External Rossby Waves in the Two-Layer Model*

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ABSTRACT

In order to clarify the extent to which the two-layer model can successfully simulate the remote tropospheric response to localized stationary forcing, the structure of stationary Rossby waves in the two-layer model is compared with that in continuous models. One finds a close correspondence when the two-layer flow is supercritical in the sense of the Phillips' criterion, except for the possibility of upstream propagation in the two-layer model when the lower-layer wind is small. When the two-layer flow is subcritical, the stationary waves can be very seriously distorted. The manner in which neutral modes are spatially or temporally destabilized by damping in the two-layer model is contrasted with similar results for Charney's model.

1. Introduction

This paper is the third in a sequence (Held et al., 1985, 1986; hereafter referred to as HPP1 and HPP2, respectively) on the linear theory of stationary external Rossby waves in horizontally uniform mean flows with vertical shear. In HPP1, the structure of these waves was discussed, and their decisive contribution to the stationary far-field tropospheric response to localized thermal or orographic forcing emphasized. In HPP2, general arguments were presented concerning the effect on stationary external Rossby waves of such "dissipative" mechanisms as thermal, Ekman, and potential vorticity damping. It was shown that thermal and low-level potential vorticity damping could be destabilizing; such a possibility was illustrated with Charney's model, and its implications for stationary forced responses in the atmosphere were discussed. Here we consider how faithfully the two-layer model reproduces the essential features of the continuous models treated in the previous two papers. We feel that the two-layer model will continue to play a central role in studies of midlatitude dynamics, particularly as more attention is focused on such complex issues as the interaction between transient and stationary eddies (e.g., Hendon and Hartmann, 1985; Pierrehumbert, 1984, 1986). To evaluate such studies, it is important to have a clear understanding of the relationship between stationary waves in two-layer and continuous models, including the effects of dissipation on these waves.

Section 2 in this paper reviews the structure of the two-layer model's stationary modes in the case of both subcritical and supercritical shear. As in the two previous papers, our interest is primarily in positive shears and westerly winds. Lindzen et al. (1968) and Chen and Trenberth (1985) have argued that spurious vertical trapping of long waves in the two-layer model could lead to false resonances and severe distortion of the stationary wave field. Our view of the value of the two-layer model is considerably more positive; when the shears are supercritical, the tropospheric wave trains emanating from a localized source can often be quite accurately represented in a two-layer model.

An important feature of the external stationary neutral mode in the two-layer model is its *upstream* group velocity when the shear is supercritical and the lower-layer wind is small. [McCartney (1975) and Fandry and Leslie (1984) are incorrect when they state that the group velocity is always positive when winds and shears are positive, but as they only work with subcritical shears their results are unaffected.] Supercritical shear is the case of relevance to the extratropical troposphere, and upstream group velocities clearly can have a profound effect on the stationary wave pattern forced by localized sources. This phenomenon is not simply an artifact of layer models; in section 3 we describe a modification of the Charney profile in which the surface temperature gradient is removed and show that this continuous profile has similar properties. Nonetheless, if the two-layer model is meant to represent the more typical atmospheric profile with a maximum temperature gradient at the ground, the model will perform poorly when it allows upstream

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group velocities. In our view, it is this possibility of upstream propagation, rather than spurious vertical reflection, that most threatens to distort the stationary wave response in the two-layer model. In section 3 we also describe how the change in sign of the group velocity of the stationary external mode, as the surface wind decreases in two-layer or continuous models, leads to the possibility of overreflection of horizontally propagating waves even in the absence of critical layers.

Dissipative destabilization has been reported in a number of two-layer studies: for the case of Ekman pumping, see Holopainen (1961) and Wiin-Nielsen et al. (1967); and for thermal damping, see the latter paper, Haltiner (1967), and Pedlosky (1975). There can be significant differences between the effects of a given dissipative mechanism in two-layer and continuous models. For example, in the two-layer model, Ekman pumping destabilizes some neutral modes, while in Charney's model it does not. In section 4 we examine the manner in which both neutral modes in the two-layer model are affected by the addition of Ekman pumping, thermal damping, and potential vorticity mixing. As in HPP2, the modal pseudomomentum or wave action emerges as a simple unifying concept in studying the effects of small amounts of dissipation on neutral modes.

2. The stationary external mode

We consider two horizontally unbounded, incompressible homogeneous fluid layers of equal mean depth on a beta-plane, bounded above and below by rigid horizontal surfaces. Quantities referring to the upper and lower layers are given subscripts 1 and 2, respectively. The Boussinesq quasi-geostrophic equations, when linearized for unforced motions about mean winds U_j that have no horizontal dependence, become

$$\partial q'_j/\partial t + U_j \partial q'_j/\partial x + Q_{jy}\partial\psi'_j/\partial x = 0, \quad (j = 1, 2). \tag{2.1}$$

The eddy potential vorticities q'_j and mean state potential vorticity gradients Q_{jy} are given by

$$q'_j = \nabla^2\psi'_j + (-1)^j(\psi'_1 - \psi'_2)/(2\lambda^2),$$
$$Q_{jy} = \beta - (-1)^j(U_1 - U_2)/(2\lambda^2), \tag{2.2}$$

where λ is the Rossby radius of deformation.

Writing stationary normal mode solutions in vector form,

$$\begin{pmatrix}\psi'_1\\ \psi'_1\end{pmatrix} = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} e^{i(kx+ly)} = \Psi e^{i(kx+ly)},$$

reduces (2.1) to the matrix eigenvalue problem

$$\begin{pmatrix} Q_{1y}/U_1 - 1/(2\lambda^2) & 1/(2\lambda^2) \\ 1/(2\lambda^2) & Q_{2y}/U_2 - 1/(2\lambda^2)\end{pmatrix}\Psi = K^2\Psi. \tag{2.3}$$

where, as usual, $K^2 = k^2 + l^2$. In (2.3) it has been assumed that the layer winds are not zero. Considered as an eigenvalue problem for K^2, the symmetry of the matrix in (2.3) implies that all eigenvalues are real. Modes corresponding to positive K^2 are horizontally propagating and therefore relevant for the far-field response to localized forcing. It is convenient to nondimensionalize velocities with $\beta\lambda^2$ (the critical value of $\hat{U} = (U_1 - U_2)/2$ for inviscid normal mode instability) and lengths by λ. We carry out this nondimensionalization without change in notation; in particular, the critical shear for instability becomes $\hat{U} = 1$, and the upper- and lower-layer potential vorticity gradients, respectively, become $1 + \hat{U}$ and $1 - \hat{U}$. Solving the eigenvalue problem for K^2 then gives the two-layer dispersion relation

$$K_\pm{}^2 = \{K_1{}^2 + K_2{}^2 - 1 \pm ([K_1{}^2 - K_2{}^2]^2 + 1)^{1/2}\}/2, \tag{2.4a}$$

where

$$K_1{}^2 = (1 + \hat{U})/U_1, \quad K_2{}^2 = (1 - \hat{U})/U_2. \tag{2.4b}$$

Aspects of this dispersion relation have been discussed by Pedlosky (1979) and Fandry and Leslie (1984), among others.

