The *Two-Fifths* and *One-Fifth* Rules for Rossby Wave Breaking in the WKB Limit

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ABSTRACT

A stationary Rossby wave, sinusoidal in longitude, is slowly switched on, and the meridional propagation of the resulting wave front through a shear flow is examined. Initially the flow is westerly everywhere and therefore free of critical layers. The transition from reversible to irreversible behavior as the wave amplitude is increased is described. It is shown that under slowly varying conditions in an inviscid quasi-linear model, a steady state is obtained if, and only if, the mean flow is decelerated by less than *two-fifths* of its initial value as a result of the passage of the wave front. If this passage causes a larger mean flow reduction, a pile-up of wave activity in the shear layer culminates in the generation of a critical layer, qualitatively as in Dunkerton's model of gravity wave–mean flow interaction. This qualitative picture is shown to be preserved in the quasi-linear model when the slowly varying assumption breaks down.

Fully nonlinear calculations show that these quasi-linear results are only part of the story. Once the mean flow is decelerated by two-fifths of its initial value in the fully nonlinear model, rapid wave breaking and irreversible mixing occur in the shear layer. But more slowly developing wave breaking also occurs for wave amplitudes that are too small to produce the two-fifths deceleration. Overturning of contours can be shown to occur in the quasi-linear slowly varying model once the mean flow has been decelerated by *one-fifth* of its initial value, and this appears to be the critical value for wave breaking to occur in the nonlinear integrations.

1. Introduction

Imagine a very small amplitude stationary Rossby wave propagating equatorward through a mean flow on which the wave has no critical layers. Think of the wave as being generated at a northern boundary by slowly switching on a sinusoidal corrugation, as shown in Fig. 1. Under these conditions the wave front will travel at some well-defined group velocity. Behind this front, a steady wave will be superimposed on a mean flow which has been decelerated slightly. This mean flow modification is reversible (in the absence of dissipation); if the source is turned off, the mean flow will return to its initial value as the wave activity relaxes to zero.

Now suppose that the initial wave amplitude is increased to the point that this linear picture breaks down. One possibility is that the linear picture does not break down qualitatively until the wave decelerates the mean flow to zero, creating a critical layer. Irreversible mixing would certainly follow once a critical layer formed. (By "irreversible" in this context we simply mean the rapid generation of an enstrophy cascade to small scales where scale-selective diffusion is assumed to exist.) Given this scenario, it would be useful to know just how strong the wave, or how weak the initial mean wind, need be for a critical layer to be induced. With this question in mind, we analyze a slowly varying and inviscid, quasi-linear (wave-mean flow interaction) model. This part of our study can be thought of as the Rossby wave analogue to Dunkerton's (1981) analysis for internal gravity waves. We also use a numerical quasi-linear model to investigate cases in which the slowly varying assumption is not accurate.

Conclusions based on quasi-linear models are necessarily tentative. Therefore, we integrate a fully nonlinear model and watch for deterioration of the quasi-linear predictions due to wave-wave interactions. We ask if wave-wave interactions come into play before the appearance of a critical layer and if the *onset* of irreversibility in the fully nonlinear model is distinct from that in the quasi-linear model. All of our calculations are inviscid in the region of the shear flow. Only the initial behavior following the passage of the wave

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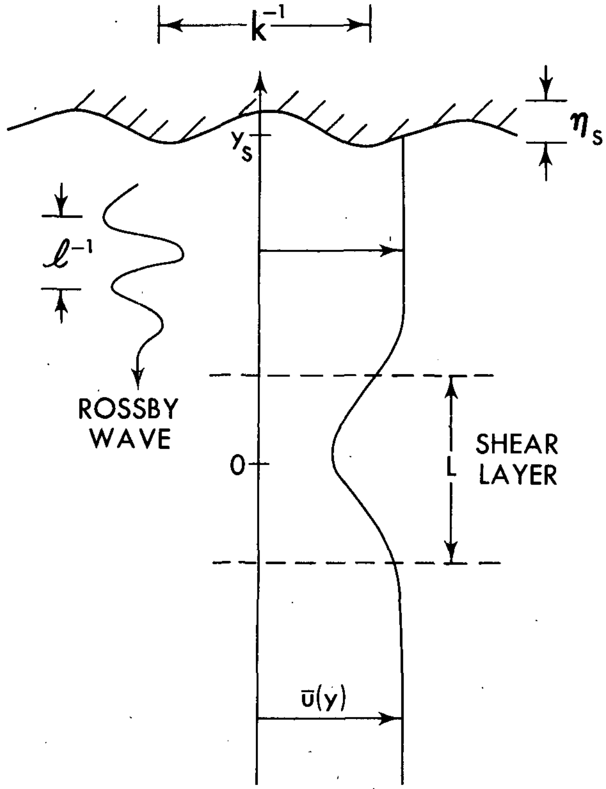

FIG. 1. Schematic of a Rossby wave forced at the northern boundary and propagating equatorward into a shear zone.

front is considered; no attempt is made to follow the solution through the breaking stage.

The plan for the paper is as follows. The mathematical framework is described in section 2. In sections 3 and 4 we present some quasi-linear results in which the slowly varying assumption is first invoked and then relaxed. In section 5, we consider full nonlinearity.

2. Mathematical formulation

As shown schematically in Fig. 1, we consider a two-dimensional non-divergent Rossby wave propagating meridionally through a stable mean zonal flow $U(y)$, $-\infty < y \leq y_s$. The wave may be regarded as generated by flow past a slowly switched-on sinusoidal corrugation at $y = y_s$. This configuration approximates the nearly horizontal propagation of large-scale stationary waves into the tropics.

