Analytic Prediction of the Properties of Stratified Planetary Surface Layers[1]

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ABSTRACT

By considering the complex of one-point, turbulent moment equations for velocity, pressure and temperature, it appears possible to predict some properties of diabatic, density-stratified planetary layers using empirical information obtained from laboratory turbulence data in the absence of density stratification. In this paper attention is focused on the near-surface, constant-flux layer. The results, like the empirical input, are simple and, hopefully, will be instructive and useful in the formulation of improved and possibly more complicated models in the future.

1. Introduction

Prediction of the behavior of two-dimensional turbulent boundary layers normally associated with engineering is now well in hand. One approach is to utilize the more-or-less conventional eddy viscosity or mixing length hypothesis which, however, must be extended beyond Prandtl's specific rule, $l \propto z$, to include the outer portion of the boundary layer. The author and others have evolved such schemes which when combined with numerical computation have considerable predictive power; i.e., boundary layer flows with pressure gradients, heat transfer, mass transpiration, large Mach number, etc., may be predicted with no empirical adjustment. The constants that do enter such models—which are called Mean Velocity Field closure models (MVF)—are determined once and for all from flat plate turbulent layers with zero pressure gradient, heat transfer, mass transpiration, Mach number, etc.

MVF models, however, do not have the capability of predicting the effects of stratification in a gravity field or other body-force like effects without extensive empirical modification. However, there does exist hope for so-called Mean Turbulent Field (MTF) closure models which include two subset models, the Mean Turbulent Energy (MTE) model and the Mean Reynolds Stress model (MRS). For neutral layers, MTE models involve the solution of the turbulent kinetic energy equation in addition to the equations for the horizontal velocity components. MRS models call for the solution of equations for all components of the Reynolds stress tensor. For stratified flow, additional equations for the heat flux vector components are included. Clearly, MRS models present a formidable computational task. Nevertheless, research to determine their ability to predict real data seems worthwhile after which analytical simplifications might be found.

Although some of the individual closure assumptions of MTF models are still very much on trial, the basic framework is not new. This author has recently contributed to a review article (Mellor and Herring, 1973) covering in some detail much of the past and recent history of MTF models as applied to neutral boundary layers and this will not be repeated. Here we note only that MTF models go back to Prandtl and Wieghardt (1945) and Kolmogoroff (1942). However, the key hypothesis of Rotta (1951) marked the beginning of a rational, but still empirical, approach to MTF modeling.

Derivations of the so-called KEYPS equation generally involve the turbulent energy equation together with quite a few assumptions. In an attempt to understand the effects of stratification, Monin (1965) has investigated the full Reynolds stress equations and discussed certain modeling assumptions. No calculations or comparisons with data were, however, forthcoming. Recently, Donaldson and Rosenbaum (1968) have proposed some closure hypotheses and performed calculations using a one-dimensional, unsteady MRS model to simulate clear air turbulence. However, there were no quantitative data available for comparison.

In the present paper we also propose MRS closure relations which are derived under the constraining assumption that the constitutive coefficients involved in each relation are isotropic. Note that the Reynolds stress tensor itself is not isotropic, a fact which is important in describing stratified flow. The closure assumptions involve empirical information related directly to well-defined turbulence structure parameters and obtained from laboratory measurements of neutral turbulent flows.

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The equations are presented in considerable generality so that, *in principle*, they can be integrated to simulate, for example, a complete planetary boundary layer. Here, however, we restrict attention to the constant-flux surface, region, thus avoiding—for the time—the considerable computational effort required for the complete layer. However, this is a logical first step since it is possible to compare directly with the constant-flux data of Businger *et al.* (1971) in the form of Monin-Obukhoff similarity variables.

2. The basic equations

The equations of motion for the mean velocity U_j and mean potential temperature Θ are

$$\frac{\partial U_i}{\partial x_i}=0, \tag{1}$$

$$\frac{\partial U_j}{\partial t}+\frac{\partial}{\partial x_k}(U_kU_j+\overline{u_ku_j})+\epsilon_{jkl}f_kU_l = -\frac{\partial P}{\partial x_j}-g_j\beta\Theta+\nu\nabla^2U_j, \tag{2}$$

$$\frac{\partial\Theta}{\partial t}+\frac{\partial}{\partial x_k}(U_k\Theta+\overline{u_k\theta})=\alpha\nabla^2\Theta, \tag{3}$$

where P is the mean kinematic pressure, $g_j=(0,0,-g)$ the gravity vector, $f_j=(0,f_y,f)$ the Coriolis parameter (the vertical component f will not be subscripted), $\beta=-(\partial\rho/\partial T)_p/\rho$ the coefficient of thermal expansion, ν the kinematic viscosity, and α the kinematic heat conductivity (or thermal diffusivity). The overbars represent ensemble averages and the lower case terms, u_k and θ, are the fluctuating components of the velocity and temperature and are governed by

$$\frac{\partial u_i}{\partial x_i}=0, \tag{4}$$

$$\frac{\partial u_j}{\partial t}+\frac{\partial}{\partial x_k}(U_ku_j+U_ju_k+u_ku_j-\overline{u_ku_j})+\epsilon_{jkl}f_ku_l = -\frac{\partial p}{\partial x_j}-g_j\beta\theta+\nu\nabla^2u_j, \tag{5}$$

$$\frac{\partial\theta}{\partial t}+\frac{\partial}{\partial k_k}(\Theta u_k+U_k\theta+u_k\theta-\overline{u_k\theta})=\alpha\nabla^2\theta. \tag{6}$$

