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# AN ALTERNATIVE FORMULATION OF CONTROLLED ROUNDING

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## ABSTRACT

In this report we introduce a technique for controlled rounding which allows a non-zero multiple of the base to either increase or decrease by the value of the base. In the more standard definition of controlled rounding a non-zero multiple of the base can increase but not decrease. We display the step-by-step procedure for finding controlled roundings under this new definition and contrast this procedure with the more standard methods.


## I. INTRODUCTION

The objective of this report is to introduce a technique for controlled rounding that allows a non-zero multiple of the base to either increase or decrease by the value of the base. In the more standard definition of controlled rounding a multiple of the base can increase by the value of the base, but not decrease, see Cox and Ernst (1982) and Cox (1987). In this section we define what is meant by a controlled rounding in general and contrast the two formulations with regards to non-zero multiples of the base. In the solution of a controlled rounding problem there is an underlying network; and for each of the two procedures the networks employed are slightly different both with respect to arcs and costs. In Section II the two methods are worked through step-by-step to show how they differ, and examples of each procedure are provided to contrast performance.

We begin by stating the problem, establishing notation and terminology, and defining a controlled rounding. Given a positive integer $\underline{b}$ and an <u>additive table</u> of non-negative integers

$$A = \begin{array}{c|cccc} a_{00} & a_{01} & a_{02} & \cdots & a_{0C} \\ \hline a_{10} & a_{11} & a_{12} & \cdots & a_{1C} \\ a_{20} & a_{21} & a_{22} & \cdots & a_{2C} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_{R0} & a_{R1} & a_{R2} & \cdots & a_{RC} \end{array} ,$$

that is,

$$\sum_{j=1}^{C} a_{ij} = a_{i0} \qquad i=1,\ldots,R$$

$$\sum_{i=1}^{R} a_{ij} = a_{0j} \qquad j=1,\ldots,C$$

$$a_{ij} \geq 0 \qquad i=1,\ldots,R \text{ and } j=1,\ldots,C$$

a <u>controlled rounding</u>, B of A, is an additive table

$$B = \begin{array}{c|cccc} b_{00} & b_{01} & b_{02} & \cdots & b_{0C} \\ \hline b_{10} & b_{11} & b_{12} & \cdots & b_{1C} \\ b_{20} & b_{21} & b_{22} & \cdots & b_{2C} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ b_{R0} & b_{R1} & b_{R2} & \cdots & b_{RC} \end{array}$$

where

$$\sum_{j=1}^{C} b_{ij} = b_{i0} \qquad i=1,\ldots,R,$$

$$\sum_{i=1}^{R} b_{ij} = b_{0j} \qquad j=1,\ldots,C,$$

$b_{ij} = bm_{ij}$ for $m_{ij}$ a non-negative integer, and

(1) $|b_{ij} - a_{ij}| \leq b$ for $i=0,\ldots,R$ and $j=0,\ldots,C$.

Replacing (1) above by

(2) $|b_{ij} - a_{ij}| < b$,

we say B is a zero-restricted controlled rounding of A. Replacing (1) by

(3) $$\begin{cases} |b_{ij} - a_{ij}| \leq b & \text{if } a_{ij} > 0 \\ b_{ij} = 0 & \text{if } a_{ij} = 0, \end{cases}$$

we say B is a weakly zero-restricted controlled rounding of A.

All two dimensional tables have zero-restricted controlled roundings, and under many circumstances one prefers a zero-restricted controlled rounding rather than one that is not zero-restricted, for example, in order to obtain an unbiased controlled rounding, see Cox (1987). When seeking a controlled rounding which is minimal with respect to some measure of closeness, zero-restricted controlled roundings will not suffice, as will be discussed in Section III.

The definitions above differ from those more commonly considered -- see Cox and Ernst (1982) and Cox (1987). Under the definition of controlled rounding as in Cox and Ernst, instead of (1) above, one has

(1') $b_{ij} = [a_{ij}/b]b$ or $[a_{ij}/b]b + b$.

where [x] represents the greatest integer less than or equal to x. The notion of zero-restricted controlled rounding remains the same, but for a counterpart of weakly zero-restricted controlled rounding, one replaces (3) by

$$(3') \qquad b_{ij} = \begin{cases} [a_{ij}/b]b \text{ or } [a_{ij}/b]b + b & \text{if } a_{ij} > 0 \\ 0 & \text{if } a_{ij} = 0. \end{cases}$$

That is, under the usual definition of controlled rounding, a mulitple of the base $a_{ij}$ can remain as either $a_{ij}$ or go to $a_{ij}$+b. For the definition above, a non-zero multiple of the base, $a_{ij}$, can go to either $a_{ij}$ - b or $a_{ij}$ + b. Of course, if zero is to change, it can only go to b since a controlled rounding is always positive. The added definition, weakly zero-restricted, recognizes the special role of zero -- zero is always to remain fixed, but a non-zero multiple of the base can either increase or decrease by the value of the base.

We first go though a step by step procedure for finding a traditional controlled rounding conforming to (1'), (2), or (3') -- incorporating a step in which the problem is reduced to the solution of a zero-one network flow problem. That procedure is then contrasted with one for finding a controlled rounding under the new definition conforming to (1), (2) or (3). In the latter case, we also incorporate the step of solving an alternative zero-one network flow problem. For a discussion of the two dimensional problem and many of the steps in Section II, we refer the reader to Cox and Ernst (1982) and Cox (1987). For a discussion of network flows as will be used in this report, see Gondran and Minoux (1984).

## II. STEP-BY-STEP PROCEDURE FOR SOLVING THE TWO DIMENSIONAL CONTROLLED ROUNDING PROBLEM

### A. Controlled rounding under definition that $b_{ij} = [a_{ij}/b]b$ or $[a_{ij}/b]b + b$

First reduce the problem modulo the base. That is, write

$$A = bD + R$$

where D and R are RxC matrices and

$$a_{ij} = bd_{ij} + r_{ij}$$

where $0 \leq r_{ij} < b$ for $i=1,\ldots,R$ and $j=1,\ldots,C$.

Define

$$d_{i0} = \sum_{j=1}^{C} d_{ij} \qquad i=1,\ldots,R$$

$$d_{0j} = \sum_{i=1}^{R} d_{ij} \qquad j=1,\ldots,C$$

$$d_{00} = \sum_{i=1}^{R} \sum_{j=1}^{C} d_{ij} = \sum_{i=1}^{R} d_{i0} = \sum_{j=1}^{C} d_{0j}$$

and

$$r_{i0} = \sum_{j=1}^{C} r_{ij} \qquad i=1,\ldots,R$$

$$r_{0j} = \sum_{i=1}^{R} r_{ij} \qquad j=1,\ldots,C$$

$$r_{00} = \sum_{i=1}^{R} \sum_{j=1}^{C} r_{ij} = \sum_{i=1}^{R} r_{i0} = \sum_{j=1}^{C} r_{0j} .$$

With these definitions, one obtains additivity of the following system of tables -- including the marginal positions:

$$
\begin{array}{c|cccc}
a_{00} & a_{01} & a_{02} & \cdots & a_{0C} \\
\hline
a_{10} & a_{11} & a_{12} & \cdots & a_{1C} \\
a_{20} & a_{21} & a_{22} & \cdots & a_{2C} \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
a_{R0} & a_{R1} & a_{R2} & \cdots & a_{RC}
\end{array}
= (b)
\begin{array}{c|cccc}
d_{00} & d_{01} & d_{02} & \cdots & d_{0C} \\
\hline
d_{10} & d_{11} & d_{12} & \cdots & d_{1C} \\
d_{20} & d_{21} & d_{22} & \cdots & d_{2C} \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
d_{R0} & d_{R1} & d_{R2} & \cdots & d_{RC}
\end{array}
+
\begin{array}{c|cccc}
r_{00} & r_{01} & r_{02} & \cdots & r_{0C} \\
\hline
r_{10} & r_{11} & r_{12} & \cdots & r_{1C} \\
r_{20} & r_{21} & r_{22} & \cdots & r_{2C} \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
r_{R0} & r_{R1} & r_{R2} & \cdots & r_{RC}
\end{array} .
$$

If S is a (zero-restricted, weakly zero-restricted) controlled rounding of the last table on the right, the sum

$$B = bD+S$$

is a (zero-restricted, weakly zero-restricted) controlled rounding of A. Thus, our objective is to form a controlled rounding of R, and we observe $r_{ij} < b$ for $i=1,\ldots,R$ and $j=1,\ldots,C$.

Following Cox and Ernst, "fold-in" the RxC table to form an R+1 by C+1 table C by adding a slack row and slack column. That is,

$$
C =
\begin{array}{c|ccccc}
c_{00} & c_{01} & c_{02} & \cdots & c_{0c} & c_{0,C+1} \\
\hline
c_{10} & c_{11} & c_{12} & \cdots & c_{1C} & c_{1,C+1} \\
c_{20} & c_{21} & c_{22} & \cdots & c_{2C} & c_{2,C+1} \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
c_{R0} & c_{R1} & c_{R2} & \cdots & c_{RC} & c_{R,C+1} \\
c_{R+1,0} & c_{R+1,1} & c_{R+1,2} & \cdots & c_{R+1,C} & c_{R+1,C+1}
\end{array}
$$

where

$$c_{ij} = r_{ij} \qquad i=1,\ldots,R \text{ and } j=1,\ldots,C$$

$$c_{i,C+1} = [(\sum_{j=1}^{C} c_{ij})/b]b + b - \sum_{j=1}^{C} c_{ij} \qquad i=1,\ldots,R$$

$$c_{R+1,j} = [(\sum_{i=1}^{R} c_{ij})/b]b + b - \sum_{i=1}^{R} c_{ij} \qquad j=1,\ldots,C$$

$$c_{R+1,C+1} = \sum_{i=1}^{R} \sum_{j=1}^{C} c_{ij} - [(\sum_{i=1}^{R} \sum_{j=1}^{C} c_{ij})/b]b$$

$$c_{i0} = [(\sum_{j=1}^{C} c_{ij})/b]b + b$$

$$= \sum_{j=1}^{C+1} c_{ij} \qquad i=1,\ldots,R$$

$$c_{0j} = [(\sum_{i=1}^{R} c_{ij})b/]b + b$$

$$= \sum_{i=1}^{R+1} c_{ij} \qquad j=1,\ldots,C$$

$$c_{R+1,0} = \sum_{j=1}^{C+1} c_{R+1,j}$$

$$c_{0,C+1} = \sum_{i=1}^{R+1} c_{i,C+1}$$

$$c_{00} = \sum_{i=1}^{R+1} \sum_{j=1}^{C+1} c_{ij} = \sum_{i=1}^{R+1} c_{i0} = \sum_{j=1}^{C+1} c_{0j} .$$

Note that at this stage all marginal values of C are multiples of b and [x] refers to the greatest integer less than or equal to x.