Given a nondimensional total streamfunction Ψ:

$$\Psi(x, y, t) = -\int \bar{u}(y, t)dy + \epsilon\psi(x, y, t), \quad (1)$$

we obtain predictive equations for the perturbation vorticity $\zeta = \psi_{yy} + \delta\psi_{xx}$ and the mean zonal flow $\bar{u}$:

$$\zeta_t + \bar{u}\zeta_x + \gamma v + \epsilon[J(\psi, \zeta) - \overline{J(\psi, \zeta)}] = 0, \quad (2a)$$

$$\bar{u}_t = -\epsilon^2(\overline{uv})_y \quad (2b)$$

where $u = -\psi_y$, $v = \psi_x$, and $\gamma = \beta - \bar{u}_{yy}$. Dimensionless quantities are related to their dimensional (primed) counterparts through

$$x = kx', \quad y = y'/L, \quad t = kU_st', \quad \bar{u} = \bar{u}'/U_s,$$
$$\psi = \psi'/\phi_s, \quad \beta = \beta'L^2/U_s,$$
$$\epsilon = \phi_s/LU_s, \quad \delta = k^2L^2 \quad (3)$$

where U_s is the initial value of the wind at the northern boundary y_s, $\phi_s = \eta_sU_s$ is the dimensional amplitude of the forced wave at y_s, k is the zonal wavenumber, L is the length scale for the zonal mean shear, while ϵ and δ are the nondimensional measures of the strength and zonal length scale of the wave, respectively. An overbar refers to the zonal mean.

The initial condition and northern boundary condition are

$$\Psi = -\int U(y)dy, \quad t = 0, \quad (4)$$

$$\psi = F(t)\cos(x), \quad y = y_s \quad (5)$$

where $F(t)$ is a monotonic switch-on function such that $F(t \leq 0) = 0$ and $F(t \geq T) = 1$ with $T \gg 1$. We assume that the mean flow is constant for some distance south of the source, before the shear zone is encountered. Under these conditions a southward propagating wave is generated which near y_s has a characteristic meridional wavenumber $|l| = [(\gamma/\bar{u}) - \delta]^{1/2}$ and group velocity $G = -2\gamma|l|/(l^2 + \delta)^2$. The switch-on implies amplitude modulation over a distance $|G|T$.

3. A slowly varying quasi-linear model

We first simplify the problem arbitrarily by allowing the wave to modify $\bar{u}$ through the eddy momentum flux while neglecting the generation of higher harmonics [i.e., we ignore the bracketed term in (2a)]. For such a "quasi-linear" model,

$$(2\gamma)^{-1}\epsilon^2\partial_t\overline{\zeta^2} = -\bar{u}_t. \quad (6)$$

If the fractional change in γ is negligible, we obtain the familiar result

$$A(y, t) \equiv \epsilon^2\overline{\zeta^2}/(2\gamma) = U(y) - \bar{u}(y, t), \quad (7)$$

where A is the "wave activity" [$A(y, 0) = 0$]. This approximation is certainly valid if the wave amplitude is sufficiently small, but it can also remain valid for O(1) changes in $\bar{u}$ if the fractional change in γ remains small. Assuming that the mean flow modification occurs on the scale of the mean flow itself, as is the case in the WKB solution described below, then

$$\frac{\Delta\gamma}{\gamma} = \frac{\bar{u}_{yy}}{\beta - \bar{u}_{yy}}\frac{\Delta\bar{u}_{yy}}{\bar{u}_{yy}} \approx \frac{\bar{u}_{yy}}{\beta - \bar{u}_{yy}}\frac{\Delta\bar{u}}{\bar{u}} \quad (8)$$

where Δ denotes the change due to the passage of the front. To justify the neglect of changes in γ when $\Delta\bar{u}/$

$\bar{u}$ is order unity, we must evidently assume that $\bar{u}_{yy} \ll \beta$.

We now assume that the meridional wavelength is much smaller than the width of the shear layer, so that the WKB approximation can be applied. Assuming slow amplitude modulation in space and time, we have

$$A_t + (GA)_y = 0 \tag{9}$$

where G is the local meridional group velocity. (Note that the assumption $\gamma \approx \beta$ is also required to formally justify the WKB approximation; if $\bar{u}_{yy}$ were of the same order as β, then $l^2 \approx \gamma/\bar{u} \approx L^{-2}$, which would be inconsistent with the assumption of scale separation.) If the passage of the wave front leaves in its wake a steady wave on a modified mean flow, then GA is independent of y behind the front, where

$$G = -2\gamma|l|^{-3} = -2\gamma^{-1/2}\bar{u}^{3/2} \tag{10}$$

Here $l = (\gamma/\bar{u})^{1/2}$ is the local meridional wavenumber, and we have made the long wave (small δ) approximation.

Thinking of the flux of wave activity near the source G_sA_s as being prescribed, then

$$A(y) = \frac{G_sA_s}{G(y)} \propto \bar{u}^{-3/2}. \tag{11}$$

The incident flux G_sA_s is a function of ϵ. For a given $\epsilon = \epsilon_1$, we plot in Fig. 2 the relationship (11) in the steady state between A and $\bar{u}$. We also plot in the same

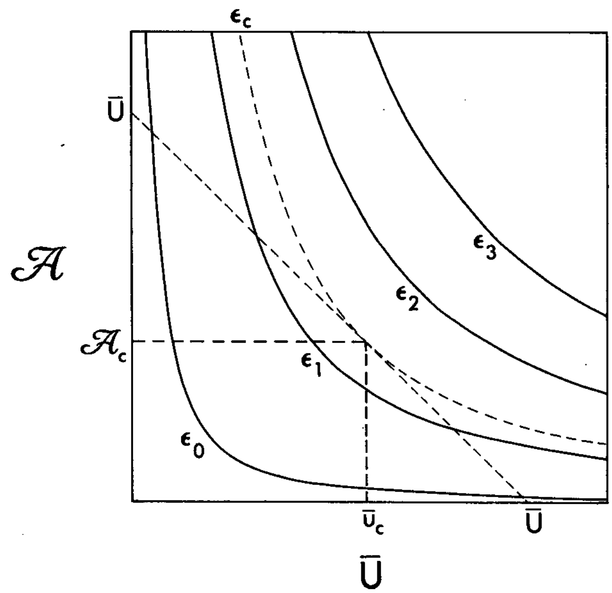

FIG. 2. Graphical derivation of A_c and $\bar{u}_c$. The curved lines represent the relationship between A and $\bar{u}$, based on slowly varying theory, that must exist in a steady state, for different forcing amplitudes ϵ. The straight line represents the relation between A and $\bar{u}$ that exists at all times for a flow that evolves from the initial wind U. The dashed curve marks the critical amplitude ϵ_c above which no steady states can evolve from the initial wind U.

figure the linear relation (7) between A and $\bar{u}$ that must be satisfied by any solution that evolves from the given initial mean flow U. Intersections between these two curves represent possible steady states for the given ϵ and U. For $\epsilon = \epsilon_1$, there are two such intersections in the figure but only the one with smaller A is accessible from an initial state with $A = 0$ and $\bar{u} = U$.