Since the mean equations (1), (2) and (3) involve the Reynolds stress and heat conduction moments, $\overline{u_iu_j}$, $\overline{u_i\theta}$, we obtain their governing equations from (5) and (6), using (4) so that

$$\frac{\partial\overline{u_iu_j}}{\partial t}+\frac{\partial}{\partial x_k}\left[U_k\overline{u_iu_j}+\overline{u_ku_iu_j}-\nu\frac{\partial}{\partial x_k}\overline{u_iu_j}\right]+\frac{\partial}{\partial x_j}\overline{pu_i}+\frac{\partial}{\partial x_i}\overline{pu_j}+f_k(\epsilon_{jkl}\overline{u_lu_i}+\epsilon_{ikl}\overline{u_lu_j})$$
$$=-\overline{u_ku_i}\frac{\partial U_j}{\partial x_k}-\overline{u_ku_j}\frac{\partial U_i}{\partial x_k}-\beta(g_j\overline{u_i\theta}+g_i\overline{u_j\theta})+\overline{p\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)}-2\nu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}}, \tag{7}$$

$$\frac{\partial\overline{u_j\theta}}{\partial t}+\frac{\partial}{\partial x_k}\left[U_k\overline{\theta u_j}+\overline{u_ku_j\theta}-\alpha\overline{u_j\frac{\partial\theta}{\partial x_k}}-\nu\overline{\theta\frac{\partial u_j}{\partial x_k}}\right]+\frac{\partial}{\partial x_j}\overline{p\theta}+\epsilon_{jkl}f_k\overline{u_l\theta}$$
$$=-\overline{u_ju_k}\frac{\partial\Theta}{\partial x_k}-\overline{\theta u_k}\frac{\partial U_j}{\partial x_k}-\beta g_j\overline{\theta^2}+\overline{p\frac{\partial\theta}{\partial x_j}}-(\alpha+\nu)\overline{\frac{\partial u_j}{\partial x_k}\frac{\partial\theta}{\partial k_k}}. \tag{8}$$

Eq. (8) involves $\overline{\theta^2}$, an equation for which is obtained from (6) so that

$$\frac{\partial\overline{\theta^2}}{\partial t}+\frac{\partial}{\partial x_k}\left[U_k\overline{\theta^2}+\overline{u_k\theta^2}-\alpha\frac{\partial\overline{\theta^2}}{\partial x_k}\right]=-2\overline{u_k\theta}\frac{\partial\Theta}{\partial x_k}-2\alpha\overline{\frac{\partial\theta}{\partial x_k}\frac{\partial\theta}{\partial x_k}}. \tag{9}$$

3. Modeling assumptions

The primary contribution of Rotta (1951) was to suggest an assumption for the term, $\overline{p(\partial u_i/\partial x_j+\partial u_j/\partial x_i)}$, which he called the "energy redistribution term" since one of its functions is to partition energy among the three energy components while not contributing to the total. [Upon contraction, the term drops out of Eq. (7)]. Based on integral relations obtained from Eq. (5)—in the neutral case where $g_j\beta\theta$ is negligible—Rotta suggested that the term might be reasonably made proportional to $\overline{u_iu_j}$ and $\partial U_i/\partial x_j$. Thus,

$$\overline{p\left(\frac{\partial u_i}{\partial x_i}+\frac{\partial u_j}{\partial x_i}\right)}=C_{ijkm}\overline{u_ku_m}+C'_{ijkm}\frac{\partial U_k}{\partial x_m}.$$

Here we assume that the constitutive coefficients are isotropic tensors; that is,

$$C_{ijkm}=C_1\delta_{ij}\delta_{km}+C_2\delta_{ik}\delta_{jm}+C_3\delta_{jk}\delta_{im}.$$

From continuity we obtain $C_1=(C_2+C_3)/3$; similar reasoning applies to C'_{ijkm}. We therefore obtain

$$\overline{p\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)} = -\frac{q}{3l_1}\left(\overline{u_iu_j}-\frac{\delta_{ij}}{3}q^2\right)+Cq^2\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right), \qquad (10)$$

where $-q/(3l_1)$ and Cq^2 have been substituted for the surviving scalar coefficients and $q\equiv(\overline{u_i^2})^{\frac{1}{2}}$. The length l_1 and the constant C must be determined empirically.

Proceeding in similar fashion, we obtain

$$\overline{p\frac{\partial\theta}{\partial x_j}}=-\frac{q}{3l_2}\overline{u_j\theta}. \qquad (11)$$

There is no term proportional to $\partial U_k/\partial x_m$ since this would involve a constitutive coefficient proportional to ϵ_{jkm} which is ruled out by coordinate reflection.[2]

An assumption for the dissipation in the present framework is

$$2\nu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}}=\frac{2}{3}\frac{q^3}{\Lambda_1}\delta_{ij} \qquad (12)$$

and therefore follows Kolmogoroff's (1941) hypothesis of local, small-scale isotropy.

Since there is no isotropic first-order tensor

$$(\alpha+\nu)\overline{\frac{\partial u_j}{\partial x_k}\frac{\partial\theta}{\partial x_k}}=0. \qquad (13)$$

However,

$$2\alpha\overline{\frac{\partial\theta}{\partial x_k}\frac{\partial\theta}{\partial x_k}}=2\frac{q}{\Lambda_2}\overline{\theta^2}. \qquad (14)$$

The diffusional terms are an enigma which is offset, somewhat, by the fact that in neutral layers at least they are not overly important terms. Following our procedure there are three possible forms for $\overline{u_ku_iu_j}$ (if we let it be proportional to $\partial\overline{u_lu_m}/\partial x_n$ through a six-order isotropic tensor and specify symmetry in k, i, j). Here we choose

$$\overline{u_ku_iu_j}=-q\lambda_1\left(\frac{\partial\overline{u_iu_j}}{\partial x_k}+\frac{\partial\overline{u_iu_k}}{\partial x_j}+\frac{\partial\overline{u_ju_k}}{\partial x_i}\right). \qquad (15)$$

For $\overline{u_ku_j\theta}$, two forms are possible. We choose

$$\overline{u_ku_j\theta}=-q\lambda_2\left(\frac{\partial\overline{u_k\theta}}{\partial x_j}+\frac{\partial\overline{u_j\theta}}{\partial x_k}\right). \qquad (16)$$

Assuming $\overline{u_k\theta^2}$ proportional to $\partial\overline{\theta^2}/\partial x_k$, we obtain

$$\overline{u_k\theta^2}=-q\lambda_3\frac{\partial\overline{\theta^2}}{\partial x_k}. \qquad (17)$$

It is questionable whether the pressure diffusional terms can be discriminated experimentally. Hanjalic and Launder (1972) assert they are small in the first place. Therefore, for the present we set

$$\overline{pu_i}=\overline{p\theta}=0 \qquad (18)$$

to complete the required modeling assumptions.