Let us take a 4x4 table and follow the steps through to this point. Letting the base b=3 and

A=

| 119 | 24 | 40 | 18 | 37 |
|---|---|---|---|---|
| 15 | 4 | 8 | 3 | 0 |
| 41 | 7 | 13 | 1 | 20 |
| 19 | 1 | 5 | 9 | 4 |
| 44 | 12 | 14 | 5 | 13 |

3D=

| 102 | 21 | 33 | 15 | 33 |
|---|---|---|---|---|
| 12 | 3 | 6 | 3 | 0 |
| 36 | 6 | 12 | 0 | 18 |
| 15 | 0 | 3 | 9 | 3 |
| 39 | 12 | 12 | 3 | 12 |

R=

| 17 | 3 | 7 | 3 | 4 |
|---|---|---|---|---|
| 3 | 1 | 2 | 0 | 0 |
| 5 | 1 | 1 | 1 | 1 |
| 4 | 1 | 2 | 0 | 1 |
| 5 | 0 | 2 | 2 | 1 |

it is easy to see that

$$A = 3D + R.$$

In addition:

C =

| 36 | 6 | 9 | 6 | 6 | 9 |
|---|---|---|---|---|---|
| 6 | 1 | 2 | 0 | 0 | 3 |
| 6 | 1 | 1 | 1 | 2 | 1 |
| 6 | 1 | 2 | 0 | 1 | 2 |
| 6 | 0 | 2 | 2 | 1 | 1 |
| 12 | 3 | 2 | 3 | 2 | 2 |

Our objective is to reassign values for $c_{ij}$ for $i=1,\ldots,R+1$ and $j=1,\ldots,C+1$ from the set $\{0,3\}$ (in general, from the set $\{0,b\}$ ) to maintain additively to the marginals. Having done this, and calling the new table F (with the same marginals as C), we observe that

F =

| $c_{00}$ | $c_{01}$ | $c_{02}$ | $\cdots$ | $c_{0C}$ | $c_{0,C+1}$ |
|---|---|---|---|---|---|
| $c_{10}$ | $f_{11}$ | $f_{12}$ | $\cdots$ | $f_{1C}$ | $f_{1,C+1}$ |
| $c_{20}$ | $f_{21}$ | $f_{22}$ | $\cdots$ | $f_{2C}$ | $f_{2,C+1}$ |
| . | . | . | ... | . | . |
| . | . | . | ... | . | . |
| . | . | . | ... | . | . |
| $c_{R0}$ | $f_{R1}$ | $f_{R2}$ | $\cdots$ | $f_{RC}$ | $f_{R,C+1}$ |
| $c_{R+1,0}$ | $f_{R+1,1}$ | $f_{R+1,2}$ | $\cdots$ | $f_{R+1,C}$ | $f_{R+1,C+1}$ |

is a controlled rounding of C.

In particular, we solve the following system of equations for $f_{ij}$:

$$\sum_{j=1}^{C+1} f_{ij} = c_{i0} \quad i=1,\ldots,R+1$$

$$\sum_{i=1}^{R+1} f_{ij} = c_{0j} \quad j=1,\ldots,C+1$$

$$f_{ij} \in \{0,b\} \quad i=1,\ldots,R+1 \text{ and } j=1,\ldots,C+1.$$

This system does have a solution, (see Cox and Ernst) and an easy way to obtain one is to find a saturated flow across the complete directed bipartite network shown in Figure 1, see (Cox, Fagan, Hemmig, Greenberg (1986) and Gondran and Minoux (1984)). In this network nodes correspond to marginal constraints, all arcs flow from left to right, and the directed arc between node $n_{i0}$ and $n_{0j}$ corresponds to cell (i,j).

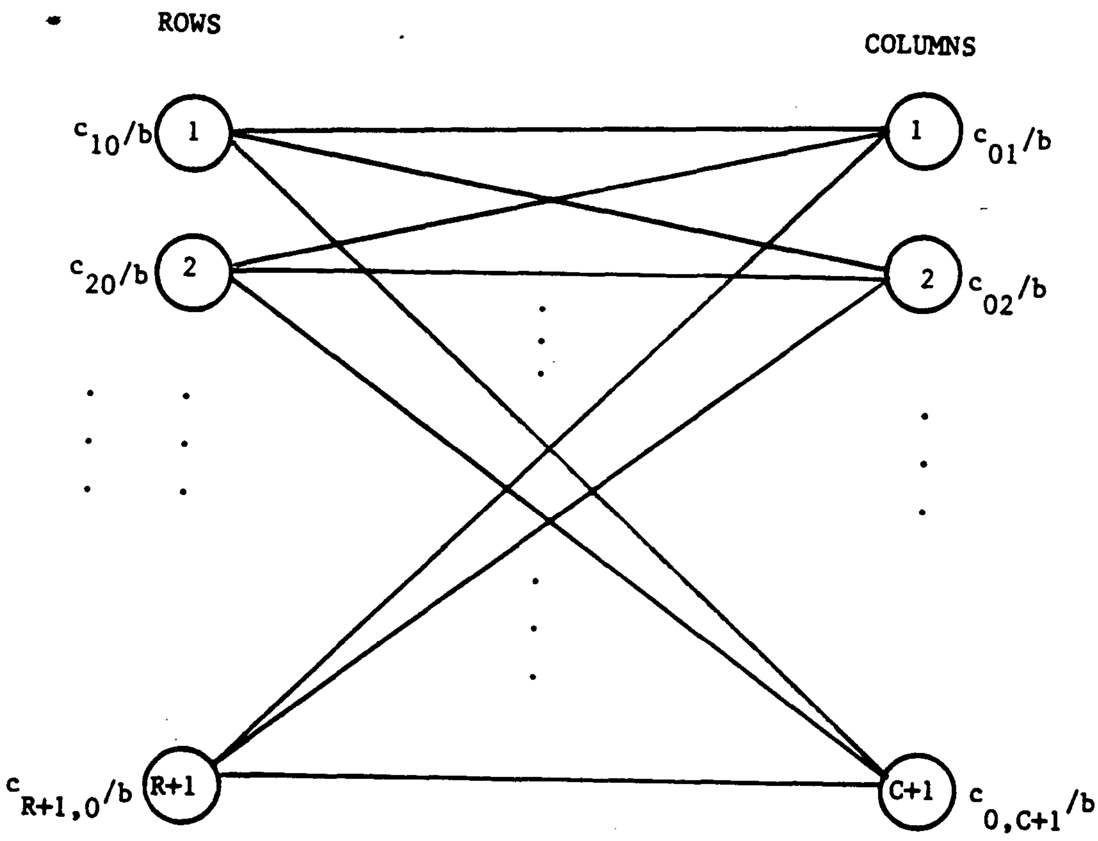


FIGURE 1

The nodes on the left correspond to sources, nodes on the right correspond to sinks, supplies and demands (marginal values) are shown alongside each source and sink respectively, each arc has upper capacity equal to one, and all arcs are directed from left to right. A saturated flow does exist, and we set $f_{ij}$ equal to the flow over arc $(n_{i0}, n_{0j})$ times $b$.

The table

$$
S = \begin{array}{c|cccc}
f_{00} & f_{01} & f_{02} & \cdots & f_{0C} \\
\hline
f_{10} & f_{11} & f_{12} & \cdots & f_{1C} \\
f_{20} & f_{21} & f_{22} & \cdots & f_{2C} \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
f_{R0} & f_{R1} & f_{R2} & \cdots & f_{RC}
\end{array}
$$

where

$$f_{i0} = \sum_{j=1}^{C} f_{ij} \qquad i=1,\ldots,R$$

$$f_{0j} = \sum_{i=1}^{R} f_{ij} \qquad j=1,\ldots,R$$

$$f_{00} = \sum_{i=1}^{R} \sum_{j=1}^{C} f_{ij} = \sum_{i=1}^{R} f_{i0} = \sum_{j=1}^{C} f_{0j}$$

is a controlled rounding of R. To see this, one need only show that

$$|r_{ij}-f_{ij}| \leq 1 \quad \text{for } i=0,\ldots,R \quad j=0,\ldots,C.$$

(a) For $i=1,\ldots,R \quad j=1,\ldots,C$:
$r_{ij} = c_{ij}$ so $|r_{ij}-f_{ij}| = |c_{ij}-f_{ij}| \leq 1$ .

(b) For i=1,...,R:

$$c_{i0} = \sum_{j=1}^{C} c_{ij} + c_{i,C+1},$$ and since

$$c_{i0} = f_{i0} + f_{i,C+1} \text{ and } \sum_{j=1}^{C} c_{ij} = r_{i0},$$

$$f_{i0} - r_{i0} = c_{i,C+1} - f_{i,C+1}.$$ Thus,

$$|f_{i0}-r_{i0}| = |c_{i,C+1} - f_{i,C+1}| \leq 1 .$$

(c) For i=1,...,R:

$|f_{0j}-r_{0j}| \leq 1$ as in (b) above .

(d) Lastly: $$c_{00} = \sum_{i=1}^{R} \sum_{j=1}^{C} c_{ij} + c_{0,C+1} + c_{R+1,0} - c_{R+1,C+1}$$ and

$$c_{00} = \sum_{i=1}^{R} \sum_{j=1}^{C} f_{ij} + c_{0,C+1} + c_{R+1,0} - f_{R+1,C+1} .$$

But $$\sum_{i=1}^{R} \sum_{j=1}^{C} f_{ij} = f_{00} \text{ and } \sum_{i=1}^{R} \sum_{j=1}^{C} c_{ij} = r_{00} ,$$

so $$c_{00} = a_{00} + c_{0,C+1} + c_{R+1,0} - c_{R+1,C+1} ,$$

and $$c_{00} = f_{00} + c_{0,C+1} - c_{R+1,0} - f_{R+1,C+1} .$$

Hence $$a_{00} - f_{00} = c_{R+1,C+1} - f_{R+1,C+1}$$

and $|a_{00}-f_{00}| \leq 1$.

In each of the cases above, it is easy to see that if $a_{ij}$ is a multiple of the base, $a_{ij}-b_{ij} = 0$ or b.