By plotting a series of such curves for different values of ϵ, it becomes evident that there exists a critical $\epsilon = \epsilon_c$ beyond which no steady states exist for the given U. Corresponding to this critical value, there is a maximum steady-state wave activity A_c and minimum mean zonal flow $\bar{u}_c$. These critical values are found by equating the $\bar{u}$-derivatives of A in Eqs. (7) and (11). From (7), $\partial A/\partial\bar{u} = -1$; while from (11), $\partial A/\partial\bar{u} = -3A/(2\bar{u})$. Therefore, $A_c = 2\bar{u}_c/3$, so that

$$A_c = 2U/5 \quad \text{and} \quad \bar{u}_c = 3U/5. \tag{12}$$

From this simple analysis we arrive at the interesting result that *steady flow is impossible if the mean wind is reduced by more than two-fifths of its initial value.*

The physical explanation for this behavior is as follows. The passing front causes a mean flow deceleration which acts to reduce the group velocity according to (10). In a shear layer, the deceleration and the reduction in group velocity are nonuniform, being largest in regions with the smallest initial winds. This causes the wave activity to pile up in such regions; indeed, for a steady state to emerge, the wave activity must increase proportionately to the decrease in group velocity so that there be no prolonged convergence of wave activity into the shear layer. However, the increase in wave activity again reduces the mean winds and the group velocity. If the "two-fifths rule" is violated, that is, if the mean flow falls below $u_c = 3U/5$, this process does not converge; instead the slowly varying theory predicts that the mean flow deceleration and the pile-up of wave activity will continue until some other dynamics intervenes, e.g., critical layer formation.

It remains to compute the critical forcing amplitude ϵ_c. Assume that the flow does reach a steady state behind the wave front. Near the source, the wave activity is then

$$A_s = \frac{\epsilon^2\overline{q_s^2}}{2\gamma_s} = \frac{\epsilon^2\gamma_s\overline{\psi_s^2}}{2\bar{u}_s^2} = \frac{\epsilon^2\gamma_s}{4\bar{u}_s^2}. \tag{13}$$

Here $\psi_s = \cos(x)$, so that $\overline{\psi_s^2} = 1/2$. Also, q_s is the eddy potential vorticity at $y = y_s$, so that $\bar{u}_s q_s = -\gamma_s\psi_s$. If the critical conditions are reached somewhere in the shear flow, then

$$A_s = A_cG_c/G_s = A_c(\bar{u}_c/\bar{u}_s)^{3/2}(\gamma_s/\gamma)^{1/2}$$
$$= (2/5)(3/5)^{3/2}U^{5/2}\bar{u}_s^{-3/2}(\gamma_s/\gamma)^{1/2}. \tag{14}$$

If the mean flow modification near the source is negligible, so that $\bar{u}_s \approx U_s$, we can evaluate A_s from (14) and substitute into (13) to obtain ϵ. This computation can be performed using the zonal flow U for each latitude; the value of ϵ_c is then determined by minimizing the resulting values of ϵ over latitude. Ignoring the variation in γ, critical conditions will always be reached first at the latitude with the smallest initial mean wind.

If the mean flow modification near the source is not negligible, then $\bar{u}_s = U_s - A_s$, and ϵ_c can be determined by solving

$$x(1-x)^{3/2} = (2/5)(3/5)^{3/2}(U/U_s)^{5/2}(\gamma_s/\gamma)^{1/2} \quad (15)$$

for $x \equiv A_s/U_s$, and then substituting into (13), i.e.,

$$\epsilon_c{}^2 = 4U_s{}^3\gamma_s{}^{-1}x(1-x)^2. \quad (16)$$

4. A numerical quasi-linear model

We use a numerical model to determine what happens in cases for which steady solutions are impossible. The initial zonal mean flow is given the form

$$U(y) = U_0 + (1 - U_0)\tanh^2(y) \quad (17)$$

with $U_0 = 0.5$. We also set $\beta = 5$ and $\delta = 0.16$. This flow is barotropically stable and there are no critical levels. The initial profile is given by the dotted line in Fig. 3b.

An absorbing "sponge" layer is introduced in the region $-3y_s < y < -y_s$ (with $y_s = 5$) by adding linear damping in the eddy vorticity tendency. More specifically, we add the term $-\lambda\zeta$, where $\lambda(y) = \sin^2[\pi(y + y_s)/4y_s]$. This damping of the wave produces a mean flow deceleration:

$$\bar{u}_t = -\epsilon^2(\overline{uv})_y = -A_t - \lambda\epsilon^2\overline{\zeta^2}/\gamma. \quad (18)$$

The second term on the right-hand side of (18) is undesirable because it prevents a steady state from developing. Therefore, we suppress this term in our calculations. The deceleration due to the wavefront $(-A_t)$ is retained within the sponge to ensure a smooth mean flow profile at the boundary of the sponge layer.

Using a standard (second-order accurate) grid point numerical model, we compute the evolution for various values of the forcing parameter ϵ. The boundary forcing is turned on slowly according to the expression

$$F(t) = \begin{cases} 0, & t \leq 0; \\ \sin^2[\pi t/(2T)], & 0 < t < T; \\ 1, & t \geq T, \end{cases} \quad (19)$$

with $T = 80$. The spectrum of the frequencies excited by the forcing is centered at zero and has a width proportional to T^{-1}; since the nondimensional zonal wavenumber is $\delta^{1/2}$, the corresponding phase speed spectrum has width $\approx (T\delta^{1/2})^{-1} = (0.4T)^{-1}$. Our choice for T ensures that the excitation of waves that have critical layers in the shear flow is entirely negligible. If T is small enough that phase speeds larger than the minimum value of $U(y)$ are significantly excited, then the flow evolution is irreversible even for infinitesimal forcing, in the sense that smaller and smaller scales are generated as the integration proceeds.