If (10)–(18) are inserted into (7)–(9) and

$$D(\ )/Dt\equiv U_k\partial(\ )/\partial x_k+\partial(\ )/\partial t,$$

we obtain

$$\frac{D\overline{u_iu_j}}{Dt}+f_k(\epsilon_{jkl}\overline{u_lu_i}+\epsilon_{ikl}\overline{u_lu_j})$$
$$=\frac{\partial}{\partial x_k}\left[q\lambda_1\left(\frac{\partial\overline{u_iu_j}}{\partial x_k}+\frac{\partial\overline{u_iu_k}}{\partial x_j}+\frac{\partial\overline{u_ju_k}}{\partial x_i}\right)+\nu\frac{\partial\overline{u_iu_j}}{\partial x_k}\right]$$
$$-\overline{u_ku_i}\frac{\partial U_j}{\partial x_k}-\overline{u_ku_j}\frac{\partial U_i}{\partial x_k}-\beta(g_j\overline{u_i\theta}+g_i\overline{u_j\theta})$$
$$-\frac{q}{3l_1}\left(\overline{u_iu_j}-\frac{\delta_{ij}}{3}q^2\right)+Cq^2\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)-\frac{2}{3}\frac{q^3}{\Lambda_1}\delta_{ij}, \qquad (19)$$

$$\frac{D\overline{u_j\theta}}{Dt}+f_k\epsilon_{jkl}\overline{u_l\theta}$$
$$=\frac{\partial}{\partial x_k}\left[q\lambda_2\left(\frac{\partial\overline{u_j\theta}}{\partial x_k}+\frac{\partial\overline{u_k\theta}}{\partial x_j}\right)+\alpha\overline{u_j\frac{\partial\theta}{\partial x_k}}+\nu\overline{\theta\frac{\partial u_j}{\partial x_k}}\right]$$
$$-\overline{u_ju_k}\frac{\partial\Theta}{\partial x_k}-\overline{\theta u_k}\frac{\partial U_j}{\partial x_k}-\beta g_j\overline{\theta^2}-\frac{q}{3l_2}\overline{u_j\theta}, \qquad (20)$$

$$\frac{D\overline{\theta^2}}{Dt}=\frac{\partial}{\partial x_k}\left[q\lambda_3\frac{\partial\overline{\theta^2}}{\partial x_k}+\alpha\frac{\partial\overline{\theta^2}}{\partial x_k}\right]-2\overline{u_k\theta}\frac{\partial\Theta}{\partial x_k}-2\frac{q}{\Lambda_2}\overline{\theta^2}. \qquad (21)$$

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[2] For example, reinstate a term proportional to $\epsilon_{jkm}\partial U_k/\partial x_m$ in (11) and carry it through the succeeding analysis. In the constant flux layer the term would yield a heat flux in the cross-stream direction. We now note that if the term $g_j\beta\theta$ is retained in Eq. (5) one finds that terms proportional to $\beta(g_i\overline{u_j\theta}+g_j\overline{u_i\theta}-\frac{2}{3}\delta_{ij}g_k\overline{u_k\theta})$ and $g_j\beta\overline{\theta^2}$ should be added to (10) and (11) respectively. For the present, these are neglected since it is our plan to utilize closure assumptions and empirical constants from neutral flows only.

4. The boundary layer approximation

If now we discuss a boundary layer where the vertical scale height is much less than the horizontal scale, Eqs. (1)–(2) and (19)–(21) may be written as follows:

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}=0 \tag{22}$$

$$\frac{DU}{Dt}-fV+\frac{\partial p}{\partial x}=\frac{\partial}{\partial z}\left(-\overline{uw}+\nu\frac{\partial U}{\partial z}\right) \tag{23a}$$

$$\frac{DV}{Dt}+fU+\frac{\partial p}{\partial y}=\frac{\partial}{\partial z}\left(-\overline{vw}+\nu\frac{\partial V}{\partial z}\right) \tag{23b}$$

$$\frac{\partial p}{\partial z}=-g\beta\Theta, \tag{23c}$$

$$\frac{D\Theta}{Dt}=\frac{\partial}{\partial z}\left(-\overline{w\theta}+\alpha\frac{\partial\Theta}{\partial z}\right) \tag{24}$$

$$\frac{D\overline{u^2}}{Dt}+2f_y\overline{uw}-2f\overline{uv}=\frac{\partial}{\partial z}\left[q\lambda_1\frac{\partial\overline{u^2}}{\partial z}+\nu\frac{\partial\overline{u^2}}{\partial z}\right]-2\overline{wu}\frac{\partial U}{\partial z}-\frac{q}{3l_1}\left(\overline{u^2}-\frac{q^2}{3}\right)-\frac{2}{3}\frac{q^3}{\Lambda_1} \tag{25a}$$

$$\frac{D\overline{v^2}}{Dt}+2f\overline{uv}=\frac{\partial}{\partial z}\left[q\lambda_1\frac{\partial\overline{v^2}}{\partial z}+\nu\frac{\partial\overline{v^2}}{\partial z}\right]-2\overline{wv}\frac{\partial V}{\partial z}-\frac{q}{3l_1}\left(\overline{v^2}-\frac{q^2}{3}\right)-\frac{2}{3}\frac{q^3}{\Lambda_1} \tag{25b}$$

$$\frac{D\overline{w^2}}{Dt}-2f_y\overline{uw}=\frac{\partial}{\partial z}\left[3q\lambda_1\frac{\partial\overline{w^2}}{\partial z}+\nu\frac{\partial\overline{w^2}}{\partial z}\right]+2\beta g\overline{w\theta}-\frac{q}{3l_1}\left(\overline{w^2}-\frac{q^2}{3}\right)-\frac{2}{3}\frac{q^3}{\Lambda_1} \tag{25c}$$