Letting

$$\begin{aligned} B &= bD+S \\ A-B &= bD+R - (bD+S) = R-S, \quad \text{so} \\ a_{ij}-b_{ij} &= r_{ij}-s_{ij} \qquad i=0,\ldots,R \quad j=0,\ldots,C, \end{aligned}$$

hence B is a controlled rounding of A when S is a controlled rounding of R. To explicitly relate the values $f_{ij}$ with the corresponding $b_{ij}$ note that if:

(a) for $i=1,\ldots,R$ $\quad j=1,\ldots,C$
$f_{ij}=1$ then $b_{ij}= [a_{ij}/b]b+b$
$f_{ij}=0$ then $b_{ij}= [a_{ij}/b]b$ ,

(b) for $i=1,\ldots,R$
$f_{i,C+1}=1$ then $b_{i0}= [a_{i0}/b]b$
$f_{i,C+1}=0$ then $b_{i0}= [a_{i0}/b]b+b$ ,

(c) for $j=1,\ldots,C$
$f_{R+1,j}=1$ then $b_{0j}= [a_{0j}/b]b$
$f_{R+1,j}=0$ then $b_{0j}= [a_{0j}/b]b+b$ ,

(d) $f_{R+1,C+1}=1$ then $b_{00}= [a_{00}/b]b+b$
$f_{R+1,C+1}=0$ then $b_{00}= [a_{00}/b]b$ .

If costs in the network are arbitrarily assigned to arcs, an arbitrary controlled rounding will be obtained. To obtain a weakly zero-restricted controlled rounding, arcs corresponding to zeros are removed. To ensure a zero-restricted controlled rounding, arcs corresponding to multiples of the base in the interior or grand total of the table are removed, and in other marginal positions they are given a lower capacity of one.

To set up the network for a sample controlled rounding problem base 3, let

$$A = \begin{array}{r|rrr} \underline{29} & \underline{7} & \underline{6} & \underline{16} \\ 21 & 7 & 5 & 9 \\ 8 & 0 & 1 & 7, \end{array} \quad \text{so} \quad 3D = \begin{array}{r|rrr} \underline{24} & \underline{6} & \underline{3} & \underline{15} \\ 18 & 6 & 3 & 9 \\ 6 & 0 & 0 & 6\ , \end{array}$$

$$R = \begin{array}{r|rrr} \underline{5} & \underline{1} & \underline{3} & \underline{1} \\ 3 & 1 & 2 & 0 \\ 2 & 0 & 1 & 1, \end{array} \quad \text{and} \quad C = \begin{array}{r|rrrr} \underline{18} & \underline{3} & \underline{6} & \underline{3} & \underline{6} \\ 6 & 1 & 2 & 0 & 3 \\ 3 & 0 & 1 & 1 & 1 \\ 9 & 2 & 3 & 2 & 2\ . \end{array}$$

The network to find a base 3 controlled rounding, F of C, is shown in Figure 2 with a flow displayed by each arc.

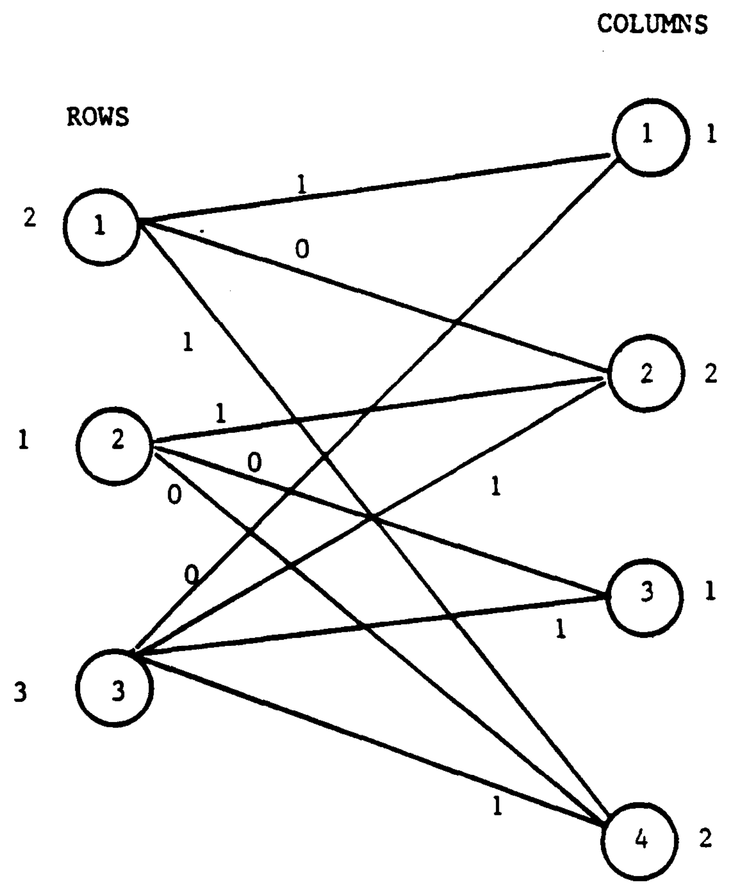


FIGURE 2

The arc corresponding to the zero cell of A, cell (2,1), has been omitted from the network to force a weakly zero-restricted controlled rounding. To force a zero-restricted controlled rounding, the arc corresponding to a non-zero multiple of the base, 9 in cell (1,3), has also been omitted and the arcs corresponding to marginal multiples of the base have a lower capacity of one. Interpreting the saturated flow through this network in terms of a base 3 controlled rounding, F of C, and from that to a zero-restricted controlled rounding, B of A, yields:

F =

| 18 | 3 | 6 | 3 | 6 |
|---|---|---|---|---|
| 6 | 3 | 0 | 0 | 3 |
| 3 | 0 | 3 | 0 | 0 |
| 9 | 0 | 3 | 3 | 3 |

B =

| 30 | 9 | 6 | 15 |
|---|---|---|---|
| 21 | 9 | 3 | 9 |
| 9 | 0 | 3 | 6 |

.

To continue with the base 3 example started earlier:

A =

| 119 | 24 | 40 | 18 | 37 |
|---|---|---|---|---|
| 15 | 4 | 8 | 3 | 0 |
| 41 | 7 | 13 | 1 | 20 |
| 19 | 1 | 5 | 9 | 4 |
| 44 | 12 | 14 | 5 | 13 |

3D =

| 102 | 21 | 33 | 15 | 33 |
|---|---|---|---|---|
| 12 | 3 | 6 | 3 | 0 |
| 36 | 6 | 12 | 0 | 18 |
| 15 | 0 | 3 | 9 | 3 |
| 39 | 12 | 12 | 3 | 12 |

R =

| 17 | 3 | 7 | 3 | 4 |
|---|---|---|---|---|
| 3 | 1 | 2 | 0 | 0 |
| 5 | 1 | 1 | 1 | 2 |
| 4 | 1 | 2 | 0 | 1 |
| 5 | 0 | 2 | 2 | 1 |

C =

| 36 | 6 | 9 | 6 | 6 | 9 |
|---|---|---|---|---|---|
| 6 | 1 | 2 | 0 | 0 | 3 |
| 6 | 1 | 1 | 1 | 2 | 1 |
| 6 | 1 | 2 | 0 | 1 | 2 |
| 12 | 3 | 2 | 3 | 2 | 2 |

.

Our objective is to obtain a controlled rounding, F of C, extract the upper left RxC subtable, derive marginals to obtain a controlled rounding S of R, and form

$$B = 3D + S$$

which will be a controlled rounding of A. Below are three controlled roundings, F of C, the corresponding controlled rounding, S of R, and finally the base 3 controlled rounding, B of A (which was our objective all along).

1.

$F_i$

| 36 | 6 | 9 | 6 | 6 | 9 |
|---|---|---|---|---|---|
| 6 | 0 | 3 | 0 | 0 | 3 |
| 6 | 3 | 0 | 0 | 3 | 0 |
| 6 | 0 | 3 | 0 | 0 | 3 |
| 6 | 0 | 0 | 3 | 3 | 0 |
| 12 | 3 | 3 | 3 | 0 | 3 |

$S_i$

| 18 | 3 | 6 | 3 | 6 |
|---|---|---|---|---|
| 3 | 0 | 3 | 0 | 0 |
| 6 | 3 | 0 | 0 | 3 |
| 3 | 0 | 3 | 0 | 0 |
| 6 | 0 | 0 | 3 | 3 |

$B_i$

| 120 | 24 | 39 | 18 | 39 |
|---|---|---|---|---|
| 15 | 3 | 9 | 3 | 0 |
| 42 | 9 | 12 | 0 | 21 |
| 18 | 0 | 6 | 9 | 3 |
| 45 | 12 | 12 | 6 | 15 |

2.

| 36 | 6 | 9 | 6 | 6 | 9 |
|---|---|---|---|---|---|
| 6 | 3 | 3 | 0 | 0 | 0 |
| 6 | 0 | 3 | 0 | 0 | 3 |
| 6 | 3 | 0 | 0 | 3 | 0 |
| 6 | 0 | 0 | 3 | 0 | 3 |

| 18 | 6 | 6 | 3 | 3 |
|---|---|---|---|---|
| 6 | 3 | 3 | 0 | 0 |
| 3 | 0 | 3 | 0 | 0 |
| 6 | 3 | 0 | 0 | 3 |
| 3 | 0 | 0 | 3 | 0 |

| 120 | 27 | 39 | 18 | 36 |
|---|---|---|---|---|
| 18 | 6 | 9 | 3 | 0 |
| 39 | 6 | 15 | 0 | 18 |
| 21 | 3 | 3 | 9 | 6 |
| 42 | 12 | 12 | 6 | 12 |

3.

| 36 | 6 | 9 | 6 | 6 | 9 |
|---|---|---|---|---|---|
| 6 | 0 | 0 | 0 | 3 | 3 |
| 6 | 0 | 3 | 0 | 0 | 3 |
| 6 | 0 | 3 | 3 | 0 | 0 |
| 6 | 3 | 3 | 0 | 0 | 0 |

| 18 | 3 | 9 | 3 | 3 |
|---|---|---|---|---|
| 3 | 0 | 0 | 0 | 3 |
| 3 | 0 | 3 | 0 | 0 |
| 6 | 0 | 3 | 3 | 0 |
| 6 | 3 | 3 | 0 | 0 |

| 120 | 24 | 42 | 18 | 36 |
|---|---|---|---|---|
| 15 | 3 | 6 | 3 | 3 |
| 39 | 6 | 15 | 0 | 18 |
| 21 | 0 | 6 | 12 | 3 |
| 45 | 15 | 15 | 3 | 12 |

One can see that each of the tables in the "B" column is a controlled rounding of A. We note that $B_1$ is a zero-restricted controlled rounding, $B_2$, is weakly zero-restricted, and $B_3$ is the only table in which a true zero is given the value 3. In all, a wide range of possible roundings exist.