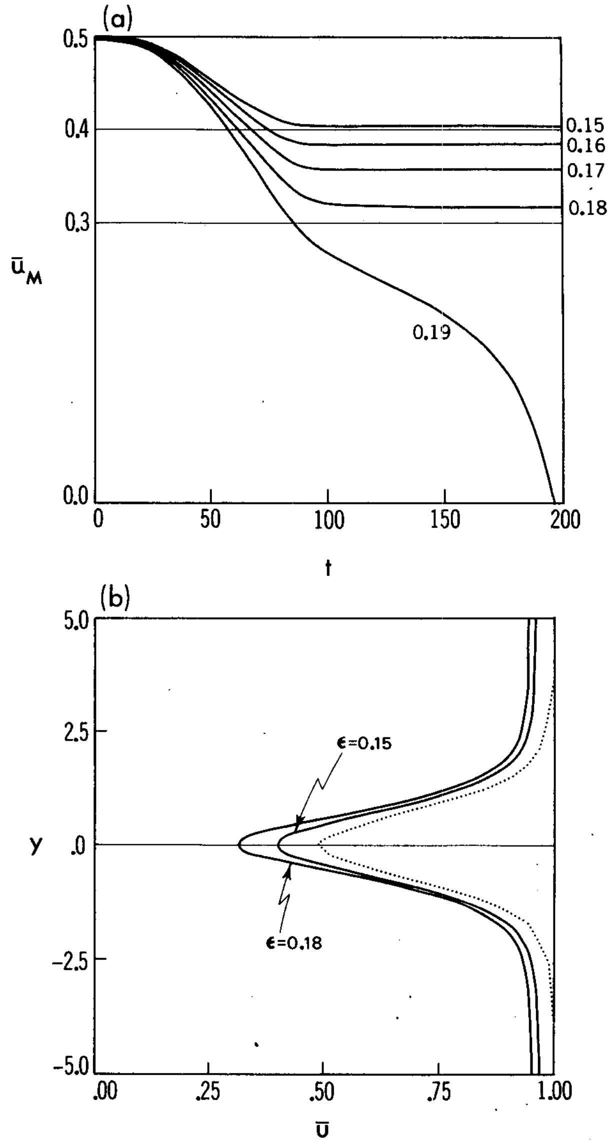

FIG. 3. (a) Minimum zonal mean wind $\bar{u}_M(t) = \min[\bar{u}(y, t)]$ for various values of ϵ (the minimum always occurs very close to $y = 0$ in these integrations). The horizontal lines mark the wind speed below which steady flow becomes impossible according to slowly varying theory ($\bar{u} = 3U/5$), and the wind speed below which overturning of vorticity contours occurs ($\bar{u} = 4U/5$). (b) The steady-state $\bar{u}$ profiles obtained for two different values of ϵ (solid lines) and the initial profile U (dotted line).

Figure 3a shows $\bar{u}(y = 0, t)$ for several values of ϵ (the minimum mean zonal flow is attained at $y = 0$ at all times in these calculations). Inspecting these curves (and others not shown), we conclude that steady flows are obtained only if $\epsilon \leq 0.186$, or equivalently, only if the mean flow is not reduced below $\bar{u} = 0.285$. Since the initial mean flow at $y = 0$ is 0.5, the WKB analysis predicts that the minimum steady-state zonal flow should be 0.3, while the corresponding estimate of ϵ_c based on (15–16) is 0.173. The predictions for the slowly varying limit are quite accurate in this case. Plots of the zonal mean flow for some of the steady solutions are included in Fig. 3b. (For $\epsilon \leq 0.18$ and $\epsilon = 0.19$ the meridional grid spacing is 1/10 and 1/30, respectively. The time step is 1/50 for all numerical integrations described in this paper.)

With this model we can also study the unsteady solutions for larger ϵ. The curve for $\epsilon = 0.19$ in Fig. 3a shows, as conjectured, that $\bar{u}$ is eventually driven to zero. Figure 4 illustrates the time development of this solution. Overturning contours (4b) develop well before the mean flow is decelerated by the critical two-fifths, a point we return to in section 5. After the critical layer forms (i.e., $\bar{u} \rightarrow 0$) in (4d), the flow evolves in a manner similar to that expected for a quasi-linear critical layer [see, e.g., Geisler and Dickinson (1974) and Haynes and McIntyre (1987)]. The profiles of the vorticity gradient γ in (4c) and (d) show a signature of partial reflection from the shear layer; the sinusoidal structure north of the shear zone is evidently induced by a standing wave component to the wave field resulting from reflections from the shear layer and the northern boundary.

If we leave the forcing on for a very long time and then turn it off, the value $\epsilon = 0.186$ marks a sharp boundary between flows that return reversibly to the original mean flow and those that do not. When the forcing is left on for shorter times, this boundary is less clear-cut. By slowly turning the source off at different times, we have found that this flow relaxes back to its initial profile as the wave recedes, even after the mean flow is reduced below $\bar{u}_c$. Indeed, by choosing the time of the turn-off carefully, we find reversible solutions with arbitrarily small positive $\bar{u}$. However, as soon as a critical layer forms, the generation of small scales is greatly accelerated so that, in practice, the flow becomes irreversible.

The accuracy of the slowly varying theory in this instance is not surprising once one evaluates the self-consistency of the WKB approximation. Given the equation $\Phi_{yy} = -l^2\Phi$ and a solution of the form $\Phi = A(y)\exp[i\theta(y)]$, we obtain the WKB approximation in the usual way by equating real and imaginary parts ($A_{yy}/A + l^2 = \theta_y^{\,2}$ and $A = \theta_y^{-1/2}$) and then assuming $A_{yy}/(Al^2) \ll 1$ so that $\theta_y^{\,2} = l^2$ and $A = l^{-1/2}$. For self-consistency, we require

$$\mu(y) \equiv |(l^{-1/2})_{yy}|/l^{3/2} \ll 1. \qquad (20)$$

For the flow described above, the maximum value of μ is approximately 0.02.