Each eigenvector Ψ is taken to be real and normalized so that $\langle\Psi, \Psi\rangle = 1$, where $\langle a, b\rangle = (a_1b_1 + a_2b_2)/2$. The amplitude ratio $\alpha = \psi_1/\psi_2$ for an eigenvector is

$$\alpha_\pm = K_1{}^2 - K_2{}^2 \pm [(K_1{}^2 - K_2{}^2)^2 + 1]^{1/2}, \tag{2.5a}$$
$$= 2(K_\pm{}^2 - K_2{}^2) + 1, \tag{2.5b}$$

where the choice of roots agrees with that in (2.4a). The positive square roots in (2.4a, 2.5a), henceforth denoted by $K_e{}^2$ and α_e, respectively, refer to the external mode, which becomes barotropic in the limit of zero shear. The squared wavenumber of the external mode lies between the $K_j{}^2$; in fact,

$$(K_1{}^2 + K_2{}^2)/2 \leq K_e{}^2 \leq \max(K_1{}^2, K_2{}^2),$$

where the inequalities are strict for positive U_2 and $\hat{U}$.

Figures 1a, b show the behavior of the two branches of K^2 as a function of U_2 for subcritical ($\hat{U} = 0.5$) and supercritical shear ($\hat{U} = 1.5$). The bold curves in the figures highlight the external mode for westerly lower-level winds. When the shear is supercritical and $U_2 > 0$, the stationary wavenumber for the external mode is bounded above by the wavenumbers corresponding to instability, but in the subcritical case, $K_e{}^2$ behaves as in barotropic models, being unbounded as U_2 approaches zero. For Charney's model, $K_e{}^2$ is always bounded above, no matter how small the vertical shear, by the wavenumbers corresponding to the Charney mode instability. This difference between super- and subcritical cases becomes important when considering the structure of stationary external Rossby waves propagating from midlatitudes into the tropics. As the waves approach the latitude at which the *low-level* wind changes from westerly to easterly, smaller and smaller horizontal scales will be generated by the two-layer

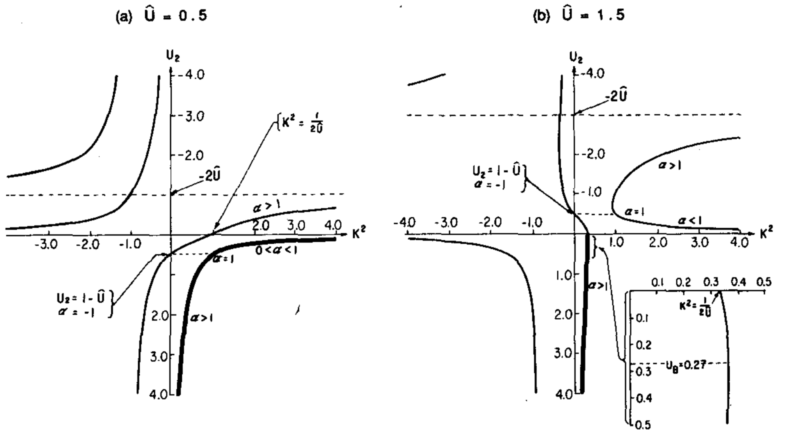

FIG. 1. (a) The two-layer dispersion relation [the two solutions to Eq. (2.4a)] for subcritical shear $\hat{U} = 0.5$. Wavenumbers are in units of λ^{-1} and winds in units of $\beta\lambda^2$. The external mode in the region of positive lower-layer wind is emphasized by a heavy line. Values of the amplitude ratio are indicated along modal curves; for the internal mode as $K^2 \rightarrow (1/2\hat{U})^{\pm}$, $\alpha \rightarrow \pm\infty$. (b) As in (a) but for supercritical shear $\hat{U} = 1.5$. The insert shows clearly the region of westward group velocity, $\partial K_e^2/\partial U_2 > 0$ for the stationary external mode at small U_2. As $K^2 \rightarrow (1/2\hat{U})^{\pm}$, $\alpha \rightarrow \pm\infty$.

model if the shear is subcritical. On the other hand, if the shear is supercritical, the scale of the external mode remains well behaved as the waves approach the lower layer's critical latitude. (This argument assumes that the transition occurs sufficiently slow for a local dispersion relation to be relevant.) The implication is that the two-layer model could seriously distort the stationary wave pattern in the subtropics when the shear is subcritical at the latitude of the low-level transition to easterlies.

Two important facts can be proved by algebraic manipulation of the dispersion relation:

1) For westerly surface winds and positive shears, the external mode is always horizontally propagating ($K_e^2 > 0$). This was noted in McCartney (1975) and found in HPP1 to be the case for linear shear (Charney) and hyperbolic tangent profiles.

2) For westerly winds in the two layers, the internal mode propagates only if the shear is subcritical and the surface wind is less than $1 - \hat{U}$. Defining $\bar{U} = (U_1 + U_2)/2$, this condition is equivalent to $\bar{U} < 1$. For the Charney profile discussed in HPP1, none of the internal modes are horizontally propagating, regardless of surface wind speed, if $r < 1$ [recall that $r = \beta N^2 H/(f^2 \partial U/\partial z)$]. In this sense the two-layer parameter $\hat{U}^{-1}$ plays the role of r.