$$\frac{D\overline{uv}}{Dt}+f_y\overline{uw}+f(\overline{u^2}-\overline{v^2})=\frac{\partial}{\partial z}\left[q\lambda_1\frac{\partial\overline{uv}}{\partial z}+\nu\frac{\partial\overline{uv}}{\partial z}\right]-\overline{wu}\frac{\partial V}{\partial z}-\overline{wv}\frac{\partial U}{\partial z}-\frac{q}{3l_1}\overline{uv} \tag{25d}$$

$$\frac{D\overline{uw}}{Dt}+f_y(\overline{w^2}-\overline{u^2})-f\overline{vw}=\frac{\partial}{\partial z}\left[2q\lambda_1\frac{\partial\overline{uw}}{\partial z}+\nu\frac{\partial\overline{uw}}{\partial z}\right]-(\overline{w^2}-Cg^2)\frac{\partial U}{\partial z}+\beta g\overline{u\theta}-\frac{q}{3l_1}\overline{uw} \tag{25e}$$

$$\frac{D\overline{vw}}{Dt}-f_y\overline{uv}+f\overline{uw}=\frac{\partial}{\partial z}\left[2q\lambda_1\frac{\partial\overline{uw}}{\partial z}+\nu\frac{\partial\overline{vw}}{\partial z}\right]-(\overline{w^2}-Cq^2)\frac{\partial V}{\partial z}+\beta g\overline{v\theta}-\frac{q}{3l_1}\overline{vw} \tag{25f}$$

$$\frac{D\overline{u\theta}}{Dt}+f_y\overline{w\theta}-f\overline{v\theta}=\frac{\partial}{\partial z}\left[q\lambda_2\frac{\partial\overline{u\theta}}{\partial z}+\alpha\overline{u\frac{\partial\theta}{\partial z}}+\nu\overline{\theta\frac{\partial u}{\partial z}}\right]-\overline{uw}\frac{\partial\Theta}{\partial z}-\overline{\theta w}\frac{\partial U}{\partial z}-\frac{q}{3l_2}\overline{u\theta} \tag{26a}$$

$$\frac{D\overline{v\theta}}{Dt}+f\overline{u\theta}=\frac{\partial}{\partial z}\left[q\lambda_2\frac{\partial\overline{v\theta}}{\partial z}+\alpha\overline{v\frac{\partial\theta}{\partial z}}+\nu\overline{\theta\frac{\partial v}{\partial z}}\right]-\overline{vw}\frac{\partial\Theta}{\partial z}-\overline{\theta w}\frac{\partial V}{\partial z}-\frac{q}{3l_2}\overline{v\theta} \tag{26b}$$

$$\frac{D\overline{w\theta}}{Dt}-f_y\overline{u\theta}=\frac{\partial}{\partial z}\left[2q\lambda_2\frac{\partial\overline{w\theta}}{\partial z}+\alpha\overline{w\frac{\partial\theta}{\partial z}}+\nu\overline{\theta\frac{\partial w}{\partial z}}\right]-\overline{w^2}\frac{\partial\Theta}{\partial z}+\beta g\overline{\theta^2}-\frac{q}{3l_2}\overline{w\theta} \tag{26c}$$

$$\frac{D\overline{\theta^2}}{Dt}=\frac{\partial}{\partial z}\left[q\lambda_3\frac{\partial\overline{\theta^2}}{\partial z}+\alpha\frac{\partial\overline{\theta^2}}{\partial z}\right]-2\overline{w\theta}\frac{\partial\Theta}{\partial z}-2\frac{q}{\Lambda_2}\overline{\theta^2}. \tag{27}$$

5. The constant-flux layer

Outside of the viscous (or "roughness") sublayer but still close to a solid surface, it may be deduced from (23a, b) and (24) that $\overline{uw}$, $\overline{vw}$, $\overline{w\theta}$ are approximately constant and equal to the total "wall" shear stress and heat flux. Without loss of generality, we can orient our coordinate system so that $\overline{vw}=0$.

We seek further simplifications. From a simple scale analysis it may be determined that the advective terms and the Coriolis terms in (25) and (26) cannot be significant *in the constant-flux layer*[3] and this should apply to atmospheric or sea surface boundary layers.

In neutral layers, it may be shown (e.g., Mellor, 1972) that together with $\overline{uw}$, $\overline{vw}$ and $\overline{w\theta}$, $\overline{u^2}$, $\overline{v^2}$, $\overline{uw}$, $\overline{u\theta}$, $\overline{v\theta}$, $\overline{w\theta}$ and $\overline{\theta^2}$ are also constant and that the diffusional terms are therefore zero. For stable or unstable layers, however, we will see that $\overline{u^2}$, etc., vary with z so that, strictly speaking, the diffusional terms should not be

[3] If the Coriolis terms are left in (25) and (26) the parameter which would appear in Eq. (41) is fz/u_τ which may be shown to be small.

neglected in a stably or unstably stratified layer. On the other hand, we note that outside the constant flux layer in neutral layers, where the diffusion terms are not zero in principle, they are nevertheless fairly small.

Energy budget measurements as reported by Wyngaard and Coté (1971) present a confusing picture. First, estimates of non-stationary advection and Coriolis terms are small in accordance with our own estimates. Furthermore, direct measurement of kinetic energy production (shear plus buoyancy production) and dissipation indicates a near balance over the entire measurement range of stable and unstable conditions. However, direct measurement of the velocity diffusion term indicates it to be significant for unstable conditions. Since production and dissipation balance, the velocity diffusion measurement is either in error (it is, after all, a triple correlation which must be differentiated) or it must be balanced by pressure-velocity diffusion which in (18) was neglected here.