We can reduce the accuracy of the slowly varying assumption by reducing the value of β. Figure 5a shows the evolution of $\bar{u}(0, y)$ for $\beta = 2$, with all the other parameters as used above. For this choice of β, $\mu(y)$ reaches values as large as 0.89. The numerically determined critical wind speed ($\approx$0.25) and the forcing amplitude ($\approx$0.42) are underestimated by the slowly varying theory ($\bar{u}_c = 0.3$; $\epsilon_c \approx 0.31$). However, we obtain the same qualitative result: for ϵ less than some

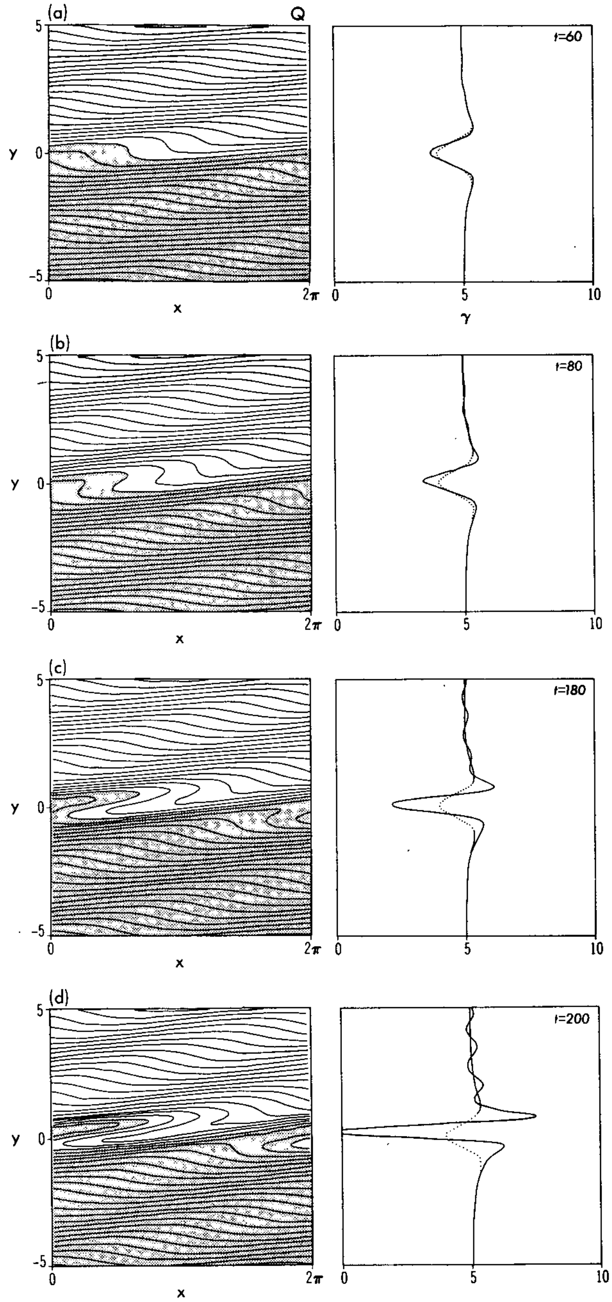

FIG. 4. Evolution of the $\epsilon = 0.19$ quasi-linear integration (see Fig. 3a). Total vorticity Q (left) and the zonal mean meridional vorticity gradient γ (right). The initial γ profile is dotted. (a) $t = 60$, $\bar{u}_M \approx 4U/5 = 0.4$, incipient overturning of a Q contour; (b) $t = 80$, $\bar{u}_M \approx 3U/5 = 0.3$; (c) $t = 180$, note the undulations in γ indicative of partial reflection; (d) $t = 200$, $\bar{u}_M \approx 0$, critical layer formation eminent.

ϵ_c a steady flow is achieved with $\bar{u}$ bounded well away from zero; for ϵ greater than ϵ_c a critical layer is induced. Figure 5b shows the steady state $\bar{u}$ and γ when $\epsilon = 0.42$.

5. Fully nonlinear integrations

It remains to examine how the inclusion of higher harmonics affects the forced wave-mean flow evolution. The numerical model is identical to that used in the calculations in section 4 except that the nonlinear bracketed terms in (2a) are now retained. The Jacobian is evaluated using spectral transforms in the zonal direction and meridional finite-differencing, with a nondimensional grid size of $\Delta y = 1/30$. The boundary and initial conditions, and the sponge layer are as before, with $U(y)$ given by (17) and with $\beta = 5$ and $\delta = 0.16$.

Our intention in this model is to generate a finite amplitude sinusoidal Rossby wave and watch it interact with the shear layer. However, the flow tends to become nonlinear and mix irreversibly in the vicinity of the source (for these large forcing amplitudes), as well as in the shear layer. This behavior may be a numerical artifact in part. We should probably use the full nonlinear boundary condition for flow past a corrugated wall, but instead continue to use the simpler condition (5). To avoid complications associated with nonlinearity near the source we use the artifice of gradually increasing the nonlinearity southward of the forcing latitude, $y_s = 5$. Specifically, in the wave equation (2a)—but not in the mean flow equation (2b)—we set $\epsilon(y) = \epsilon_0 \sin^2[\pi(5 - y)/2]$ for $3 < y \leq 5$, and $\epsilon = \epsilon_0$ for $y \leq 3$.

Sixteen zonal harmonics are retained in the model, with the wavenumber of the forcing equal to that of the lowest wavenumber. No subharmonics of the forced wave are included. A very small biharmonic diffusion $\kappa\nabla^4\zeta$ is added to the vorticity equation so as to maintain a smooth solution when a cascade to small scales occurs; κ is chosen to be 1.25×10^{-6}, so that the diffusion time across a grid space is approximately one time unit.

Figure 6, which is the nonlinear equivalent of Fig. 3a, summarizes the results of these integrations. When $\epsilon_0 < 0.15$, steady states are obtained that are nearly identical to their quasi-linear counterparts. For $\epsilon_0 \geq 0.18$, the evolution is very similar to that in the quasi-linear model up to the point that the two-fifths criterion is satisfied, at which time the wave breaks. Figure 7 shows the evolution of the absolute vorticity for the case $\epsilon_0 = 0.18$ during the period of wave breaking, $90 < t < 99$. There appears to be a distinct onset of an instability at $t \approx 96$, with wavenumber 6 developing significant amplitude. After the wave overturns, closed contours are generated by the diffusion. It is clear in this case, and for larger ϵ, that irreversible wave breaking sets in long before a critical layer is formed. However, the "two-fifths rule" based on the quasi-linear model still seems to be relevant in marking the time at which the breaking occurs.