B. Controlled rounding under defintion that $|a_{ij} - b_{ij}| \leq b$

In this section, we define a controlled rounding to have the property

$$|a_{ij} - b_{ij}| \leq b$$

where zero-restricted and weakly zero-restricted are defined as earlier.

The broad outline of this approach will be very much like that above. As before, reduce the problem by writing

$$A = bD + R$$

where D and R are RxC matrices where

$$d_{ij} = r_{ij} = 0 \quad \text{if } a_{ij} = 0$$

otherwise write

$$a_{ij} = bd_{ij} + r_{ij} \qquad 0 < r_{ij} \leq b \text{ for } i=1,\ldots,R \text{ and } j=1,\ldots,C.$$

It is important to observe that the range on $r_{ij}$ is

$$0 < r_{ij} \leq b$$

whereas in the previous procedure the range as $r_{ij}$ was

$$0 \leq r_{ij} < b.$$

Define the tables D and R exactly as earlier, and form a controlled rounding, S of R. The sum

$$B = bD + S$$

will then be a controlled rounding of A. To find S, begin by introducing a <u>slack row</u> and a <u>slack column</u> to "fold-in" R to form the R+1 by C+1 matrix C exactly as earlier with the single exception that

$$c_{R+1,C+1} = [(\sum_{i=1}^{R} c_{i,C+1})/b]b + b - \sum_{i=1}^{R} c_{i,C+1}$$

$$= [(\sum_{j=1}^{C} c_{R+1,j})/b]b + b - \sum_{j=1}^{C} c_{R+1,j} \ .$$

For the base 3 example introduced earlier:

A =

| | | | | |
|---|---|---|---|---|
| 119 | 24 | 40 | 18 | 37 |
| 15 | 4 | 8 | 3 | 0 |
| 41 | 7 | 13 | 1 | 20 |
| 19 | 1 | 5 | 9 | 4 |
| 44 | 12 | 14 | 5 | 13 |

3D =

| | | | | |
|---|---|---|---|---|
| 93 | 18 | 33 | 9 | 33 |
| 9 | 3 | 6 | 0 | 0 |
| 36 | 6 | 12 | 0 | 18 |
| 12 | 0 | 3 | 6 | 3 |
| 36 | 9 | 12 | 3 | 12 |

R =

| | | | | |
|---|---|---|---|---|
| 26 | 6 | 7 | 9 | 4 |
| 6 | 1 | 2 | 3 | 0 |
| 5 | 1 | 1 | 1 | 2 |
| 7 | 1 | 2 | 3 | 1 |
| 8 | 3 | 2 | 2 | 1 |

C =

| | | | | | |
|---|---|---|---|---|---|
| 45 | 9 | 9 | 12 | 6 | 9 |
| 9 | 1 | 2 | 3 | 0 | 3 |
| 6 | 1 | 1 | 1 | 2 | 1 |
| 9 | 1 | 2 | 3 | 1 | 2 |
| 9 | 3 | 2 | 2 | 1 | 1 |
| 12 | 3 | 2 | 3 | 2 | 2 |

In general, form a controlled rounding, F of C, where

$$f_{ij} \in \begin{cases} \{0,b,2b\} & \text{if } c_{ij} \text{ is a non-zero multiple b} \\ \{0,b\} & \text{otherwise,} \end{cases}$$

for i=1,...,R+1 and j=1,...,C+1. That is, solve the following system of equations for $f_{ij}$:

$$\sum_{j=1}^{C+1} f_{ij} = c_{i0} \qquad i=1,\dots,R+1$$

$$\sum_{i=1}^{R+1} f_{ij} = c_{0j} \qquad j=1.=,\dots,C+1$$

$$f_{ij} \in \begin{cases} \{0,b\} & \text{if } c_{ij} < b \\ \{0,b,2b\} & \text{if } c_{ij} = b \text{ ,} \end{cases}$$

for i=1,...,R+1 and j=1,...,C+1. This system does have solutions each of which corresponds to a saturated flow through the network in Figure 3. Nodes correspond to marginal constraints and arcs between node $n_{i0}$ and $n_{0j}$ correspond to cell (i,j). In this network, if $a_{mn}$ is a non-zero multiple of the base there will be two arcs between the pair of nodes $n_{m0}$ and $n_{0n}$, otherwise there is exactly one arc between those nodes. That is, if cell (m,n) of table C equals 1, that is, $c_{mn}$ = 1, then there are two arcs between row node m and column node n.

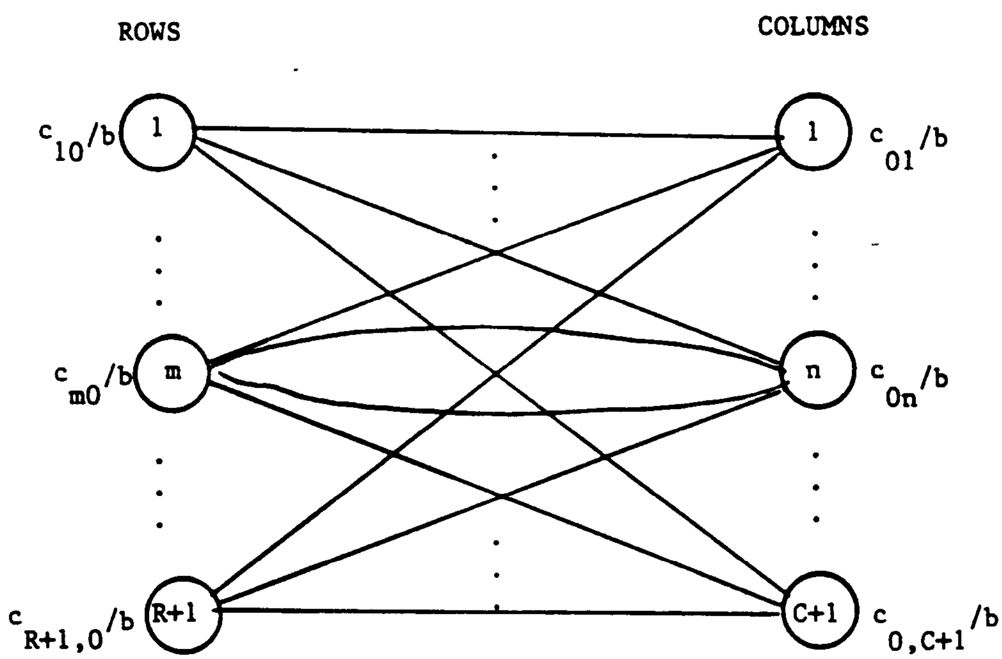


FIGURE 3

The nodes on the left correspond to sources, nodes on the right correspond to sinks, supplies and demands are shown alongside each source and sink respectively, each arc has upper capacity equal to one, and all arcs are directed from left to right. Set $f_{ij}$ equal to the total flow between nodes $n_{i0}$ and $n_{0j}$ times b. To obtain an arbitrary controlled rounding, set all costs arbitrarily. To obtain an arbitrary weakly zero-restricted controlled rounding, all arcs corresponding to true zeros are removed from the network (so $f_{ij} = 0$ for these arcs) and all other costs are set arbitrarily. To obtain a zero-restricted controlled rounding, remove all arcs corresponding to zero cells and for arcs that do not correspond to cells which are a multiple of the base, let the cost equal zero. If a cell is a non-zero multiple of the base, let one of the arcs have cost -1 and the other arc have cost +1. This assignment will encourage the flow of exactly one unit between the pair of nodes corresponding to a non-zero multiple of the base and make it equally costly to send no units as to send two units, thus encourage non-zero multiples of the base not to change. Since a zero-restricted controlled rounding always does exist, this will find such a controlled rounding.

We define the table S as before and observe that S is a controlled rounding of R (according to the new definition) by adapting the proof in Section A above and

$$B = bD + S$$

is a controlled rounding of A. One can see that for cells which are either zero or not a multiple of the base in A, if:

(a) for $i=1,\ldots,R$ $\quad j=1,\ldots,C$

$f_{ij} = 1$ then $b_{ij} = [a_{ij}/b]b+b$

$f_{ij} = 0$ then $b_{ij} = [a_{ij}/b]b$ ,

(b) for $i=1,\ldots,R$

$f_{i,C+1} = 1$ then $b_{i0} = [a_{i0}/b]b$

$f_{i,C+1} = 0$ then $b_{i0} = [a_{i0}/b]b+b$ ,

(c) for $j=1,\ldots,C$

$f_{R+1,j} = 1$ then $b_{0j} = [a_{0j}/b]b$

$f_{R+1,j} = 0$ then $b_{0j} = [a_{0j}/b]b+b$ ,

(d) $f_{R+1,C+1} = 1$ then $b_{00} = [a_{ij}/b]b+b$
$f_{R+1,C+1} = 0$ then $b_{00} = [a_{ij}/b]b$ .

For cells which are <u>non-zero multiples of the base</u>, if:

(a) for $i=1,\ldots,R$ $j=1,\ldots,C$
$f_{ij} = 2$ then $b_{ij} = a_{ij}+b$
$f_{ij} = 1$ then $b_{ij} = a_{ij}$
$f_{ij} = 0$ then $b_{ij} = a_{ij}-b$,

(b) for $i=1,\ldots,R$
$f_{i,C+1} = 2$ then $b_{i0} = a_{i0}-b$
$f_{i,C+1} = 1$ then $b_{i0} = a_{i0}$
$f_{i,C+1} = 0$ then $b_{i0} = a_{i0}+b$,

(c) for $j=1,\ldots,C$
$f_{R+1,j} = 2$ then $b_{0j} = a_{0j}-b$
$f_{R+1,j} = 1$ then $b_{0j} = a_{0j}$
$f_{R+1,j} = 0$ then $b_{0j} = a_{0j}+b$,

(d) $f_{R+1,C+1} = 2$ then $b_{00} = a_{00}+b$
$f_{R+1,C=1} = 1$ then $b_{00} = a_{00}$
$f_{R+1,C+1} = 0$ then $b_{00} = a_{00}-b$ .

To set up the network for the base 3 sample problem used earlier, let

$$A = \begin{array}{c|ccc} 29 & 7 & 6 & 16 \\ \hline 21 & 7 & 5 & 9 \\ 8 & 0 & 1 & 7, \end{array} \quad \text{so}$$

$$3D = \begin{array}{c|ccc} 21 & 6 & 3 & 12 \\ \hline 15 & 6 & 3 & 6 \\ 6 & 0 & 0 & 0, \end{array} \qquad R = \begin{array}{c|ccc} 8 & 1 & 3 & 4 \\ \hline 6 & 1 & 2 & 3 \\ 2 & 0 & 1 & 1, \end{array} \quad \text{and} \quad C = \begin{array}{c|cccc} 21 & 3 & 6 & 6 & 6 \\ \hline 9 & 1 & 2 & 3 & 3 \\ 3 & 0 & 1 & 1 & 1 \\ 9 & 2 & 3 & 2 & 2 \end{array} .$$

The network to find a controlled rounding for C is shown in Figure 4 with a flow displayed by each arc. Note that each arc is directed from left to right. For each pair of nodes having two arcs between them, assume the upper arc has cost -1 and the lower cost +1, all other arcs have cost zero.