Figure 2a shows the number of horizontally propagating stationary modes as a function of U_2 and $\hat{U}$ in the two-layer model. Figures 2b and 2c show the number of horizontally propagating stationary modes in the Charney model [$U = U_0 + \Lambda z$; N^2 = constant] and a hyperbolic tangent shear flow [$U = U_0 + (U_1 - U_0)\tanh(z/1.5H)$; N^2 = constant] as a function of the surface wind and vertical shear. (In Charney's model there are an infinite number of discrete modes, only a finite number of which can be horizontally propagating; for the hyperbolic tangent profile, there are a finite number of modes, all of which can be horizontally propagating.) For small surface winds, the behavior of the two-layer model is analogous to that of the continuous models: only one horizontally propagating mode exists for large vertical shears, and a second mode appears as one decreases the vertical shear. The two-layer model naturally misses the higher internal modes that appear at still smaller shears. When the vertical shear is identically zero, the continuous model possesses only one mode, the external mode. In this limit, the second mode of the two-layer model is clearly artificial (Lindzen et al., 1968). Charney's model has the unrealistic feature (due to the unbounded winds) that the number of horizontally propagating models does not decrease as Λ approaches 0 with U_0 fixed. In contrast, for the hyperbolic tangent profile, the internal modes disappear one by one as the shear is decreased until only the external mode remains. Note how small Λ must be before the internal modes disappear. Only in exceptionally weak vertical shears is it correct to think of the second horizontally propagating mode in the two-layer model as artificial.

Values of the amplitude ratio α are indicated along the modal curves in Fig. 1. From (2.5b) we see that for supercritical shear and positive lower layer wind, $\alpha_e > 1$, in analogy with the upper tropospheric maximum found in continuous models. For subcritical shear the external mode has greater amplitude in the lower layer when the westerly surface wind drops below $1 - \hat{U}$, i.e., when the internal mode becomes horizontally propagating. The supercritical and subcritical cases also differ dramatically in the structure of the external mode

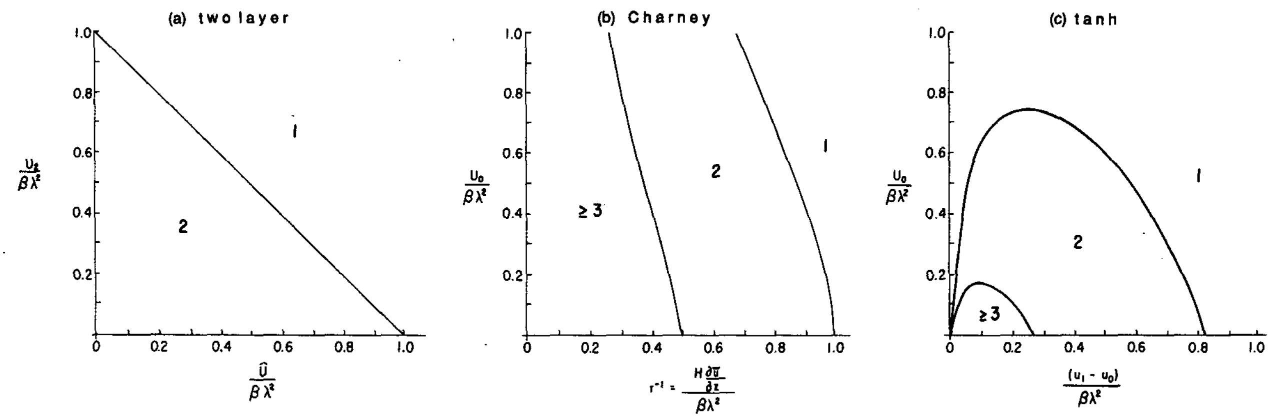

FIG. 2. (a) The number of horizontally propagating modes in (a) the two-layer model; (b) the linear shear profile (the "Charney model"), and (c) the hyperbolic tangent profile of HPP2.

for small positive surface winds: in the supercritical case the amplitude is concentrated in the upper layer, and in the subcritical case in the lower layer.

A bound on the amplitude ratio of either mode in terms of the ratio U_1/U_2 which will prove useful can be established by verifying algebraically that $U_1/U_2 - \alpha$ has, for positive shears, the same sign as U_1/U_2. For lower-layer westerlies, it follows that while the amplitude of the external mode increases with height when it is the only horizontally propagating mode, the amplitude increase is less rapid than that of the mean wind. This behavior is analogous to that found in the Charney model (see Fig. 1 in HPP1), and will play a similar role in the discussions of thermal forcing and destabilization below.

Another difference between the subcritical and supercritical cases concerns the group velocity of the stationary external mode. By Galilean invariance the group velocity of a stationary wave is given by

$$\vec{G} = -2k(\partial K^2/\partial U_2)^{-1}(k, l) \qquad (2.6)$$

(where the partial derivative is taken at fixed $\hat{U}$) so that the zonal group velocity G_x of each mode is proportional to the slope of the respective curve in Fig. 1. From the figure it can be seen that while the group velocity is finite and positive for all values of the lower-layer wind in the subcritical case, in the supercritical case the external mode's group velocity becomes infinite for a weak westerly wind whose value depends on the shear. (In Charney's model, this infinite group velocity is realized only in the limit of vanishing surface wind; in the other continuous models considered in HPP1 the group velocity was always bounded.) For still smaller lower-layer winds the group velocity has a westward component, a result that can be obtained from (2.3):

$$-\partial K_e^2/\partial U_2 = [(K_1^2/U_1)A_+ + (K_2^2/U_2)A_-]/2, \qquad (2.7a)$$

where $A_\pm = 1 \pm (K_1^2 - K_2^2)[(K_1^2 - K_2^2)^2 + 1]^{-1/2}$. From the positive definiteness of $A_\pm$ it is clear that the crucial term is $K_2^2 = Q_{2y}/U_2$: if the potential vorticity gradient in the lower layer is positive, G_x cannot be negative; but if the shear is supercritical, G_x will always be negative for sufficiently small lower-layer wind.