Despite the confused experimental situation we will proceed here by neglecting the combined velocity and pressure diffusion term. We also neglect the diffusion of $\overline{\theta^2}$, which Wyngaard and Coté reported as small in all cases. It may now be shown that $\partial V/\partial z$, $\overline{uv}$, $\overline{v\theta}=0$. Therefore, (25a–f), (26a–c) and (27) reduce to:

$$2\overline{uw}\frac{\partial U}{\partial z}+\frac{q}{3l_1}\left(\overline{u^2}-\frac{q^2}{3}\right)+\frac{2}{3}\frac{q^3}{\Lambda_1}=0 \tag{28a}$$

$$\frac{q}{3l_1}\left(\overline{v^2}-\frac{q^2}{3}\right)+\frac{2}{3}\frac{q^3}{\Lambda_1}=0 \tag{28b}$$

$$-2\beta g\overline{w\theta}+\frac{q}{3l_1}\left(\overline{w^2}-\frac{q^2}{3}\right)+\frac{2}{3}\frac{q^3}{\Lambda_1}=0 \tag{28c}$$

$$\overline{uv}=0 \tag{28d}$$

$$(\overline{w^2}-Cq^2)\frac{\partial U}{\partial z}-\beta g\overline{u\theta}+\frac{q}{3l_1}\overline{uw}=0 \tag{28e}$$

$$\overline{vw}=0 \tag{28f}$$

$$\overline{uw}\frac{\partial \Theta}{\partial z}+\overline{\theta w}\frac{\partial U}{\partial z}+\frac{q}{3l_2}\overline{u\theta}=0 \tag{29a}$$

$$\overline{v\theta}=0 \tag{29b}$$

$$\overline{w^2}\frac{\partial \Theta}{\partial z}-\beta g\overline{\theta^2}+\frac{q}{3l_2}\overline{w\theta}=0 \tag{29c}$$

$$2\overline{w\theta}\frac{\partial \Theta}{\partial z}+\frac{2q}{\Lambda_2}\overline{\theta^2}=0. \tag{30}$$

a. Neutral case

First, we consider the case where $g=0$ [which is equivalent to letting $\zeta\rightarrow 0$ in Eqs. (41a–g)]. Let

$$-\overline{uw}\equiv u_\tau^2 \tag{31a}$$

$$\overline{u^2},\ \overline{v^2},\ \overline{w^2}\equiv u_\tau^2(\overline{u^{*2}},\overline{v^{*2}},\overline{w^{*2}}) \tag{31b, c, d}$$

$$q^*=[\overline{u^{*2}}+\overline{v^{*2}}+\overline{w^{*2}}]^{\frac{1}{2}} \tag{31e}$$

$$-\overline{\theta w}\equiv H \tag{31f}$$

$$\varphi_M\equiv\frac{\kappa z}{u_\tau}\frac{\partial U}{\partial z} \tag{32a}$$

$$\varphi_H\equiv\frac{\kappa z u_\tau}{H}\frac{\partial \Theta}{\partial z}. \tag{32b}$$

With fairly weak assumptions (Millikan, 1939; Yajnik, 1970; Mellor, 1972) about the nature of turbulent boundary layers, it may be shown that $\varphi_M(z)=$ constant from which follows the logarithmic law of the wall. Furthermore, the von Kármán constant κ is chosen empirically so that $\varphi_M=1$. It follows from (28) and (29) that l_1, l_2,Λ_1, $\Lambda_2\propto z$ so that

$$l_1=\kappa A_1 z,\quad \Lambda_1=\kappa B_1 z, \tag{33a, b}$$

$$l_2=\kappa A_2 z,\quad \Lambda_2=\kappa B_2 z. \tag{33c, d}$$

Therefore A_1, A_2, B_1, B_2, C are empirically required constants which are not completely independent as will be seen below. If we define

$$\gamma\equiv\tfrac{1}{3}-2\frac{A_1}{B_1}, \tag{34}$$

Eqs. (28a,b,c,e) may be written, after some algebra, as

$$\overline{u^2}=(1-2\gamma)q^2, \tag{35a}$$

$$\overline{v^2}=\gamma q^2, \tag{35b}$$

$$\overline{w^2}=\gamma q^2, \tag{35c}$$

$$-\overline{uw}\equiv u_\tau^2=\left[\frac{(1-3\gamma)(\gamma-C)}{2}\right]^{\frac{1}{2}}q^2, \tag{35d}$$

which we have written dimensionally to emphasize that the full Reynolds Stress tensor depends on u_τ, γ and C. In the course of the above analysis one finds that $B_1=q^{*3}$ so that (35d) may be written

$$B_1=\left[\frac{2}{(1-3\gamma)(\gamma-C)}\right]^{\frac{3}{4}},\quad A_1=B_1\frac{1-3\gamma}{6}, \tag{36a, b}$$

where we have also solved for A_1, using (34).

We now appeal to some laboratory measurements in the constant-flux region of a two-dimensional, wind-tunnel boundary layer. The measurements of Klebanoff (1955) give

$$\begin{bmatrix} \overline{u^2}, & \overline{uw}, & \overline{uv} \\ \overline{wu}, & \overline{w^2}, & \overline{wv} \\ \overline{vu}, & \overline{vw}, & \overline{v^2} \end{bmatrix} q^{-2} \approx \begin{bmatrix} 0.57, & -0.16, & 0 \\ -0.16, & 0.15, & 0 \\ 0\ , & 0\ , & 0.28 \end{bmatrix},$$

where some judgment is required to estimate the values appropriate to the constant-flux region. Some recent measurement by So and Mellor (1972) and by Champagne *et al.* (1970) yield

$$\begin{bmatrix} 0.53, & -0.16, & 0 \\ -0.16, & 0.20, & 0 \\ 0, & 0, & 0.27 \end{bmatrix} \text{ and } \begin{bmatrix} 0.48, & -0.17, & 0 \\ -0.17, & 0.24, & 0 \\ 0, & 0, & 0.28 \end{bmatrix}$$

respectively. The latter case is not a boundary layer but a grid-generated, near-homogeneous, constant shear flow downstream of the grid where $\overline{u_iu_j}/q^2$ has approached the above constant values. It turns out that Eqs. (34) and (35a, b, c, d) apply equally well to this flow where, however, z in (32a) and (33a, b) must be redefined as a characteristic dimension of the flow which is invariant in the cross-flow direction.