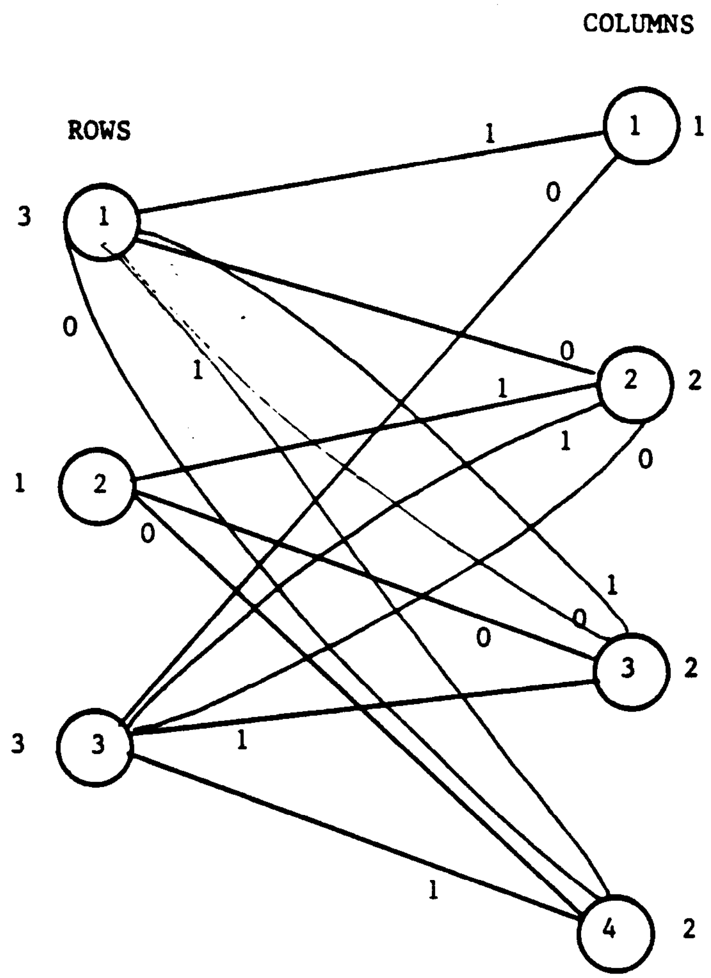


FIGURE 4

FIGURE 4

The arc corresponding to the zero cell of A, cell (2,1), has been omitted from this network to force a weakly zero-restricted controlled rounding. To ensure a zero-restricted controlled rounding, the pairs of nodes corresponding to cells with a non-zero multiple of the base in A; namely,

cell (0,2) with value 6
cell (1,0) with value 21
cell (1,3) with value 9,

have two arcs between them; one with cost -1 and the other with cost +1. Interpreting the saturated flow through this network in terms of a base 3 controlled rounding, F of C, and then a base 3 zero-restricted controlled rounding, B of A, yields:

F =

| 21 | 3 | 6 | 6 | 6 |
|---|---|---|---|---|
| 9 | 3 | 0 | 3 | 3 |
| 3 | 0 | 3 | 0 | 0 |

B =

| 30 | 9 | 6 | 15 |
|---|---|---|---|
| 21 | 9 | 3 | 9 |
| 9 | 0 | 3 | 6 |

Contining with the base 3example used earlier, let

A =

| 119 | 24 | 40 | 18 | 37 |
|---|---|---|---|---|
| 15 | 4 | 8 | 3 | 0 |
| 41 | 7 | 13 | 1 | 20 |
| 19 | 1 | 5 | 9 | 4 |
| 44 | 12 | 14 | 5 | 13 |

3D =

| 93 | 18 | 33 | 9 | 33 |
|---|---|---|---|---|
| 9 | 3 | 6 | 0 | 0 |
| 36 | 6 | 12 | 0 | 18 |
| 12 | 0 | 3 | 6 | 3 |
| 36 | 9 | 12 | 3 | 12 |

R =

| 26 | 6 | 7 | 9 | 4 |
|---|---|---|---|---|
| 6 | 1 | 2 | 3 | 0 |
| 5 | 1 | 1 | 1 | 2 |
| 7 | 1 | 2 | 3 | 1 |

C =

| 45 | 9 | 9 | 12 | 6 | 9 |
|---|---|---|---|---|---|
| 9 | 1 | 2 | 3 | 0 | 3 |
| 6 | 1 | 1 | 1 | 2 | 1 |
| 9 | 1 | 2 | 3 | 1 | 2 |
| 9 | 3 | 2 | 2 | 1 | 1 |
| 12 | 3 | 2 | 3 | 2 | 2 |

As above, our objective is to obtain a controlled rounding, F of C, extract the upper left RxC subtable with derived marginals which will be controlled rounding, S of R, and form

$$B = 3D + S$$

which will be a base 3 controlled rounding of A. Below are three controlled roundings, F of C, the corresponding controlled rounding, S of R, and finally the controlled rounding, B of A.

1'.

$F_i$

| 45 | 9 | 9 | 12 | 6 | 9 |
|---|---|---|---|---|---|
| 9 | 0 | 3 | 3 | 0 | 3 |
| 6 | 3 | 0 | 0 | 3 | 0 |
| 9 | 0 | 3 | 3 | 0 | 3 |
| 9 | 3 | 0 | 3 | 3 | 0 |
| 12 | 3 | 3 | 3 | 0 | 3 |

$S_i$

| 24 | 6 | 6 | 9 | 6 |
|---|---|---|---|---|
| 6 | 0 | 3 | 3 | 0 |
| 6 | 3 | 0 | 0 | 3 |
| 6 | 0 | 3 | 3 | 0 |
| 9 | 3 | 0 | 3 | 3 |

$B_i$

| 120 | 24 | 39 | 18 | 39 |
|---|---|---|---|---|
| 15 | 3 | 9 | 3 | 0 |
| 42 | 9 | 12 | 0 | 21 |
| 18 | 0 | 6 | 9 | 3 |
| 45 | 12 | 12 | 6 | 15 |

2'.

$F_i$

| 45 | 9 | 9 | 12 | 6 | 9 |
|---|---|---|---|---|---|
| 9 | 3 | 3 | 3 | 0 | 0 |
| 6 | 0 | 3 | 0 | 0 | 3 |
| 9 | 3 | 0 | 3 | 3 | 0 |
| 9 | 3 | 0 | 3 | 0 | 3 |
| 12 | 0 | 3 | 3 | 3 | 3 |

$S_i$

| 27 | 9 | 6 | 9 | 3 |
|---|---|---|---|---|
| 9 | 3 | 3 | 3 | 0 |
| 3 | 0 | 3 | 0 | 0 |
| 9 | 3 | 0 | 3 | 3 |
| 6 | 3 | 0 | 3 | 0 |

$B_i$

| 120 | 27 | 39 | 18 | 36 |
|---|---|---|---|---|
| 18 | 6 | 9 | 3 | 0 |
| 39 | 6 | 15 | 0 | 18 |
| 21 | 3 | 3 | 9 | 6 |
| 42 | 12 | 12 | 6 | 12 |

3'.

$F_i$

| 45 | 9 | 9 | 12 | 6 | 9 |
|---|---|---|---|---|---|
| 9 | 0 | 0 | 3 | 3 | 3 |
| 6 | 0 | 3 | 0 | 0 | 3 |
| 9 | 0 | 3 | 6 | 0 | 0 |
| 9 | 6 | 3 | 0 | 0 | 0 |
| 12 | 3 | 0 | 3 | 3 | 3 |

$S_i$

| 27 | 6 | 9 | 9 | 3 |
|---|---|---|---|---|
| 6 | 0 | 0 | 3 | 3 |
| 3 | 0 | 3 | 0 | 0 |
| 9 | 0 | 3 | 6 | 0 |
| 9 | 6 | 3 | 0 | 0 |

$B_i$

| 120 | 24 | 42 | 18 | 36 |
|---|---|---|---|---|
| 15 | 3 | 6 | 3 | 3 |
| 39 | 6 | 15 | 0 | 18 |
| 21 | 0 | 6 | 12 | 3 |
| 45 | 15 | 15 | 3 | 12 |

Note that the tables $B_i$ are exactly those derived in the preceeding section so all remarks in that section apply; the tables $F_i$ and $S_i$, however are different from those earlier. Table $B_1$' is a zero-restricted rounding, $B_2$' is weakly zero-restricted, and $B_3$' is neither. The objective of this little exercise was to show how this alternative definition plays out and to show (as one would expect) that every controlled rounding under the previous definition is a controlled rounding in the new definition. In fact,

$$d_{ij} \text{ (old definition)} = d_{ij} + 1 \text{ new definition)}$$

and

$$f_{ij} \text{ (old definition)} = f_{ij} - 1 \text{ (new definition)}$$

for those cells, $a_{ij}$ which are non-zero multiples of the base for $i = 1,\ldots,R$ and $j = 1,\ldots,C$.

In the next few examples we show controlled roundings which conform to the new definition but not the old. We employ the same tables A, D, R and C used earlier.

$F_i$ — $S_i$ — $B_i$

4.

$F_i$

| 45 | 9 | 9 | 12 | 6 | 9 |
|---|---|---|---|---|---|
| 9 | 0 | 0 | 3 | 0 | 6 |
| 6 | 0 | 3 | 0 | 3 | 0 |
| 9 | 0 | 3 | 3 | 0 | 3 |
| 9 | 3 | 3 | 0 | 3 | 0 |
| 12 | 6 | 0 | 6 | 0 | 0 |

$S_i$

| 24 | 3 | 9 | 6 | 6 |
|---|---|---|---|---|
| 3 | 0 | 0 | 3 | 0 |
| 6 | 0 | 3 | 0 | 3 |
| 6 | 0 | 3 | 3 | 0 |
| 9 | 3 | 3 | 0 | 3 |

$B_i$

| 117 | 21 | 42 | 15 | 39 |
|---|---|---|---|---|
| 12 | 3 | 6 | 3 | 0 |
| 42 | 6 | 15 | 0 | 21 |
| 18 | 0 | 6 | 9 | 3 |
| 45 | 12 | 15 | 3 | 15 |

5.