Another form of (2.7a) can be derived directly by differentiating both sides of the matrix equation (2.3) with respect to U_2 and then taking the scalar product with the normalized external eigenvector:

$$-\partial K_e^2/\partial U_2 = (q_{1e}^2/Q_{1y} + q_{2e}^2/Q_{2y})/2. \qquad (2.7b)$$

We refer to the expression on the right-hand side of (2.7b) as the vertically averaged pseudomomentum and denote it by P. (Note: our choice of sign in defining the pseudomomentum is not conventional. Edmon et al., 1980, discuss the connection between P, which they call "wave activity," and Eliassen–Palm flux.) The zonal group velocity has the same sign as P: from (2.7b) and (2.6) it follows that

$$G_x P = 2k^2, \qquad (2.8)$$

as in continuous models (see HPP2).

In none of the cases considered in HPP1 were the group velocity or pseudomomentum of the external mode negative. In the two-layer model, the value U_B of the lower-layer wind at which the sign change occurs is shown in Fig. 3 as a function of shear. The region of parameter space in which the pseudomomentum and group velocity are negative is not negligible; for a supercritical flow with $\hat{U} = 2$, upstream propagation occurs for $U_2 < 0.33$, or 3.3 m s^{-1} if we choose $\beta\lambda^2 = 10$ m s^{-1}. Given that it is not found in Charney's model or the continuous models analyzed in HPP1, this upstream development in the two-layer model should be considered artificial when the shear is supercritical and lower-layer winds are weak westerlies.

When the external mode is the only horizontally propagating mode, the problem of determining the steady far-field response to localized forcing reduces to

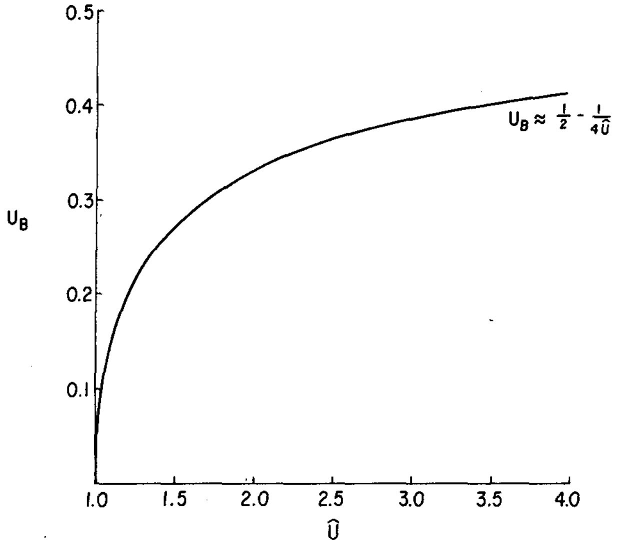

FIG. 3. The value U_B of the lower-layer wind at which the group velocity of the external mode changes sign, as a function of supercritical shear.

one of determining the projection of the forcing on the external mode. In terms of the meridional velocity $v = (v_1, v_2) = (\partial\psi_1/\partial x, \partial\psi_2/\partial x)$, the total response to orography and diabatic heating satisfies the (nondimensional) equation

$$\begin{pmatrix} \nabla^2 - 1/2 + K_1{}^2 & 1/2 \\ 1/2 & \nabla^2 - 1/2 + K_2{}^2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -Q/U_1 \\ Q/U_2 - \partial h/\partial x \end{pmatrix}. \quad (2.9)$$

where heating is modeled as a transfer rate Q of mass between layers and h is bottom topography. The equation satisfied by the projection $\langle v, v_e\rangle$ on the external mode can be deduced by taking the inner product of both sides of (2.9) with v_e. The symmetry of the matrix M in (2.9) implies that $\langle Mv, v_e\rangle = \langle v, Mv_e\rangle = (\nabla^2 + K_e{}^2)\langle v, v_e\rangle$; writing the external mode as $v_e = (v_{e1}, v_{e2})$, we have

$$(\nabla^2 + K_e{}^2)\langle v, v_e\rangle = -[v_{e2}\partial h/\partial x - Q(v_{e2}/U_2 - v_{e1}/U_1)]/2. \quad (2.10)$$

The expression multiplying Q in (2.10), $Q_{\rm eff} = (v_{e2}/U_2 - v_{e1}/U_1)$, is the two-layer analogue of the heating efficiency factor $-\partial(v_e U^{-1})/\partial z$ in continuous models (see HPP1). The sign of $Q_{\rm eff}$ is the same as that of $(U_1/U_2 - \alpha_e)$, and hence of U_1/U_2; for positive shears and surface winds $Q_{\rm eff}$ is positive, as in the continuous models.

Equation (2.8) takes the form of the barotropic vorticity equation linearized about the "equivalent barotropic wind" $U_e = K_e{}^{-2}$. This is the wind which yields the correct stationary wavelength when substituted into the familiar Rossby formula ($U_e = \beta K_e{}^{-2}$ in dimensional form). By averaging the two equations in (2.1), one can show that

$$\left.\begin{aligned} U_e &= (\psi_{e1}U_1 + \psi_{e2}U_2)/(\psi_{e1} + \psi_{e2}) \\ &= U_2 + \hat{U}(\alpha_e - 1)/(\alpha_e + 1) \end{aligned}\right\}. \quad (2.11)$$

For supercritical shears, α_e is larger than 1, so that U_e is greater than the average wind $\bar{U}$, consistent with the upper tropospheric equivalent winds found in HPP1. As U_2 approaches 0 with fixed supercritical shear, U_e approaches U_1.

3. A modified Charney profile

It is natural to ask whether the behavior of the group velocity of stationary waves in the two-layer model is shared by any continuous models. We demonstrate that it is: analogous behavior is found in a model obtained by modifying the Charney profile so as to remove the temperature gradient near the surface.

We work in log-pressure coordinates. Writing the mean state in the form $U_0 + U(z)$, with $U(0) = U_0$ and $U'(0) = \Lambda$, and assuming that N^2 is constant, we can nondimensionalize (x, y, z, U) by $(NH/f_0, NH/f_0, H, \Lambda_0 H)$ where H is the scale height, and put the potential vorticity gradient in the form

$$Q_y = r + U'(z) - U''(z) - 2U'(z)\delta(z)$$

where $r = \beta N^2 H/(f_0{}^2\Lambda_0)$. The eigenvalue problem takes the form $L\eta = K^2\eta$, where $\eta = e^{-z/2}\psi$ must vanish at great heights, and $L = d^2/dz^2 - [1/4 - Q_y/U + \delta(z)]$.