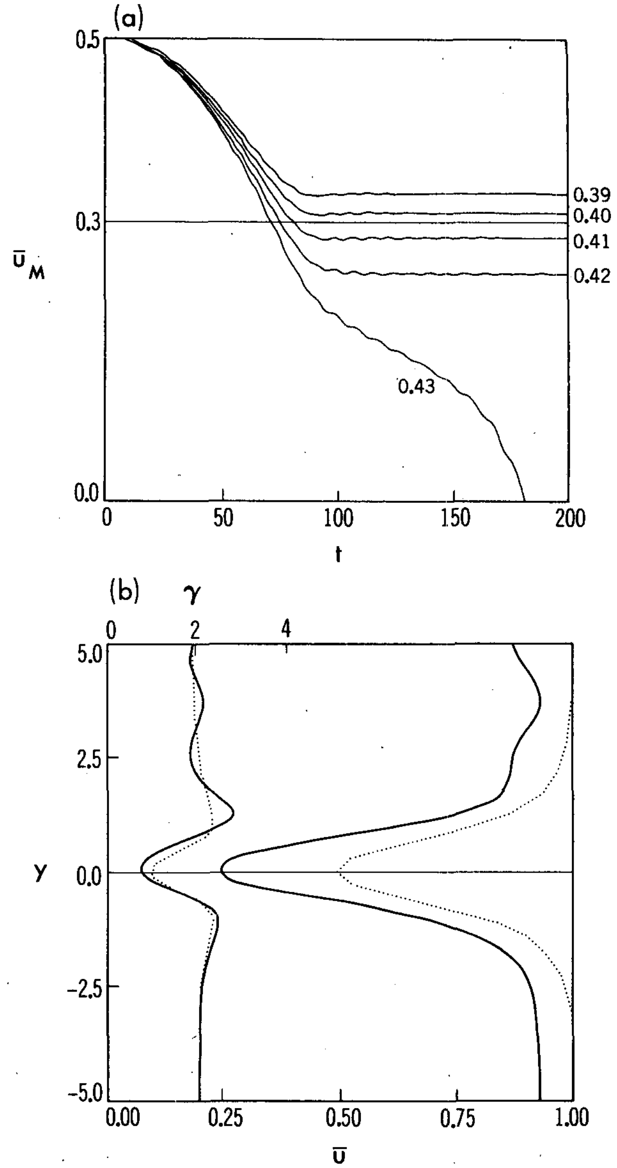

FIG. 5. (a) As in Fig. 3a, but for $\beta = 2$. (b) Steady-state zonal mean $\bar{u}$ and γ when $\epsilon = 0.42$.

We have not extended the integration very far beyond this point, since we are focusing on the onset of irreversible mixing. As the integration proceeds, the amplitude of the solution builds up between the source and the shear zone because of reflection from the breaking region, so that the evolution then depends on the location of the source as well as the manner in which we have artificially turned off the nonlinearity near the source.

Nonlinear integrations with $0.15 < \epsilon < 0.18$ have the distinction of producing irreversible wave breaking without the mean flow being decelerated by two-fifths of its initial value. The time required for this breaking

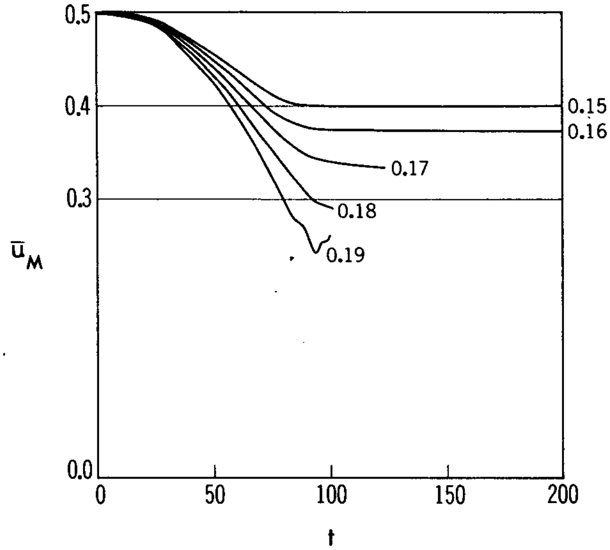

FIG. 6. As in Fig. 3a, but for the fully nonlinear model.

to develop increases as ϵ decreases. For $\epsilon = 0.17$, it occurs at $t \approx 120$, and for $\epsilon = 0.16$ at $t \approx 190$. The development of the irreversible wavebreaking with $\epsilon = 0.16$ is illustrated in Fig. 8. The overturning occurs in a region of smaller zonal extent than that in Fig. 7, but it has the same qualitative character. As in the case with larger wave amplitude, wavenumber 6 seems to be preferentially excited. In fact, in this case a steady solution is obtained that closely resembles the quasi-linear solution if four or fewer harmonics of the forced wave are retained in the model.

By $t = 200$, the case with $\epsilon = 0.15$, has not yet broken although close inspection shows a hint of developing time dependence. Integrations with $\epsilon < 0.15$ show no instability by $t = 200$. While we cannot rule out the possibility that very slow instabilities will eventually develop in these cases, we suspect that there is something special about the $\epsilon = 0.15$ case, for it is in this case that a north–south oriented absolute vorticity contour is generated. The implication is that instabilities of the sort seen in Figs. 7 and 8 develop (on time scales of interest, at least) only on locally overturned contours. That a locally reversed potential vorticity gradient can drastically effect subsequent evolution has been aptly demonstrated by Haynes (1987). [In this regard we note the close similarity between our Figs. 7 and 8 and Hayne's Fig. 6 (corresponding to his large ϵ simulation).]

In the slowly varying quasi-linear model, one can show that an overturning contour develops as soon as the mean flow is decelerated by *one-fifth* of its initial value. Incipient overturning of a contour implies that the meridional gradient of the absolute vorticity field vanishes, which translates into the criterion $2|\tilde{q}_y| = \bar{q}_y$ for a disturbance of the form $q = \tilde{q}\exp(ix) + (cc)$. But for a stationary slowly varying wave,

$$|\tilde{q}_y|^2 = l^2|\tilde{q}|^2 = |\tilde{q}|^2\bar{q}_y/\bar{u} = A\bar{q}_y{}^2/\bar{u} \quad (21)$$

Therefore, overturning first occurs when $A = \bar{u}/4$. Using (7), this condition reduces to $A = U/5$ or, equivalently, $\bar{u} = 4U/5$.