$F_i$

| 45 | 9 | 9 | 12 | 6 | 9 |
|---|---|---|---|---|---|
| 9 | 3 | 3 | 0 | 0 | 3 |
| 6 | 3 | 3 | 0 | 0 | 0 |
| 9 | 3 | 0 | 6 | 0 | 0 |
| 9 | 0 | 0 | 3 | 3 | 3 |
| 12 | 0 | 3 | 3 | 3 | 3 |

$S_i$

| 27 | 9 | 6 | 9 | 3 |
|---|---|---|---|---|
| 6 | 3 | 3 | 0 | 0 |
| 6 | 3 | 3 | 0 | 0 |
| 9 | 3 | 0 | 6 | 0 |
| 6 | 0 | 0 | 3 | 3 |

$B_i$

| 120 | 27 | 39 | 18 | 36 |
|---|---|---|---|---|
| 15 | 6 | 9 | 0 | 0 |
| 42 | 9 | 15 | 0 | 18 |
| 21 | 3 | 3 | 12 | 3 |
| 42 | 9 | 12 | 6 | 15 |

6.

| 45 | 9 | 9 | 12 | 6 | 9 |
|---|---|---|---|---|---|
| 9 | 0 | 3 | 6 | 0 | 0 |
| 6 | 0 | 3 | 0 | 0 | 3 |
| 9 | 0 | 0 | 6 | 0 | 3 |
| 9 | 3 | 0 | 0 | 3 | 3 |
| 12 | 6 | 3 | 0 | 3 | 0 |

| 24 | 3 | 6 | 12 | 3 |
|---|---|---|---|---|
| 9 | 0 | 3 | 6 | 0 |
| 3 | 0 | 3 | 0 | 0 |
| 6 | 0 | 0 | 6 | 0 |
| 6 | 3 | 0 | 0 | 3 |

| 117 | 21 | 39 | 21 | 36 |
|---|---|---|---|---|
| 18 | 3 | 9 | 6 | 0 |
| 39 | 6 | 15 | 0 | 18 |
| 18 | 0 | 3 | 12 | 3 |
| 42 | 12 | 12 | 3 | 15 |

7.

| 45 | 9 | 9 | 12 | 6 | 9 |
|---|---|---|---|---|---|
| 9 | 3 | 3 | 0 | 0 | 3 |
| 6 | 3 | 0 | 3 | 0 | 0 |
| 9 | 3 | 3 | 0 | 0 | 3 |
| 9 | 0 | 3 | 3 | 3 | 0 |
| 12 | 0 | 0 | 6 | 3 | 3 |

| 27 | 9 | 9 | 6 | 3 |
|---|---|---|---|---|
| 6 | 3 | 3 | 0 | 0 |
| 6 | 3 | 0 | 3 | 0 |
| 6 | 3 | 3 | 0 | 0 |
| 9 | 0 | 3 | 3 | 3 |

| 120 | 27 | 42 | 15 | 36 |
|---|---|---|---|---|
| 15 | 6 | 9 | 0 | 0 |
| 42 | 9 | 12 | 3 | 18 |
| 18 | 3 | 6 | 6 | 3 |
| 45 | 9 | 15 | 6 | 15 |

8.

| 45 | 9 | 9 | 12 | 6 | 9 |
|---|---|---|---|---|---|
| 9 | 0 | 0 | 0 | 3 | 6 |
| 6 | 0 | 3 | 3 | 0 | 0 |
| 9 | 3 | 3 | 3 | 0 | 0 |
| 9 | 3 | 3 | 3 | 0 | 0 |
| 12 | 3 | 0 | 3 | 3 | 3 |

| 27 | 6 | 9 | 9 | 3 |
|---|---|---|---|---|
| 3 | 0 | 0 | 0 | 3 |
| 6 | 0 | 3 | 3 | 0 |
| 9 | 3 | 3 | 3 | 0 |
| 9 | 3 | 3 | 3 | 0 |

| 120 | 24 | 42 | 18 | 36 |
|---|---|---|---|---|
| 12 | 3 | 6 | 0 | 3 |
| 42 | 6 | 15 | 3 | 18 |
| 21 | 3 | 6 | 9 | 3 |
| 45 | 12 | 15 | 6 | 12 |

Note that $B_4$ throught $B_7$ are all weakly zero-restricted and in $B_8$ the zero cell (1,4) becomes a 3. In $B_4$ all marginal multiples of the base decrease and interior multiples of the base do not change. In $B_5$ all marginal multiples of the base increase or stay the same while non-zero multiples of the base both increase and decrease in the interior. The remaining tables illustrate additional variations.

## III. MINIMIZING A MEASURE OF CLOSENESS

A. Controlled rounding under definition that $b_{ij} = [a_{ij}/b]b$ or $[a_{ij}/b]b+b$

Given a two way table A, Cox and Ernst (1982) seek a zero-restricted controlled rounding, B of A, which minimizes the objective function

$$G_1 = \sum_{i=1}^{R} \sum_{j=1}^{C} |a_{ij}-b_{ij}|^p$$

for $1 \leq p \leq \infty$ . In this note, we confine our attention to the cases $1 \leq p < \infty$ . For R as defined earlier it suffices to find a (zero-restricted) controlled rounding, S of R, to minimize

$$\sum_{i=1}^{R} \sum_{j=1}^{C} |r_{ij}-s_{ij}|^p$$

and form

$$B = bD + S$$

to obtain a (zero-restricted) controlled rounding which minimizes

$$\sum_{i=1}^{R} \sum_{j=1}^{C} |a_{ij}-b_{ij}|^p$$

since

$$a_{ij} - b_{ij} = r_{ij} - s_{ij} \quad i=0,\ldots,R \quad j=0,\ldots,C.$$

Thus all computations can be done over R and since $r_{ij}<b$ for $i=1,\ldots,R$ and $j=1,\ldots,C$ we can divide all entries of R by b and assume without loss of generality that $b=1$ and $r_{ij}<1$ for $i=1,\ldots,R$ and $j=1,\ldots,C$. By folding in R to form C, it suffices to minimze:

$$F_1' = \sum_{i=1}^{R} \sum_{j=1}^{C} |c_{ij}-f_{ij}|^p$$

subject to

(4) $\sum_{j=1}^{C+1} f_{ij} = c_{i0}$ $\qquad i=1,\ldots,R+1$

(5) $\sum_{i=1}^{R+1} f_{ij} = c_{0j}$ $\qquad j=1,\ldots,C+1$

(6) $f_{ij}\varepsilon\{0,1\}$ $\quad i=1,\ldots,R+1 \qquad j=1,\ldots,C+1$

(7) $f_{ij} = c_{ij}$ if $[c_{ij}]=c_{ij}$ $\qquad i=1,\ldots,R+1 \quad j=1,\ldots,C+1$

to obtain a zero-restricted controlled rounding of C which minimizes $F_1'$. One derives from table F as in Section II a zero-restricted controlled rounding, B of A, which minimizes

$$\sum_{i=1}^{R} \sum_{j=1}^{C} |a_{ij}-b_{ij}|^P .$$

As shown in Cox and Ernst, the controlled rounding which minimizes $F_1'$ will also minimize

$$F_1 = \sum_{i=1}^{R} \sum_{j=1}^{C}((1-c_{ij})^P-(c_{ij})^P)f_{ij} ,$$

and conversly. However, $F_1$ is linear in the $f_{ij}$ so we can take advantage of integer <u>linear</u> techniques. In particular, use the network in Figure 1 in which all arcs have upper capacity one and,

(a) if $c_{ij} = 0$, remove arc $(n_{i0},n_{0j})$
(b) if $c_{ij} = 1$, set lower capacity on arc $(n_{i0},n_{0j})$ equal to 1
(c) the cost $d_{ij}$ over arc $(n_{i0},n_{0j})$ is

$$d_{ij} = \begin{cases} (1-c_{ij})^P-(c_{ij})^P & i=1,\ldots,R \quad j=1,\ldots,C \\ 0 \quad \text{otherwise} . \end{cases}$$

The minimal cost saturated flow over this network will yield a controlled rounding, F of C, and hence a zero-restricted controlled rounding, B of A, which minimizes, $F_1$ and hence $G_1$.

The notion of closeness can be extended to include marginal positions. One can seek a zero-restricted controlled rounding, B of A, to minimize the objective function:

$$G_2 = \sum_{i=0}^{R} \sum_{j=0}^{C} |a_{ij} - b_{ij}|^P .$$

Employing arguments as above, forming R and finding a zero-restricted controlled rounding, S of R, to minimize

$$\sum_{i=0}^{R} \sum_{j=0}^{C} |r_{ij} - s_{ij}|^P$$

will yield a zero-restricted controlling of A,

$$B = bD + S,$$

which minimizes $G_2$. Folding-in R to obtain C, a zero-restricted controlled rounding, F of C, which minimizes

$$F_2' = \sum_{i=1}^{R+1} \sum_{j=1}^{C+1} |c_{ij} - f_{ij}|^P$$

will produce, as in Section II, a zero-restricted controlled rounding, S of R, which minimizes

$$\sum_{i=0}^{R} \sum_{j=0}^{C} |r_{ij} - s_{ij}|^P .$$

This is proved in the Appendix by showing that the two objective functions

$$\sum_{i=1}^{R+1}\sum_{j=1}^{C+1}|c_{ij} - f_{ij}|^P \text{ and } \sum_{i=0}^{R}\sum_{j=0}^{C}|r_{ij} - s_{ij}|^P$$

differ by a constant when R, C, S, and F are related as in Section II. By using techniques as in Cox and Ernst (1982) one can show that to find the zero-restricted controlled rounding, F of C, which minimizes $F_2'$ it suffices to find a controlled rounding which minimizes

$$F_2 = \sum_{i=1}^{R+1}\sum_{j=1}^{C+1}((1-c_{ij})^P-(c_{ij})^P)f_{ij} \ .$$

To find such a controlled rounding, use the network of Figure 1, as before, in which all arcs have upper capacity one and where now:

(a) if $c_{ij} = 0$, remove arc $(n_{i0},n_{0j})$,
(b) if $c_{ij} = 1$, set lower capacity on arc $(n_{i0},n_{0j})$ equal to 1,
(c) let the cost over arc $(n_{i0},n_{0j})$ be equal to $((1-c_{ij})^P-(c_{ij})^P)$ for $i=1,\ldots,R+1$, $j=1,\ldots,C+1$.

The minimal cost saturated flow over this network will yield a zero-restricted controlled rounding, F of C, which minimizes $F_2$ and (as in Section II) a zero-restricted controlled rounding, B of A, which minimizes $G_2$.