The profile which we will refer to henceforth as the "modified Charney profile" is given in nondimensional form by

$$U(z) = \begin{cases} \kappa z^2/2 + U_0, & z < z_c = \kappa^{-1} \\ z - (2\kappa)^{-1} + U_0, & z \geq z_c, \end{cases}$$

(see Fig. 4). For this profile, the δ-function term in Q_y vanishes because there is no shear at the ground, and instead we have

$$Q_y(z) = \begin{cases} r - \kappa + \kappa z, & z < z_c \\ 1 + r, & z \geq z_c. \end{cases}$$

Lindzen and Farrell (1980) discussed a similar modification of a piecewise linear profile, inserting a region near the ground of constant Q_y rather than constant curvature. There is now a critical parameter for instability, for if $\kappa < r$, or equivalently if $z_c > 1/r$, the curvature is sufficiently spread that Q_y is everywhere positive. As z_c shrinks to zero the Charney profile is recovered (as in HPP1, we ignore the "non-Doppler" term in the lower boundary condition). Figure 5 shows the convergence, for fixed $r = 0.5$, of the modified external mode dispersion relation to the Charney relation as z_c approaches 0. (For each U_0, the eigenvalue problem is solved by a combination of shooting and New-

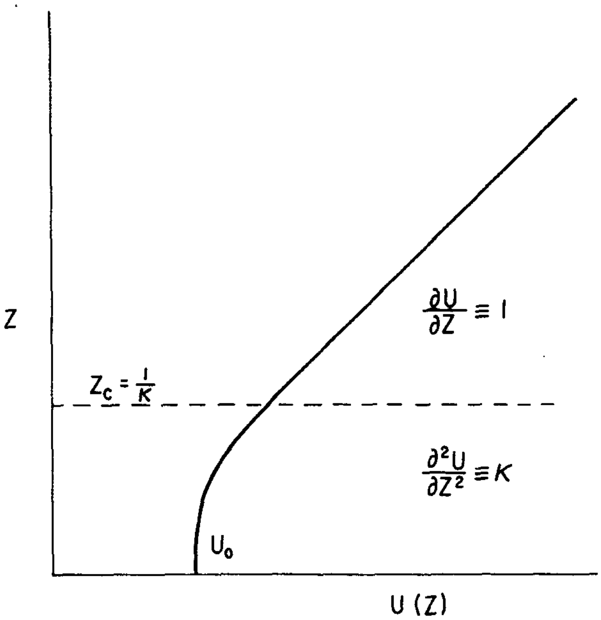

FIG. 4. The modified Charney profile.

ton's method similar to that described in HPP1). As the figure shows, for $z_c > 0$ the external mode has similar behavior for small surface winds to that seen in Fig. 1b: there is a region of negative group velocity which shrinks as z_c approaches 2, the critical value for instability when $r = 0.5$. All of the internal modes are horizontally trapped for $r = 1$ in the Charney model, and this is also true in all the cases shown in the figure except $z_c = 1.7$. Unlike the two-layer case, the transition

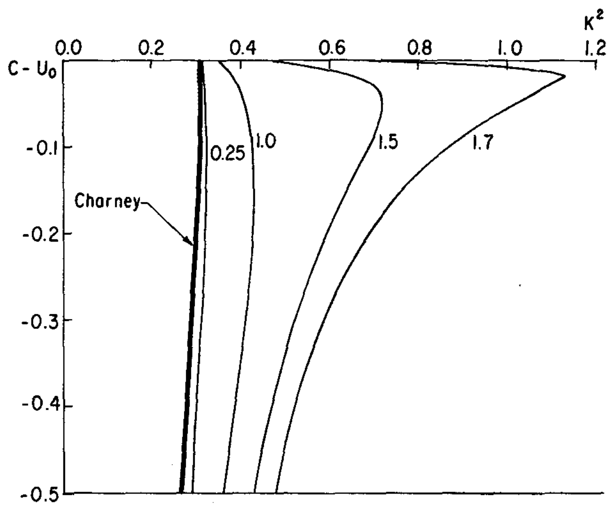

FIG. 5. The convergence of the external mode dispersion relation for the modified profile to the external mode (bold curve) for the Charney profile, as z_c approaches zero. Wavenumbers are in units of $(NH/f_0)^{-1}$ and surface winds in units of ΛH. The nondimensional beta parameter r is 0.5 in all cases.

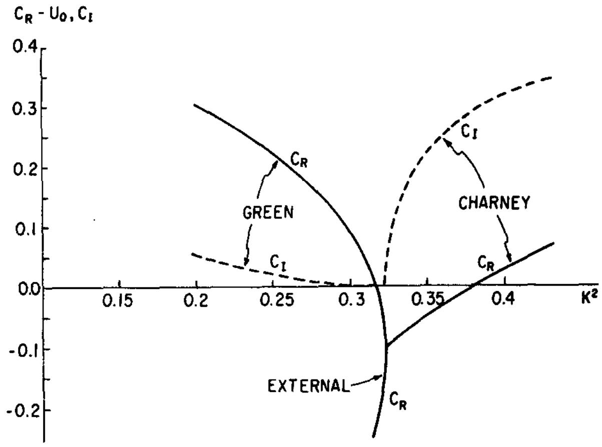

FIG. 6. The external Green, and Charney modes near the branch point for the modified Charney profile with $z_c = 0.25$, $r = 0.5$. Relative phase speeds $c_R - U(0)$ are shown with solid lines, and values of c_I with dashed lines; speeds are in units of ΛH.

from a trapped to propagating first internal mode does not occur exactly at the critical z_c, but at a smaller value: when $r = 0.5$, the first internal mode is trapped only for $z_c \leqslant 1.65$. Further calculations show that the negative group velocity is still present if a small amount of shear at the ground is allowed. Similarly, the jump discontinuity in Q_y can be removed by using smoother profiles.