We now see that (35b, c) permits only equal values of $\overline{v^2}$ and $\overline{w^2}$, whereas the data indicated more or less unequal values. Possibly, this is a defect in the isotropy assumption involved in (10), but, hopefully not a serious impediment to our ultimate goal. Taking an average we then obtain $\gamma \approx 0.23$. Using an average value of $\overline{uw}/q^2$ in (35d), we find that $C \approx 0.056$. From (36a, b) we obtain $A_1 \approx 0.78$ and $B_1 \approx 15.0$.

A quantity of interest is the ratio of the velocity and temperature dissipation terms. From (12), (14), (33b) and (33d) we obtain

$$2\nu\overline{\left(\frac{\partial u_i}{\partial x_k}\right)^2} \Big/ 2\alpha\overline{\left(\frac{\partial \theta}{\partial x_k}\right)^2} = \frac{\overline{q^2}}{\overline{\theta^2}}\frac{B_2}{B_1}.$$

For high-Reynolds-number, decaying turbulent fields, similarity considerations yield (Hinze, 1959) $B_2/B_1 = \frac{2}{3}$ which agrees apparently with the measurements of Gibson and Schwartz (1965); however, this was deduced from a rather short decay history. The boundary layer measurements of Johnson (1959) give the result $B_2/B_1 \approx 0.36$. Thus there exists considerable uncertainty in B_2, i.e., $5.5 < B_2 < 10.0$.

Finally, a quantity of interest that may be extracted from (28e) and (29c) is the turbulent Prandtl number

$$\mathrm{Pr}_t \equiv -\overline{uw}\frac{\partial \Theta}{\partial z} \Big/ \left(-\overline{\theta w}\frac{\partial U}{\partial z}\right) = \frac{A_1}{A_2}\frac{\gamma - C}{\gamma}.$$

It is fairly well established experimentally that $0.7 < \mathrm{Pr}_t < 0.85$. For the neutral condition of the data to be discussed below it was found that $\mathrm{Pr}_t = 0.74$. Therefore, using this value and $\gamma = 0.23$, $C = 0.056$, we obtain $A_2 = 0.79$.

b. Stratified boundary layers

At this point we make the seemingly naive assumption that (33a, b, c, d) and the constants A_1, A_2, B_1, B_2, C obtained for neutral layers apply equally to stable and unstable layers. To cast the complete equations in non-dimensional form we now define a Monin-Obukhoff length scale

$$L \equiv \frac{u_\tau^3}{\kappa g \beta H}, \tag{39}$$

and

$$\zeta \equiv \frac{z}{L}, \tag{40a}$$

$$\overline{\theta^{*2}} \equiv \frac{u_\tau^2 \overline{\theta^2}}{H^2}, \tag{40b}$$

$$\overline{u^*\theta^*} \equiv \frac{\overline{u\theta}}{H}. \tag{40c}$$

Thus, (28a, b, c, e), (29a, c) and (30) may be written

$$-2\varphi_M + \frac{q^*}{3A_1}\left(\overline{u^{*2}} - \frac{q^{*2}}{3}\right) + \frac{2}{3}\frac{q^{*3}}{B_1} = 0, \tag{41a}$$

$$\frac{q^*}{3A_1}\left(\overline{v^{*2}} - \frac{q^{*2}}{3}\right) + \frac{2}{3}\frac{q^{*3}}{B_1} = 0, \tag{41b}$$

$$2\zeta + \frac{q^*}{3A_1}\left(\overline{w^{*2}} - \frac{q^{*2}}{3}\right) + \frac{2}{3}\frac{q^{*3}}{B_1} = 0, \tag{41c}$$

$$(\overline{w^{*2}} - Cq^{*2})\varphi_M - \zeta\overline{u^*\theta^*} - \frac{q^*}{3A_1} = 0, \tag{41d}$$

$$-\varphi_H - \varphi_M + \frac{q^*}{3A_2}\overline{u^*\theta^*} = 0, \tag{41e}$$

$$\overline{w^{*2}}\varphi_H - \zeta\overline{\theta^{*2}} - \frac{q^*}{3A_2} = 0, \tag{41f}$$

$$-2\varphi_H + 2\frac{q^*}{B_2}\overline{\theta^{*2}} = 0. \tag{41g}$$

For some advantage in solving the equations, (41a–g) can be reduced to

$$q^{*3} = B_1(\varphi_M - \zeta), \tag{42a}$$

$$\overline{\theta^{*2}} = \frac{B_2}{q^*}\varphi_H, \tag{42b}$$

$$\varphi_M[\gamma - C - (6A_1\zeta + 3A_2\zeta)/q^{*3}] - \varphi_H[3A_2\zeta/q^{*3}] = 1/(3A_1 q^*), \quad (42c)$$

$$\varphi_H[\gamma - (6A_1\zeta + B_2\zeta)/q^{*3}] = 1/(3A_2 q^*). \quad (42d)$$

The solutions to these equations are shown in Figs. 1 and 2 in comparison with the data of Businger *et al.* (1971) for $A_1 = 0.78$, $B_1 = 15.0$, $A_2 = 0.79$, $C = 0.056$. Since there exists uncertainty, B_2 had to be chosen so that calculated values did fit the φ_H data on the stable side; nevertheless, the chosen value, $B_2 = 8.0$, lies in the range $5.5 < B_2 < 10.0$ estimated from neutral data.[4] By absorbing it into the definitions of φ_M, φ_H and ζ, we are not required to choose a value of κ. However, the value of $\kappa = 0.35$ was chosen by Businger *et al.*, so that the experimental $\varphi_M = 1$ at the neutral point, $\zeta = 0$. This differs somewhat from the more commonly accepted value, $\kappa = 0.40$.

In the process of obtaining the φ_M, φ_H solution, other interesting quantities are obtained including the turbulent energy components shown in Fig. 3. Note the relative decrease in the vertical component of the turbulent energy as stability increases.