The objective functions $G_1$ and $G_2$ are not equivalent, and in fact, if

A =

| | | | |
|---|---|---|---|
| 8/3 | 1 | 2/3 | 1 |
| 2/3 | 0 | 1/3 | 1/3 |
| 1 | 1/3 | 1/3 | 1/3 |
| 1 | 2/3 | 0 | 1/3 , |

$B_1$ =

| | | | |
|---|---|---|---|
| 2 | 1 | 0 | 1 |
| 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 1 |
| 1 | 1 | 0 | 0 , |

$B_2$ =

| | | | |
|---|---|---|---|
| 3 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 , |

then under an integer (base 1) controlled rounding $B_1$ minimizes $G_1$ and $B_2$ minimizes $G_2$, however, $B_1$ does not minimize $G_2$ nor does $B_2$ minimize $G_1$ as can be verified by direct computation.

Only <u>zero-restricted controlled roundings</u> of a table A were considered by Cox and Ernst in minimizing either of the objective functions $G_1$ or $G_2$. Since zero-restricted controlled roundings are a proper subset of all controlled roundings, it is reasonable to expect that given a table A the minimum of $G_1$ or $G_2$ over all controlled roundings will be strictly less that over only zero-restricted controlled roundings. This is the case as is shown by the example below. Let

A =

| 3 | 3/4 | 3/4 | 3/4 | 3/4 |
|---|---|---|---|---|
| 3/4 | 3/4 | 0 | 0 | 0 |
| 3/4 | 0 | 3/4 | 0 | 0 |
| 3/4 | 0 | 0 | 3/4 | 0 |
| 3/4 | 0 | 0 | 0 | 3/4 |

$B_3$ =

| 4 | 1 | 1 | 1 | 1 |
|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 | 0 |
| 1 | 0 | 0 | 1 | 0 |
| 1 | 0 | 0 | 0 | 1 |

and note that $B_3$ is a (non-zero-restricted) controlled rounding of A. One can compute:

$$G_1(B_1) = 4(1/4)^P \qquad G_2(B_1) = 12(1/4)^P + 1.$$

For any zero-restricted controlled rounding of A, the grand total, $a_{00}$, must remain as 3, so exactly one column marginal and exactly one row marginal must equal 0. Thus, exactly one diagonal in A must equal zero. By a row and column exchange, the table

$B_4$ =

| 3 | 0 | 1 | 1 | 1 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 | 0 |
| 1 | 0 | 0 | 1 | 0 |
| 1 | 0 | 0 | 0 | 1 |

is the unique zero-restricted controlled rounding of A. Thus the minima of $G_1$ and $G_2$ over all zero-restricted controlled roundings are realized at $B_4$, and

$$G_1(B_4) = 3(1/4)^P + (3/4)^P \qquad G_2(B_4) = 9(1/4)^P + 3(3/4)^P .$$

It is clear that

$$G_1(B_3) < G_1(B_4)$$

for all $p \geq 1$ and

$$G_2(B_3) < G_2(B_4)$$

for p = 1,2,3. Thus, in general, the minimum of $G_1$ and $G_2$ are not realized over zero-restricted controlled roundings. Note however that controlled roundings, $B_3$ and $B_4$ are both are weakly zero-restricted. It remains an open question as to whether $G_1$ and $G_2$ do achieve their minima over weakly zero-restricted controlled roundings.

To formulate a linear approach to address non-zero-restricted controlled roundings, B of A, with respect to minimizing a distance function, we can focus on either $G_1$ or $G_2$ -- the analyses are comparable. Below we examine $G_2$ and couch the discussion in terms of $F_2$ over C. The goal is to find a controlled rounding, B of A, to minimize:

$$G_2 = \sum_{i=1}^{R+1} \sum_{j=1}^{C+1} |b_{ij}-a_{ij}|^P .$$

This objective is equivalent to minimizing the linear function

$$F_2 = \sum_{i=1}^{R+1} \sum_{j=1}^{C+1} ((1-c_{ij})^P-(c_{ij})^P)f_{ij}$$

subject to (4), (5), and (6). This is most easily done by finding a minimal cost saturated flow over the network in Figure 1 where the cost on arc $(n_{i0},n_{0j})$ is

$$(1-c_{ij})^P-(c_{ij})^P$$

for i=1,..., R+1 and j=1,...,C+1.

Note that a <u>zero</u> in C corresponds to an internal multiple of the base in A and a <u>one</u> in C corresponds to a marginal multiple of the base in A. To ensure a weakly zero-restricted controlled rounding of A, remove arcs from the network that correspond to true zeros in A (i.e., $a_{ij}$ = 0). As noted above, however, it is not clear that such a step is needed as controlled roundings of A that minimize $G_1$ or $G_2$ may always be weakly zero-restricted.

B. Controlled rounding under definition that $|b_{ij} - a_{ij}| \leq b$

In this section, we examine the distance measure $G_2$ under the extended definition of controlled rounding. A comparable analysis can be carried out for $G_1$. If A is a table, one seeks a controlled rounding, under the extended definition, B of A, which minimizes

$$G_2 = \sum_{i=0}^{R} \sum_{j=0}^{C} |b_{ij}-a_{ij}|^p$$

for $1 \leq p < \infty$. Since every controlled rounding under the extended definition is also a controlled rounding under the Cox-Ernst definition the minimum of $G_2$ under the extended definition is less than or equal to the minimum under the earlier definition. Strict inequality holds as can be seen by considering the table

$$A = \begin{array}{c|cccc} 1 & 1/4 & 1/4 & 1/4 & 1/4 \\ \hline 1/4 & 1/4 & 0 & 0 & 0 \\ 1/4 & 0 & 1/4 & 0 & 0 \\ 1/4 & 0 & 0 & 1/4 & 0 \\ 1/4 & 0 & 0 & 0 & 1/4 \end{array}$$

which has minima for $G_1$ and $G_2$ realized by the weakly zero-restricted controlled rounding under the extended definition,

$$B = \begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} .$$

As in Section A above, to find a controlled rounding, B of A, to minimize $G_2$, it suffices to find a controlled rounding, S of R, to minimize

$$\sum_{i=0}^{R} \sum_{j=0}^{C} |r_{ij}-s_{ij}|^p .$$

To find the controlled rounding, S of R, to minimize this function it suffices to find a controlled rounding, F of C, to minimize

$$E_2'' = \sum_{i=1}^{R+1} \sum_{j=1}^{C+1} |c_{ij}-f_{ij}|^P$$

as can be seen by modifying the proof of the Proposition in the Appendix to fit the extended definition of controlled rounding. For the extended definition of rounding, as we pass from A to R and from R to C observe that $r_{ij} \geq 1$ and a <u>zero</u> in R corresponds to a true zero in A and a <u>one</u> in the interior of R corresponds to a non-zero multiple of the base in A. Letting

$$T = \{(i,j)\varepsilon(R+1)x(C+1)|c_{ij} \neq 1\}$$
$$S = \{(i,j)\varepsilon(R+1)x(C+1)|c_{ij} = 1\} ,$$

the objective is to minimize:

$$E_2' = \sum_{(i,j)\varepsilon T} |c_{ij}-f_{ij}|^P + \sum_{(i,j)\varepsilon S} |1-(g_{ij}+h_{ij})|^P$$

subject to

(8) $$\sum_{(i,j)\varepsilon T} f_{ij} + \sum_{(i,j)\varepsilon S} (g_{ij}+h_{ij}) = c_{i0} \qquad i=1,\ldots,R+1$$

(9) $$\sum_{(i,j)\varepsilon T} f_{ij} + \sum_{(i,j)\varepsilon S} (g_{ij}+h_{ij}) = c_{0j} \qquad j=1,\ldots,C+1$$

(10) $$f_{ij}, g_{ij}, h_{ij} \varepsilon \{0,1\} \quad i=1,\ldots,R+1 \qquad j=1,\ldots,C+1 .$$

(Note that the sum $g_{ij}+h_{ij}$ for $(i,j)\varepsilon S$ corresponds to the two arcs in Figure 3 between $(n_{i0},n_{0j})$ for $(i,j)\varepsilon S$).

We can replace the second summand in $E_2'$ by

$$\sum_{(i,j)\varepsilon S} |1-(g_{ij}-h_{ij})|^P$$

if we add to (8)-(10) the additional constraint

(11) $h_{ij} \leq g_{ij}$ for $i=1,\ldots,R+1$ $j=1,\ldots,C+1$

since

$$|1-(g_{ij}+h_{ij})|=|1-(g_{ij}-h_{ij})|$$

subject to (10) and (11). But

$$1-(g_{ij}-h_{ij})\varepsilon\{0,1\}$$

subject to (10) and (11) so

$$|1-(g_{ij}-h_{ij})|^P=1-(g_{ij}-h_{ij})$$

subject to (10) and (11). Thus,

$$E_2 = \sum_{(i,j)\varepsilon T}((1-c_{ij})^P-(c_{ij})^P)f_{ij}+\sum_{(i,j)\varepsilon S}(1+h_{ij}-g_{ij})$$

$$=\sum_{(i,j)\varepsilon T}((1-c_{ij})^P-(c_{ij})^P)f_{ij}+\sum_{(i,j)\varepsilon S}(h_{ij}-g_{ij})+|S|$$

subject to (8)-(11) is equivalent to $E_2'$ subject to (8)-(10), where $|S|$ is the cardinality of the set S.

Note that $h_{ij}$ and $g_{ij}$ are symetric with respect to (8) and (9) and the value of $E_2'$ will always be less for

$$h_{ij} = 0 \text{ and } g_{ij} = 1$$

than for

$$h_{ij} = 1 \text{ and } g_{ij} = 0.$$

Thus, the minimum of $E_2'$ subject to (8)-(11) is identical to the minimum of $E_2$ subject to (8)-(10). That is, condition (11) is <u>not needed</u>.

Hence, to find a controlled rounding, B of A, under the extended definition to minimize $G_2$ it suffices to find the minimum of $E_2$ subject to (8)-(10). To find the minimum of $E_2$ subject to (8)-(10) employ the network in Figure 3 and assign costs as follows

(a) for $(i,j) \varepsilon$ T the cost on $(n_{i0}, n_{0j})$ is equal to $(1-c_{ij})^P-(c_{ij})^P$

(b) for $(i,j) \varepsilon S$ the cost on one of the arcs between $n_{i0}$ and $n_{0j}$ is equal to -1 and on the other arc equal to +1.

A minimum cost saturated flow over this network will yield a controlled rounding, F or C, leading to a controlled rounding, B of A, as in Section II. By deleting all arcs from this network corresponding to zero cells in C, one obtains the weakly zero-restricted controlled rounding that minimizes $G_2$.