For the two-layer dispersion relation, an instability emerges just at the point where $\partial K^2/\partial U_0 = 0$. Similar behavior occurs here; Fig. 6 shows the dispersion relation for the modified profile near the point in question when $z_c = 0.25$ (and $r = 0.5$). Denoting by c_B the phase speed at which the external mode branches into the Green and Charney modes, one finds it clear that the effect of the modification has been to move c_B below $U(0)$ (compare Fig. 1 of HPP2). Figure 6 also illustrates the remark of Miles (1964) that for the Charney profile the Green mode seems at its onset to be qualitatively different from the Charney mode. Here we see that the modified Charney mode still becomes unstable at the bifurcation point, while the Green mode becomes unstable only as it encounters a critical layer. The figure further suggests that only the part of the two-layer external mode dispersion relation with positive pseudomomentum exhibits behavior parallel to that of the external mode in the Charney model. The part with negative pseudomomentum is more akin to the unstable Green mode.

We know from the work of Geisler and Dickinson (1975) that inclusion of the so-called "non-Doppler" term in the lower boundary condition in Charney's model also introduces a very small region of instabilities without critical layers; i.e., $c_B - U(0)$ becomes slightly negative. It is now clear that the "non-Doppler" term is not required for $C_B - U(0)$ to be negative.

A curious consequence of the change in sign of the pseudomomentum, or group velocity, of neutral modes is the possibility that meridionally propagating waves may be overreflected even in the absence of critical layers. As an example, consider the following (admittedly somewhat contrived) two-layer scattering problem: north and south of a shear zone the basic state winds are asymptote to positive constants, but to the north, the lower-layer wind, U_2, is sufficiently large that the pseudomomentum P of the external mode is positive, while to the south it is small enough that P is negative. The vertical shear is a fixed supercritical value and all nonconservative effects are assumed absent. Numerical calculations show that a stationary external mode wave train incident from the north is overreflected from the shear zone, with the overreflection increasing as the meridional shear is reduced. This result may be understood as follows. Any steady solution must have the property that the convergence of the pseudomomentum (or Eliassen–Palm) flux is everywhere zero, and as the external mode is the only horizontally wavelike solution, far from the shear zone there will be only transmitted and reflected external waves. Because the transmitted wave carries away negative pseudomomentum, to avoid a convergence of pseudomomentum flux the reflected wave must export more positive pseudomomentum than imported by the incident wave; overreflection must occur. Furthermore, as the meridional shear becomes more gradual, the overreflection increases. It becomes infinite in the limit that a WKB approximation is valid, for then the change in sign of group velocity means that a single WKB wavefunction can represent both the transmitted and reflected waves, with no incident wave present. A similar argument can clearly be made for the modified Charney model.

4. Dissipative destabilization

We turn now to the effects on the external mode of various "dissipative" mechanisms. To study these effects we introduce positive parameters γ_T, γ_{mj}, γ_{pj} ($j = 1, 2$) for thermal, mechanical, and potential vorticity damping, respectively, and modify (2.1) to

$$\partial q'_j/\partial t + U_j \partial q'_j/\partial x + Q_{jy}\partial\psi'_j/\partial x = (-1)^{j+1}\gamma_T\hat{\psi}' - \gamma_{pj}q'_j - \gamma_{mj}\nabla^2\psi'_j \quad (4.1)$$

where $\hat{\psi}' = (\psi'_1 - \psi'_2)/2$ and $j = 1, 2$. For mechanical damping, we restrict attention to a lower Ekman layer ($\gamma_{m1} = 0$, $\gamma_{m2} = \gamma_{EK} \geq 0$).

We consider modes with space–time dependence $\exp[i(kx + ly - \omega t)]$, hold the meridional wavenumber at a constant real value for simplicity, and examine the behavior of $\omega = \omega_R + i\omega_I$ as a function of real k and γ (temporal destabilization) and the behavior of $k = k_r + ik_I$ as a function of real ω and γ (spatial destabilization). The latter is relevant for the effects of dissipation on the amplitude of stationary ($\omega = 0$) wave trains.

Using overbars to denote zonal averages, (4.1) yields a conservation law analogous to (4) in HPP2:

$$\partial P/\partial t = M - \overline{(\gamma_{p1}q'_1 - \gamma_T\hat{\psi}')q'_1}/Q_{1y} - \overline{(\gamma_{EK}\nabla^2\psi'_2 + \gamma_{p2}q'_2 + \gamma_T\hat{\psi}')q'_2}/Q_{2y} \quad (4.2)$$

where M is the momentum flux convergence $-\partial(\overline{u'_1v'_1} + \overline{u'_2v'_2})/\partial y$, and P is the pseudomomentum. For normal modes on a horizontally uniform flow, M vanishes and $\partial P/\partial t = 2\omega_I P$, so if we fix k and focus on one dissipative process with strength proportional to γ, (4.2) is of the form

$$2\omega_I(\gamma)P(\gamma) = \gamma D(\gamma)$$

where D is the quadratic quantity on the right-hand side of (4.2) appropriate to the choice of γ. Inviscid unstable modes as well as marginally stable neutral modes have zero pseudomomentum. For modes with nonzero P, expanding about a neutral point we have to lowest order in γ

$$(\partial\omega_I/\partial\gamma)_{\gamma=0} = D(0)/[2P(0)]. \quad (4.3a)$$

To establish the connection with the spatial destabilization problem, we note that implicit differentiation of the dispersion relation $\omega = \Omega(k, \gamma)$ gives

$$-(\partial\Omega/\partial k)_\gamma(\partial k/\partial\gamma)_\omega = (\partial\Omega/\partial\gamma)_k$$

(cf. Gaster, 1962). But the first expression on the left evaluated at $\gamma = 0$ is just the zonal group velocity of the neutral mode, so taking imaginary parts results in

$$-G_x(\partial k_I/\partial\gamma)_{\gamma=0} = (\partial\omega_I/\partial\gamma)_{\gamma=0}. \quad (4.3b)$$

Since spatial growth in the direction of propagation occurs when k_I and G_x have opposite signs, the conclusion from (4.3a, b) is that positive $D(0)/P(0)$ implies spatial as well as temporal destabilization.