Eqs. (42a, b, c, d) may be examined for their asymptotic behavior. Without going into detail and using

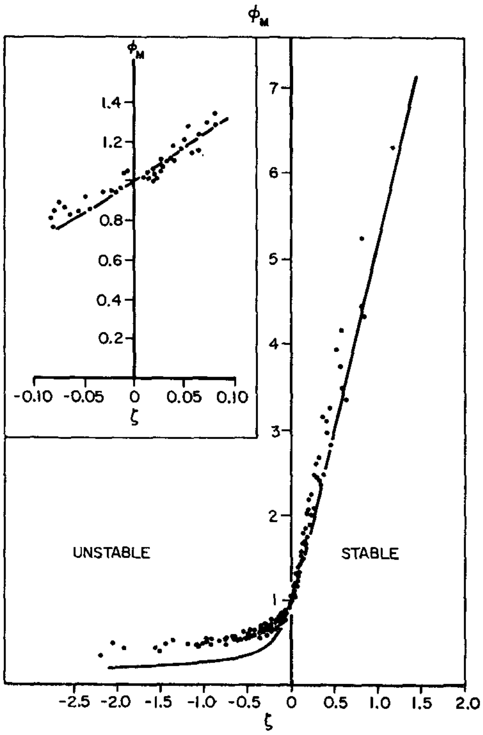

FIG. 1. Comparison of the velocity profile data of Businger *et al.* (1971) with calculated values (solid curve). Insert is detail near $\zeta = 0$.

[4] Varying B_2 affects the calculated φ_H only on the stable side and does not affect φ_M significantly. A variation, $\delta B_2/B_2 = \pm\frac{1}{8}$, produces a variation, $\delta\varphi_H/\varphi_H \approx \pm 0.3$, on the stable side.

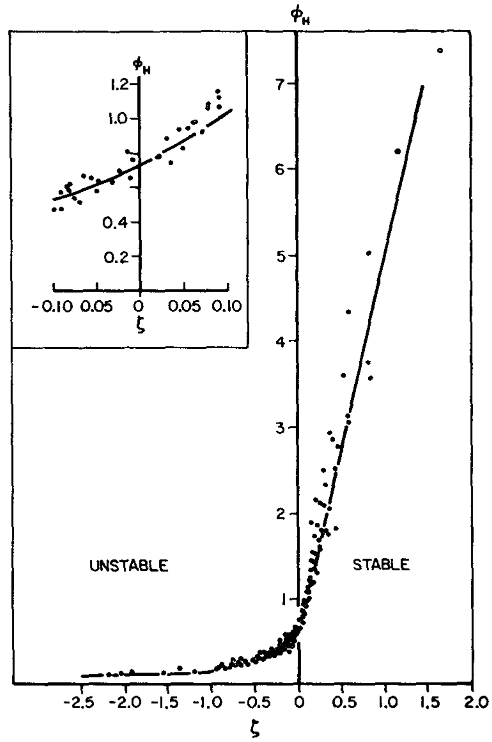

FIG. 2. Comparison of the temperature profile data of Businger *et al.* (1971) with calculated values (solid curve). Insert is detail near $\zeta = 0$.

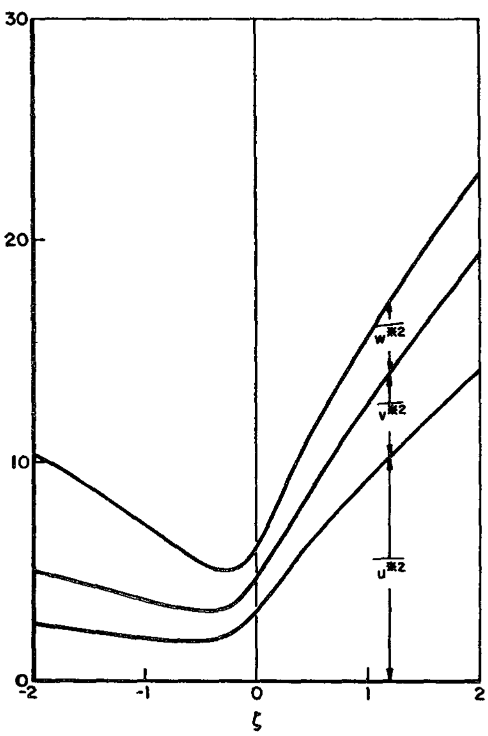

FIG. 3. Variation of calculated components of the turbulent energy.

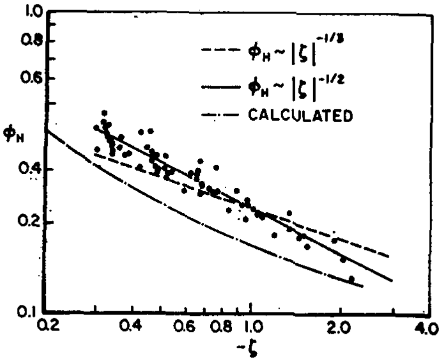

FIG. 4. Temperature gradient data under unstable conditions (from Businger *et al.*, 1971).

the aforementioned values of A_1, B_1, A_2, B_2 and C, it may be determined that

$$\left.\begin{aligned} \varphi_M &\approx 4.68\zeta \\ \varphi_H &\approx 4.70\zeta \end{aligned}\right\} \zeta \to \infty, \quad \begin{aligned} &(43a) \\ &(43b) \end{aligned}$$

$$\left.\begin{aligned} \varphi_M &\approx 0.239(-\zeta)^{-\frac{1}{3}} \\ \varphi_H &\approx 0.164(-\zeta)^{-\frac{1}{3}} \end{aligned}\right\} \zeta \to -\infty. \quad \begin{aligned} &(44a) \\ &(44b) \end{aligned}$$

The later result has been obtained by a number of researchers including Prandtl (1932) and may be deduced from the requirement that, as $\zeta \to -\infty$, u_τ should no longer be a factor in the determination of heat flux.

As discussed by Businger *et al.*, the data shown in Fig. 4 indicates $\varphi_H \approx (-\zeta)^{\frac{1}{2}}$ in the range $0 < -\zeta < 2$. However, this is consistent with our calculated result which, although a bit low, agrees well with the $\frac{1}{3}$-power law only after $-\zeta > 3$. Other data assembled by Businger (1972) for much larger values of $-\zeta$ are shown in Fig. 5. The data and the calculated values agree quite well and the asymptotic $\frac{1}{3}$-power law seems secure.