### C. Concluding Remarks

In this section, we considered controlled roundings, B of A, to minimize either of the objective functions $G_1$ or $G_2$, either zero-restricted or not zero-restricted, and using either the Cox-Ernst definition of controlled rounding or the definition introduced here. Similiarities have been examined, and examples have been provided to exhibit differences. The open question remains as to whether the minimum of $G_1$ or $G_2$ under any of the scenarios above can be achieved with a weakly zero-restricted controlled rounding.

## IV. SUMMARY

In this report we extend the definition of controlled rounding to allow a non-zero multiple of the base to either increase or decrease by the value of the base. Under the more standard definition of controlled rounding (Cox, Ernst) a non-zero multiple of the base can increase but not decrease. We exhibited step-by-step method for finding two dimensional controlled roundings under this new definition and contrasted the methods and results with the usual definition.

Procedures were developed to find controlled roundings of a table which minimize a measure of closeness-of-fit which can be applied under either of the two definitions for not-necessarily zero-restricted controlled roundings.

Examples were provided to show that the "closest" controlled rounding of a table need not be zero-restricted and in fact, may be a rounding under the new definition but not the standard definition. The notion of weakly zero-restricted controlled rounding has been introduced -- under which a non-zero multiple of the base can change while zeros must remain zeros. It is an open question as to whether each of the measures of closeness examined in this report can be optimized over all controlled roundings by a weakly zero-restricted controlled rounding.

The extended definition of controlled rounding and weakly zero-restricted introduced here for two dimensional tables can be applied to three (and higher) dimensional tables, see Fagan, Greenberg, and Hemmig (1988).

# APPENDIX

We freely use here the notation and conventions established in the body of the text. Let

$$\alpha = \begin{array}{c|cccc} \alpha_{00} & \alpha_{01} & \alpha_{02} & \cdots & \alpha_{0C} \\ \hline \alpha_{10} & \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1C} \\ \alpha_{20} & \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2C} \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ \alpha_{R0} & \alpha_{R1} & \alpha_{R2} & \cdots & \alpha_{RC} \end{array}$$

be an arbitrary RxC two-way table having a base 1 controlled rounding as defined in Cox and Ernst (1982)

$$\beta = \begin{array}{c|cccc} \beta_{00} & \beta_{01} & \beta_{02} & \cdots & \beta_{0C} \\ \hline \beta_{10} & \beta_{11} & \beta_{12} & \cdots & \beta_{1C} \\ \beta_{20} & \beta_{21} & \beta_{22} & \cdots & \beta_{2C} \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ \beta_{R0} & \beta_{R1} & \beta_{R2} & \cdots & \beta_{RC} \end{array} .$$

To find a linear expression for the distance measure

$$\sum_{i=0}^{R} \sum_{j=0}^{C} |\beta_{ij}-\alpha_{ij}|^{p}$$

for $1 \leq p < \infty$, we follow along the lines of Cox and Ernst. Letting

$$D = \{(i,j)|\beta_{ij}=[\alpha_{ij}]\} \text{ and } U = \{(i,j)|\beta_{ij}=[\alpha_{ij}]+1\}$$

and

$$\bar{\alpha}_{ij} = \alpha_{ij}-[\alpha_{ij}],$$

then

$$\sum_{i=0}^{R} \sum_{j=0}^{C} |\beta_{ij}-\alpha_{ij}|^{p} = \sum_{D} (\bar{\alpha}_{ij})^{p} + \sum_{U} (1-\bar{\alpha}_{ij})^{p}$$

$$= \sum_{i=0}^{R} \sum_{j=0}^{C} (\bar{\alpha}_{ij})^P (1+[\alpha_{ij}]-\beta_{ij}) + \sum_{i=0}^{R} \sum_{j=0}^{C} (1-\bar{\alpha}_{ij})^P (\beta_{ij}-[\alpha_{ij}])$$

$$= \sum_{i=0}^{R} \sum_{j=0}^{C} ((1-\bar{\alpha}_{ij})^P - (\bar{\alpha}_{ij})^P)\beta_{ij}$$

$$+ \sum_{i=0}^{R} \sum_{j=0}^{C} ((\bar{\alpha}_{ij})^P - (1-\bar{\alpha}_{ij})^P)[\alpha_{ij}] + \sum_{i=0}^{R} \sum_{j=0}^{C} (\bar{\alpha}_{ij})^P .$$

Similarly,

$$\sum_{j=1}^{R} \sum_{j=1}^{C} |\beta_{ij}-\alpha_{ij}|^P = \sum_{i=1}^{R} \sum_{j=1}^{C} (1-\bar{\alpha}_{ij})^P - (\bar{\alpha}_{ij})^P \beta_{ij}$$

$$+ \sum_{i=1}^{R} \sum_{j=1}^{C} ((\bar{\alpha}_{ij})^P - (1-\bar{\alpha}_{ij})^P)[\alpha_{ij}] + \sum_{i=1}^{R} \sum_{j=1}^{C} (\bar{\alpha}_{ij})^P .$$

(If $\alpha_{ij} < 1$ for $i=1,\ldots,R$ $j=1,\ldots,C$ then $\bar{\alpha}_{ij} = \alpha_{ij}$ and $[\alpha_{ij}]=0$, and this last equation reduces to equation (15) in Cox and Ernst.) Thus, if $\alpha$ and $\beta$ are RxC tables

$$(1) \qquad \sum_{i=1}^{R} \sum_{j=1}^{C} |\beta_{ij}-\alpha_{ij}|^P = \sum_{i=1}^{R} \sum_{j=1}^{C} ((1-\bar{\alpha}_{ij})^P - (\bar{\alpha}_{ij})^P)\beta_{ij} + K_1$$

and if $\alpha$ and $\beta$ are (R+1) by (C+1) tables

$$(2) \qquad \sum_{i=1}^{R+1} \sum_{j=1}^{C+1} |\beta_{ij}-\alpha_{ij}|^P = \sum_{i=1}^{R+1} \sum_{j=1}^{C+1} ((1-\bar{\alpha}_{ij})^P - (\bar{\alpha}_{ij})^P)\beta_{ij} + K_2$$

where $K_1$ and $K_2$ are constants.

Given an RxC table A, write the table sum

$$A = D + R$$

where $d_{ij} = [a_{ij}]$ and so $0 \le r_{ij} < 1$ $i=1,\ldots,R$ $j=1,\ldots,C$. If C is the "fold-in" of R then

(3) $c_{ij} = r_{ij}$ $\quad i=1,\ldots,R \quad j=1,\ldots,C$

(4) $c_{i,C+1} = [r_{i0}]+1-r_{i0}$ $\quad i=1,\ldots,R$

(5) $c_{R+1,j} = [r_{0j}]+1-r_{0j}$ $\quad j=1,\ldots,C$

(6) $c_{R+1,C+1} = r_{00} - [r_{00}]$

and if F is a controlled rounding of C, then F induces a controlled rounding, S of R, as in Section II of the text. Furthermore, every controlled rounding, S or R, can "fold-in" to a controlled rounding, F of C, where

(7) $f_{ij} = s_{ij}$ $\quad i=1,\ldots,R \quad j=1,\ldots,C$

(8) $f_{i,C+1} = [r_{i0}]+1-s_{i0}$ $\quad i=1,\ldots,R$

(9) $f_{R+1,j} = [r_{0j}]+1-s_{0j}$ $\quad j=1,\ldots,C$

(10) $f_{R+1,C+1} = s_{00}-[r_{00}]$ .

That is, there is a one-one onto correspondence between controlled roundings of R and those of C where the correspondence is through the mapping of "fold-in" as indicated.

Proposition: Given R and C as above and the correspondence between controlled roundings of R and of C, the controlled rounding S of R which minimizes

$$H_1(S) = \sum_{i=0}^{R} \sum_{j=0}^{C} |s_{ij}-r_{ij}|^P$$

corresponds to the controlled rounding F of C which minimizes

$$H_2(F) = \sum_{i=1}^{R+1} \sum_{j=1}^{C+1} |f_{ij}-c_{ij}|^p$$

for $1 \le p < \infty$.

Proof: Let S be a controlled rounding of R, F a controlled rounding of C, and assume S and F correspond as above. Note that

$$\sum_{i=1}^{R+1} \sum_{j=1}^{C+1} ((1-c_{ij})^p - (c_{ij})^p)f_{ij} = \sum_{i=1}^{R} \sum_{j=1}^{C} ((1-c_{ij})^p-(c_{ij})^p)f_{ij}$$

$$+ \sum_{i=1}^{R}((1-c_{i,C+1})^p-(c_{i,C+1})^p)f_{i,C+1}+\sum_{j=1}^{C}((1-c_{R+1,j})^p-(c_{R+1,j})^p)f_{R+1,j}$$

$$+ ((1-c_{R+1,C+1})^p-(c_{R+1,C+1})^p)f_{R+1,C+1}$$

$$= \sum_{i=1}^{R} \sum_{j=1}^{C}((1-\bar{r}_{ij})^p-(\bar{r}_{ij})^p)s_{ij}+\sum_{i=1}^{R}((\bar{r}_{i0})^p-(1-\bar{r}_{i0})^p)([r_{i0}]+1-s_{i0})$$

$$+ \sum_{j=1}^{C}((\bar{r}_{i0})^p-(1-\bar{r}_{i0})^p)([r_{0j}]+1-s_{0j})+((1-\bar{r}_{00})^p-(r_{00})^p)(s_{00}-[r_{00}])$$

$$= \sum_{i=0}^{R} \sum_{j=0}^{C} ((1-\bar{r}_{ij})^p-(\bar{r}_{ij})^p)s_{ij}+ K_3$$

employing relations (3)-(10) where $K_3$ is a constant. Thus

$$\sum_{i=1}^{R+1} \sum_{j=1}^{C+1}|f_{ij}-c_{ij}|^p = \sum_{i=1}^{R+1} \sum_{j=1}^{C+1}((1-c_{ij})^p-(c_{ij})^p)f_{ij}+ K_2$$

$$= \sum_{i=0}^{R} \sum_{j=0}^{C}((1-\bar{r}_{ij})^p-(\bar{r}_{ij})^p)s_{ij}+ K_2 + K_3$$

$$= \sum_{i=0}^{R} \sum_{j=0}^{C}|s_{ij}-r_{ij}|^p-K_1 + K_2 + K_3 ,$$

using (1), (2) and the operations above.

Thus, if S and F correspond as defined above, $H_1(S)$ and $H_2(F)$ differ by a constant $-K_1 + K_2 + K_3$. Hence the rounding S of R which minimizes $H_1(S)$ corresponds to the controlled rounding F of C which minimizes $H_2(F)$, so the proof is complete.

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