We have confirmed that the steady solutions obtained in the quasi-linear numerical integrations of section 4 contain overturning contours (much like the solution shown in Fig. 4b) when $0.15 < \epsilon < 0.185$. As seen in Fig. 3, in these solutions the mean flow at $y = 0$ has been decelerated by an amount between one-fifth and two-fifths of its initial value $[U(0) = 0.5]$. By modifying the argument leading to (15)—replacing the factor $(2/5)(3/5)^{3/2}$ by $(1/5)(4/5)^{3/2}$—we confirm that overturning should commence at the value $\epsilon \approx 0.15$.

Our interpretation of the nonlinear integrations is as follows. If ϵ is sufficiently large to decelerate the mean flow by two-fifths of its initial value, the runaway increase in wave amplitude predicted by the quasi-linear model quickly results in irreversible wave breaking, long before the mean flow is decelerated to the point of critical layer formation. For somewhat smaller values of ϵ, for which the mean flow deceleration is between one- and two-fifths of the initial flow, we presume that there are solutions of the fully nonlinear model which closely resemble their quasi-linear counterparts, and which, therefore, evolve into steady states in which the meridional potential vorticity gradient is locally reversed. However, these solutions are unstable, and the instabilities eventually produce irreversible wave breaking once again.

It is interesting to ask if these considerations are still relevant if a critical latitude exists for the incident wave in the initial mean flow. Suppose, in particular, that, $\bar{u} = \Lambda y$ so that there is a critical latitude for a stationary wave at $y = 0$. Ignoring coefficients of order unity, wave breaking occurs when $A \approx \bar{u}$ or equivalently when η, the meridional particle displacement, is of the same order as the inverse of the local wavenumber l, i.e.,

$$\psi/\bar{u} \sim (\bar{u}/\beta)^{1/2} \quad \text{or} \quad \psi^2 \sim \bar{u}^3/\beta = \Lambda^3 y^3/\beta.$$

Therefore, breaking occurs for $y \approx Y_{\text{BREAK}}$, where

$$Y_{\text{BREAK}}^3 \approx \beta\psi^2/\Lambda^3$$

However, the WKB assumption breaks down for this shear flow where $ly < 1$, i.e.,

$$y < Y_{\text{WKB}} = \Lambda/\beta.$$

Note that

$$(Y_{\text{BREAK}}/Y_{\text{WKB}})^3 \sim \beta^4\psi^2/\Lambda^6 = (Y_{\text{NL}}/Y_{\text{WKB}})^4, \quad (22)$$

where $Y_{\text{NL}} = (\psi/\Lambda)^{1/2}$. Here Y_{NL} is the critical layer width, or the size of the particle displacements near y

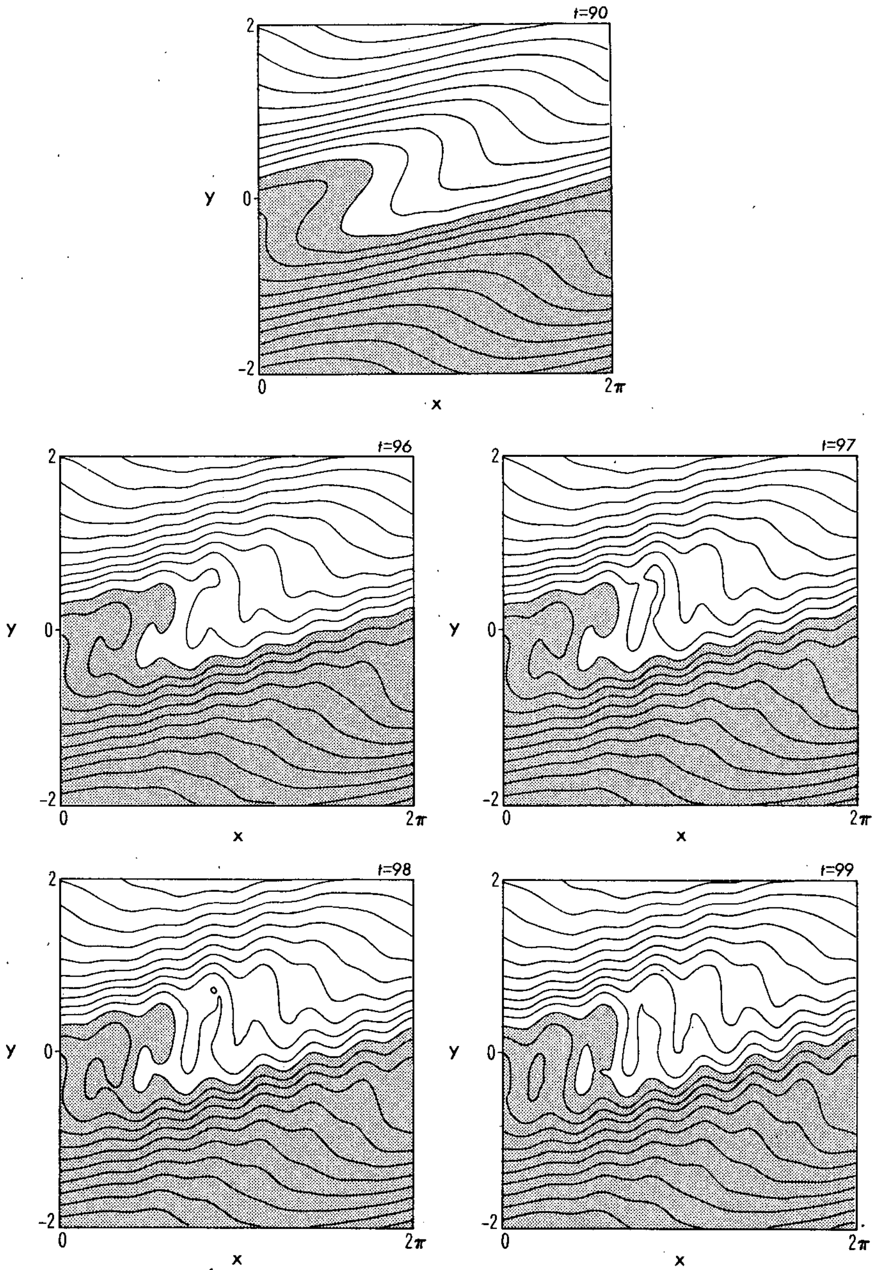

FIG. 7. Evolution of the total vorticity field at the time of breaking in the fully nonlinear model for $\epsilon = 0.18$.

