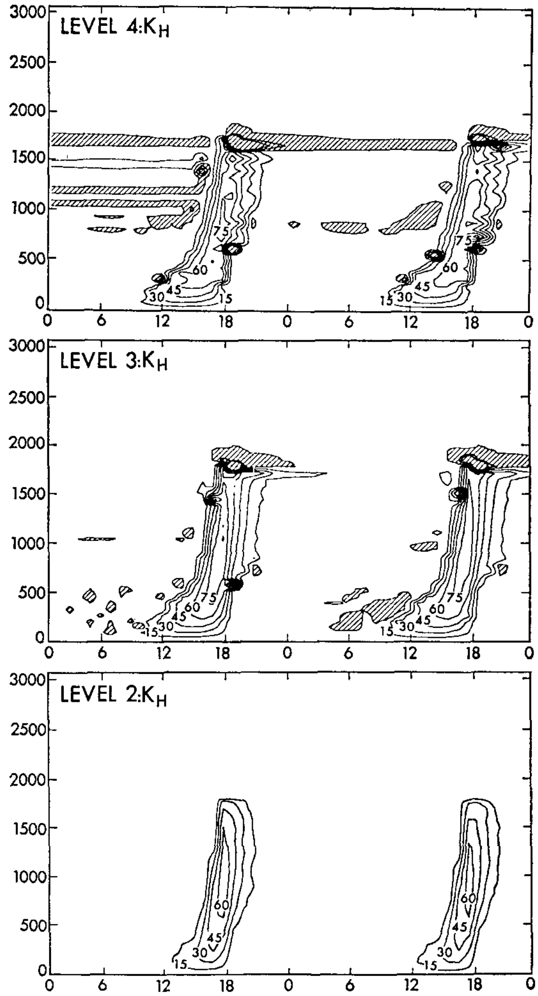

FIG. 13. Heat exchange coefficient: cross-hatched areas indicate negative values.

values of eddy viscosities have been reported by Eskinazi and Erian (1969) in locally inbedded regions where the stress and velocity gradients are small.

In Levels 3 and 4, K_H may, in principal, be negative and is so, as seen in local regions in Fig. 13.

The components, $\overline{u^2}$, $\overline{v^2}$ and $\overline{w^2}$ should, of course, be positive definite. This turns out to be true in practice everywhere in the Level 2 calculation and very nearly so for Levels 3 and 4 where, however, small negative values appeared between 0700–0800 (after a discontinuity in the tendency of wall temperature) at a couple of grid points.

6. Conclusion

A hierarchy of turbulent models has systematically been derived from the same empirical base. Although we have not presented all of the numerical results or diagnostic numbers, an examination of these numbers

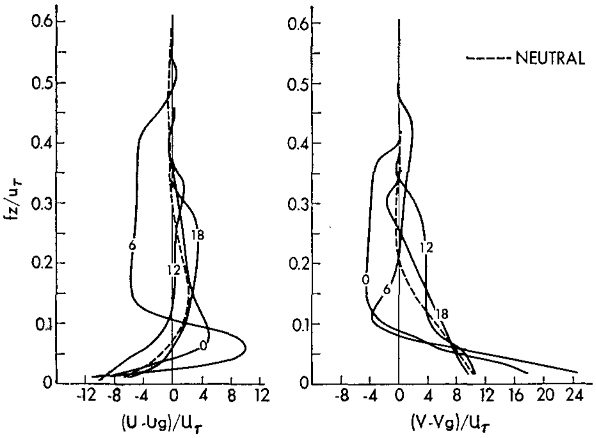

FIG. 14. Mean velocity profiles in similarity form resolved in directions parallel and normal to the surface shear stress. The time-varying profiles and the neutral stationary profile are taken from Level 4 calculations.

indicates that the ordering system used in discriminating among the various model levels is in fact valid.

For the numbers that have been presented and the particular problem we have solved, it may readily be concluded that the very simple Level 2 model is quite adequate although there is some advantage in going to Level 3. There appears to be very little incentive to add the eight additional differential equations required by Level 4.

All models display the remarkable property whereby turbulence is nearly extinguished (and precisely so for Level 2) at a gradient Richardson number of 0.21.

Velocity profiles in similarity form are shown in Fig. 14 along with a stationary neutral profile. This plot, in particular, indicates sources of difficulty in interpreting experimental data; for example, the identification of "pure" neutral stationary Ekman layer data has proven difficult. Furthermore, the determination of the height of the layer and the geostrophic velocity might be deceptive if data up to say 1500 m were all that were available. Comparison of data and calculations will be forthcoming in the future where the calculations themselves should provide guidelines for the proper location of boundary conditions.

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