Another interesting way of presenting results is in the form

$$\frac{-\overline{uw}}{\kappa^2 z^2 \left(\dfrac{\partial U}{\partial z}\right)^2} = \frac{1}{\varphi_M{}^2}, \tag{44a}$$

$$\frac{-\overline{w\theta}}{\kappa^2 z^2 \left(\dfrac{\partial U}{\partial z}\right)\left(\dfrac{\partial \Theta}{\partial z}\right)} = \frac{1}{\varphi_M \varphi_H}, \tag{44b}$$

$$\frac{\overline{u^2}, \overline{v^2}, \overline{w^2}}{\kappa^2 z^2 \left(\dfrac{\partial U}{\partial z}\right)^2} = \frac{\overline{u^{*2}}, \overline{v^{*2}}, \overline{w^{*2}}}{\varphi_M{}^2}, \tag{44c}$$

as a function of Richardson number

$$\mathrm{Ri} \equiv \frac{\beta g \dfrac{\partial \Theta}{\partial z}}{\left(\dfrac{\partial U}{\partial z}\right)^2} = \frac{\zeta \varphi_H}{\varphi_M{}^2}. \tag{45}$$

The computed quantities are plotted in Fig. 4. We obtain a critical value Ri = 0.21 beyond which the flow

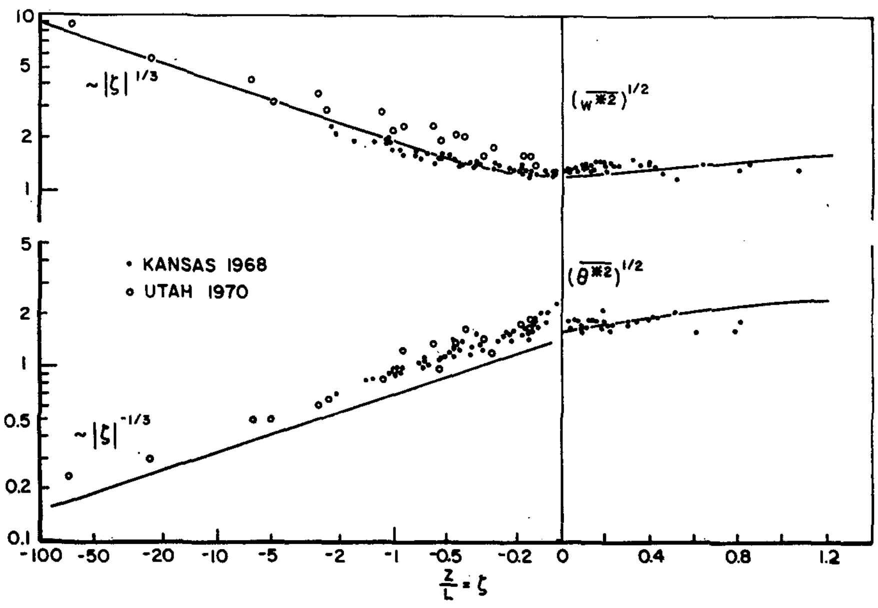

FIG. 5. Vertical velocity and temperature variance from data assembled by Businger (1972). The Kansas data for $(\overline{w^{*2}})^{\frac{1}{2}}$ are from Haugen *et al.* (1971) and for $(\overline{\theta^{*2}})^{\frac{1}{2}}$ from Wyngaard *et al.* (1971).

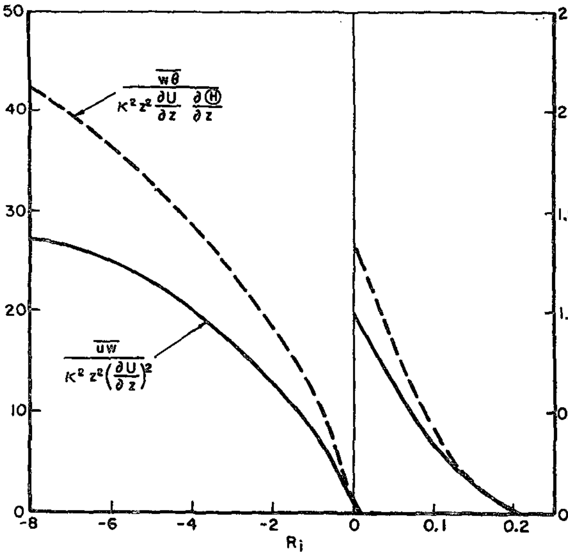

FIG. 6. Momentum and heat flux as a function of Richardson number. Note change of scale for the left and right curves.

is laminar. This limiting value is also obtained from (43a, b).

6. Discussion

A planetary boundary layer model is proposed which is here applied to the constant-flux layer. Empirical information used in the model is solely derived from neutral turbulent flows so that the intrinsic prediction power of the model may be judged.

From the experimental data explicitly considered in this paper the basic model using the simple empiricism in (33a, b, c, d) appears to be quite good. However, from the spectral data of Kaimal *et al.* (1972) it would appear that $\overline{u^2}/u_\tau^2$ and $\overline{v^2}/u_\tau^2$ do not increase on the stable side as indicated in Fig. 3.[5]

If this be the case, future investigations must deal with alterations of (10), (11), (12), (13) and (14), or the specific empirical input in the form of Eqs. (33a, b, c, d). With regard to the latter, various formulations for a length scale have been proposed (and summarized by Mellor and Herring, 1973) and might supplement (33a, b, c, d), in a way which—hopefully—would not require explicit empirical adjustment for stratification. On the other hand, it is clear that the present model compares favorably with a considerable amount of data.

[5] The author is indebted to a reviewer for pointing this out. It should be noted, however, that integral values of the spectral data, rescaled to yield $\overline{u^2}/u_\tau^2$ and $\overline{v^2}/u_\tau^2$, do not seem to be directly available in the literature.

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