For the case of lower-layer Ekman pumping we have $D(\gamma_{EK}) = \overline{q_2\nabla^2\psi_2}/Q_{2y}$. For a stationary neutral mode, $\nabla^2\psi_2 = -K^2\psi_2$ and $q_2/Q_{2y} = -\psi_2/U_2$, so that

$$\partial\omega_I/\partial\gamma_{EK} = -K^2\overline{\psi_2^2}/(2PU_2),$$

$$\partial k_I/\partial\gamma_{EK} = -\overline{\psi_2^2}/(4U_2),$$

assuming $P \neq 0$. Therefore, a mode is destabilized if PU_2 is negative (in a frame of reference moving with the wave) and stabilized if PU_2 is positive. Figure 7a, b shows the propagating neutral modes that are destabilized for subcritical and supercritical shears. Recalling that the sign of P is the same as the slope of the dispersion curve, we see that both modes in the subcritical case have positive P, so that those modes propagating westward with respect to the lower layer wind ($U_2 > 0$ in the figure) are stabilized, while those with phase speeds greater than the lower-layer wind ($U_2 < 0$) are destabilized. In the supercritical case, the external mode is stabilized when U_2 is large, but for small U_2, the change in sign of P results in destabilization. Short waves propagating eastward with respect to the lower-

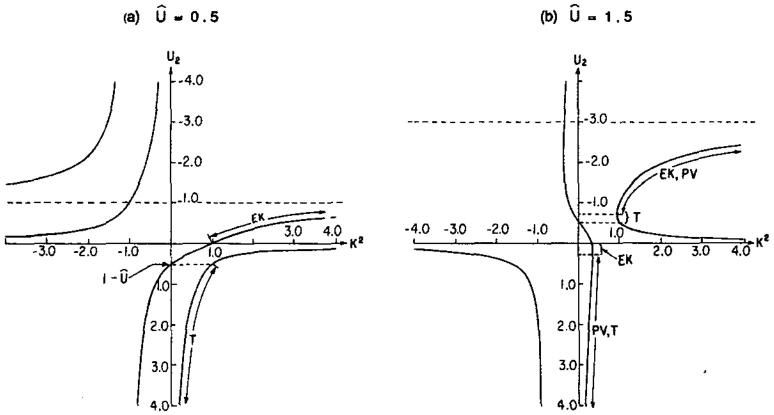

FIG. 7. The parts of the two-layer dispersion relation for which the pseudomomentum argument predicts temporal destabilization by Ekman (EK), potential vorticity (PV), and thermal (T) damping. (a) $\hat{U} = 0.5$, (b) $\hat{U} = 1.5$.

layer wind are also destabilized. There is no analogous destabilization of neutral modes by Ekman pumping in Charney's model, since there are no stationary neutral modes with P of the same sign as the surface wind. (However, there would be destabilization near the long-wave cutoff in the modified Charney model of section 3.)

From Fig. 7a, b, one can conclude that the addition of weak Ekman pumping causes the disappearance of both the critical shear for instability and the short-wave cutoff, but that a long-wave cutoff is retained. Figure 8a shows the growth rates for $\gamma_{Ek} = 0.1$ and $\hat{U} = 1.5$ after subtracting the inviscid rates. The destabilization at both the short- and long-wave inviscid cutoffs ($K^2 = 0.36, 0.93$) is evident, with that at short wavelengths continuing as $K \to \infty$.

Turning to potential vorticity damping, one can verify that if the damping is the same in each layer, then $\partial\omega_I/\partial\gamma = -1$, as would be expected on the basis of (4.1). But if we consider potential vorticity damping confined to the lower layer ($\gamma_{p2} > 0$; $\gamma_{p1} = 0$), in analogy to the low-level damping considered in HPP2, we find significant destabilization. In this case, (4.3a) again becomes (with nonzero P)

$$\partial\omega_I/\partial\gamma_{p2} = -\overline{q_2^2}/(2PQ_{2y}). \qquad (4.4)$$

Here, waves are destabilized when the product PQ_{2y} is negative. For subcritical shears this cannot happen, since Q_{2y} is positive, as is the pseudomomentum for each mode. Thus, the minimum critical shear for instability is unchanged by the addition of small lower-layer potential vorticity damping. For supercritical shears, both the short and the long-wave cutoffs are removed, as shown in Figs. 7b and 8b. Most importantly, for stationary wave theory, the external mode is destabilized when $P > 0$. As a result of low-level potential vorticity damping, such as might be generated by baroclinic transients, a stationary wave will amplify in the direction of its group velocity (as long as the lower-layer wind is not so small that $P < 0$).

Figure 9 is a plot of $-k_I$ as a function of γ_{p2} and U_2 for $\hat{U} = 1.5$. The maximum value of $k_I = -0.084$ is reached with $\gamma_{p2} = 0.24$ and $U_2 \to 0$. The nondimensional wavelength $\lambda = 2\pi/k$ of this stationary mode is 9, giving a maximum value of $|\lambda k_I| = 0.80$.

For thermal damping, we find

$$\partial\omega_I/\partial\gamma_T = \psi_2^2(\alpha - 1)(U_1/U_2 - \alpha)/(4U_1P). \qquad (4.4c)$$

The modes that are destabilized are shown in Fig. 7, as can be verified using the values of α shown in Fig. 1 and the remarks in section 2 on the sign of $Q_{eff} \propto (\alpha - U_1/U_2)$. A critical shear required for instability no longer exists due to the destabilization of the external mode when $U_2 > 1 - \hat{U}$, i.e., when winds are large enough to trap the internal mode. In the supercritical case, the external mode is destabilized when $P > 0$, and a small region near the short-wave cutoff is also destabilized. Figure 8c shows the corresponding change in growth rates for $\gamma_T = 0.1$. There is no long-wave cutoff in either the subcritical or supercritical case, while the short-wave cutoff is shifted to $K = 1$.