$= 0$, in the nonlinear critical layer valid for $Y_{NL}/Y_{WKB} \ll 1$. Our detailed predictions concerning wave breaking are dependent on the WKB approximation, and therefore require that $Y_{BREAK} > Y_{WKB}$. From (22), this will not hold if $Y_{NL} \ll Y_{WKB}$, the case for which small-amplitude nonlinear critical layer theory holds. Only if $Y_{NL} > Y_{WKB}$ can Y_{BREAK} be larger than Y_{WKB}. In this case, wave breaking will occur well before the critical layer is reached, and the analysis described here should be relevant. See Held and Phillips (1987) and Robinson (1988) for examples of waves overturning before reaching their critical layers.

6. Summary and concluding remarks

In this paper we consider the propagation of a Rossby wave, generated by slowly switching on stationary forcing, through a shear flow which is initially free of

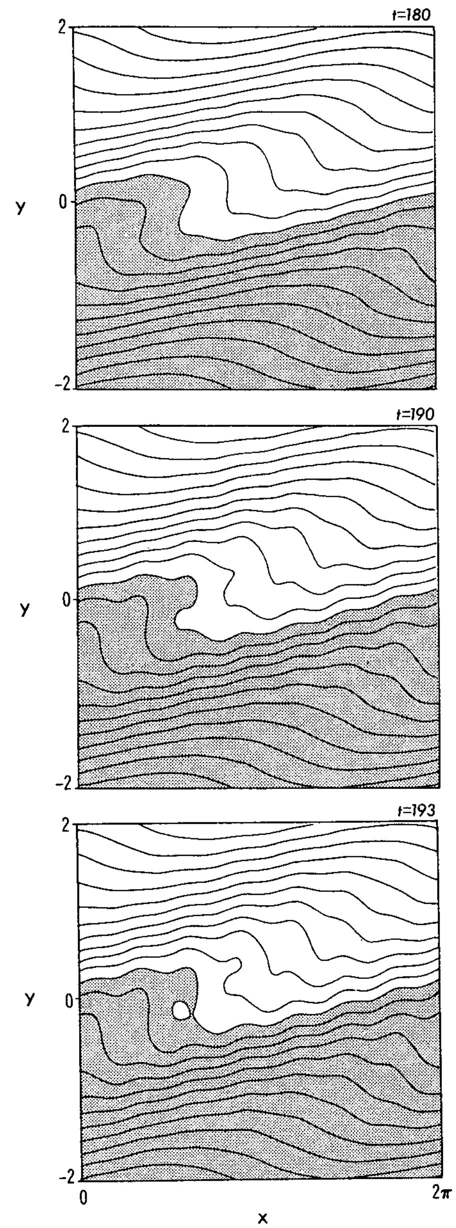

FIG. 8. As in Fig. 7, but for $\epsilon = 0.16$.

critical levels. We examine how linear theory breaks down as the wave amplitude increases, using both inviscid quasi-linear (wave-mean flow interaction) and fully nonlinear barotropic models. Two simple rules emerge from a WKB analysis of the quasi-linear model:

1) If the mean zonal flow is decelerated by less than two-fifths of its original value by the passage of the wave front, the quasi-linear model reaches a steady state; if the deceleration is greater than this critical amount, a runaway pile-up of wave activity results, due to the same group velocity feedback that Dunkerton (1981) analyzes in a gravity wave model. A numerical quasi-linear model shows that the associated mean flow deceleration culminates in the creation of a critical layer. It also shows that this qualitative behavior continues to hold for a wave that is not slowly-varying, although the required mean flow deceleration is somewhat greater than the two-fifths required in the slowly varying limit.

2) If the mean flow deceleration is greater than one-fifth but less than two-fifths, the steady state that evolves in the quasi-linear model has absolute vorticity contours that overturn locally.

Both of these results have implications for fully nonlinear integrations. When the mean flow is decelerated by two-fifths of its initial value, rapid wave breaking ensues. The mean zonal flow continues to decelerate beyond this point but, unlike the quasi-linear model, irreversible mixing is generated long before a critical layer forms. If the forcing amplitude is such that the quasi-linear model predicts a steady flow with overturning streamlines and mean flow deceleration between one- and two-fifths of the initial flow, the nonlinear model slowly develops an instability that eventually produces irreversible mixing.

We close with several additional points:

- It is not clear that an overturning streamline is critical for instability in this system, or if the growth rates simply increase rapidly at this point. The well-known instability of Rossby waves on a uniform mean flow on the beta-plane shows that this cannot be a general requirement [Lorenz (1972)]. The small aspect ratio (meridional scale/zonal scale) of the waves considered here is probably important in this regard [as it was in Haynes (1987)].
- The difference in forcing amplitude between a wave breaking when the mean flow is decelerated by one-fifth rather than two-fifths of its initial value is not very large. The mean flow deceleration is proportional to the wave amplitude squared when the local wave amplitude is prescribed, but the positive group velocity feedback causes the deceleration to be even more sensitive to the forcing amplitude. For the case analyzed in section 4, ϵ need only be increased from 0.15 to 0.185 to double the resulting deceleration from one-fifth to two-fifths.
- The relevance of these results for the atmosphere can be questioned in several ways. Breaking of vertically propagating planetary waves is of central importance to the dynamics of the middle atmosphere (McIntyre and Palmer 1983). However, these waves break not only because of propagation into regions of weak mean winds but also because of their increasing amplitude with height, due to the density decrease. More seriously, wave paths in the y–z plane can change in response to the evolving mean flow, a complication that has no analogue in our simple barotropic example. A barotropic model would appear to be more relevant to the

quasi-horizontal propagation of planetary waves from midlatitudes into the tropics. However, the neglect of the restoring forces on the mean flow, particularly the acceleration by the Hadley cell, remains an unrealistic feature. Moreover, most disturbances in the troposphere have critical latitudes; the case without critical latitudes that we have analyzed is the exception rather than the rule. Yet we believe that an understanding of the wave breaking problem posed here is a useful stepping-stone towards understanding the breaking of large amplitude waves in the presence of a critical layer and restoring forces.

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