5. Conclusions

The two-layer model's success in modeling stationary responses to localized sources depends to a great extent on the number of horizontally propagating modes, the structure of these modes, and the sign of their group velocity. When the shear is supercritical, the internal stationary mode is horizontally trapped, and there is no need to be concerned with contamination of the far-field response to localized forcing; the external mode will dominate the far-field, as it does in continuous models for typical midlatitude conditions. On the other hand, if the shear is subcritical and the lower-layer wind is small enough that the mean wind $(U_1 + U_2)/2$ is less

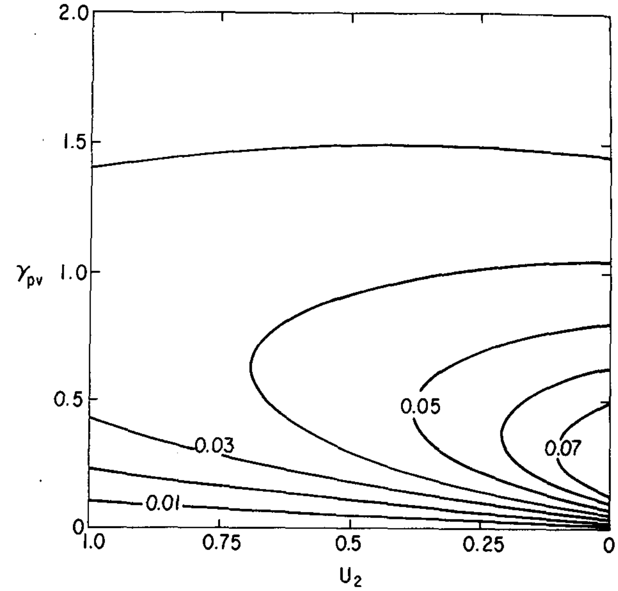

FIG. 9. Spatial destabilization by potential vorticity mixing; level curves of $-k_I$, regarded as a function of γ_{pv} and the lower-layer wind required to make the mode stationary.

than $\beta\lambda^2$, the internal mode can propagate horizontally. Even this horizontally propagating internal mode need not be considered artificial in all cases; internal modes trapped by tropospheric winds are also a possibility in continuous models if the shear is weak (but not too weak)—see Fig. 2c.

A source of error due to the structure of the external mode when the shear is subcritical is that as the lower-layer wind approaches zero, the mode becomes bottom heavy and develops arbitrarily small scales, unlike behavior in either the supercritical case or in continuous models. This can result in serious distortion as stationary external waves propagate equatorward to the low-level transition from mean westerlies to mean easterlies.

None of the continuous models considered in HPP1 had modes with negative pseudomomentum. However, the two-layer model's external mode does have negative pseudomomentum when the shear is supercritical and the lower-level wind is smaller than a value U_B dependent on the shear. Due to the corresponding change in sign of the group velocity, forced responses will show upstream propagation. This behavior has no analogue in the Charney model, and although it can be reproduced in continuous models by decreasing the surface temperature gradients near the ground, it appears to be undesirable.

When the shear is supercritical, the dispersion relation for the two-layer model in its broad character

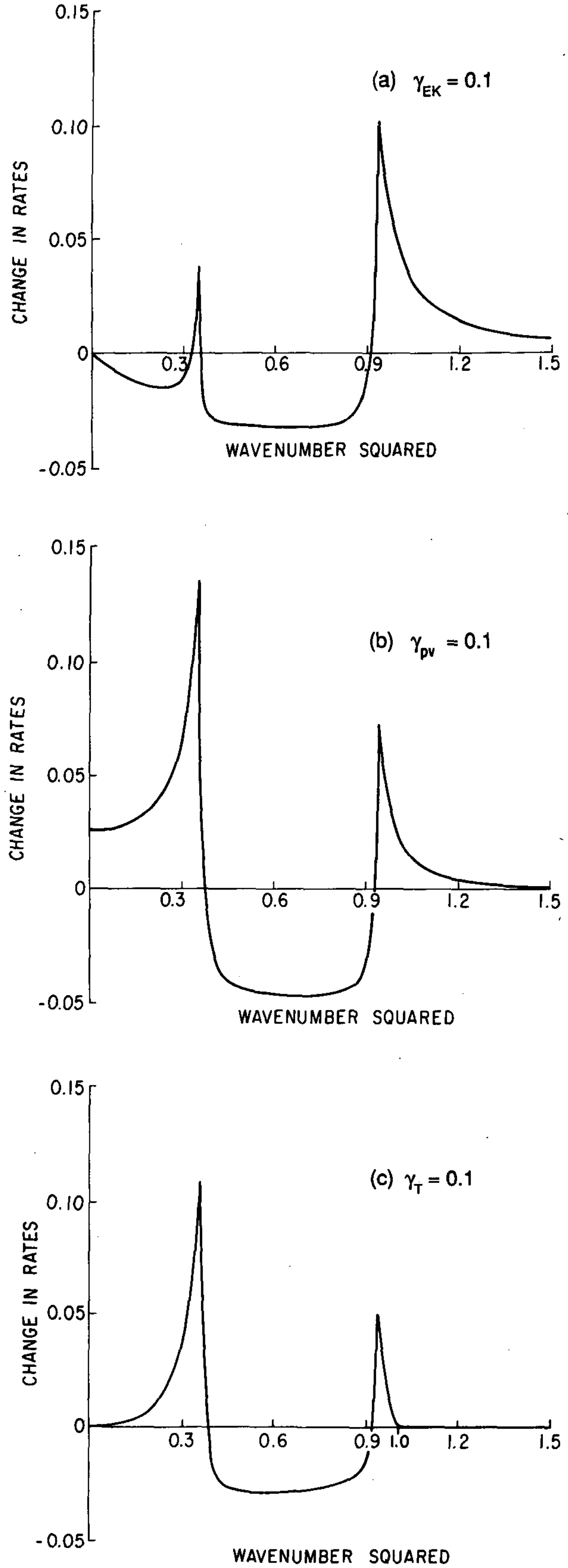

FIG. 8. (a) Temporal destabilization for $\hat{U} = 1.5$ and $\gamma_{Ek} = 0.1$; viscid minus inviscid rates; (b) as in (a) but for potential vorticity damping γ_{pv}; (c) as in (a), but for thermal damping γ_T.

resembles the Charney problem and the modified Charney problem discussed in section 3. Each has an analogue of the external, Green, and Charney modes: the external mode branches at a phase speed c_B into the Green and Charney modes. It happens that c_B is smaller than the lower-layer wind in the two-layer model, equal to the surface wind in Charney's model (ignoring the non-Doppler term), smaller than the surface wind in the modified Charney model, and larger than the surface wind in the hyperbolic tangent profile of Fig. 2c. For phase speeds less than c_B, the external modes in the two-layer and Charney models behave in a completely analogous manner. The continuation of the external mode under various forms of damping into the exponentially growing or decaying Charney mode is qualitatively identical. The external modes in each model are stabilized (both temporally and spatially) by Ekman pumping, and destabilized by thermal and low-level potential vorticity damping.

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