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EXPECTED ABSOLUTE DEPARTURE OF CHI-SQUARE
FROM ITS MEDIAN
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#### Abstract

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## EXPECTED ABSOLUTE DEPARTURE OF CHI-SQUARE FROM ITS MEDIAN

Abstract<br>We develop a formula for the expected absolute departure of $x^{2}$ from its median.<br>Key words: chi-square, median, expected absolute departure<br>Beverley D. Causey<br>Mathematical Statistician<br>Statistical Research Division<br>Bureau of the Census<br>Washington, D.C. 20233<br>May 1983

Let $D_{f}$ denote the median of $\chi_{f}^{2}$ (chi-square with $f$ degrees of freedom); we here determine $E\left(\left|\chi_{f}^{2}-D_{f}\right|\right)$. Let $D_{f}$ denote this quantity. Note that $E\left(\left|x_{f}^{2}-c\right|\right)$ is minimal for $c=D_{f}$.

We first need $D_{f}$. One easily obtains $D_{1}=Z^{2}$ with $\Phi(Z)=.75$ and $D_{2}=2 \log 2$; for $f \geqslant 3$ one may base an approximation to $D_{f}$ on the approximation to $\chi_{f}^{2}$ of Peizer and Pratt (1968): $D_{f}=f-2 / 3+.08 / f$. To obtain $D_{f}$ exactly, in essence, as well as to obtain $E_{f}$, we make use of the following, familiar results (obtainable from Kennedy and Gentle 1980). For $k \geqslant 1$ :

$$
\begin{align*}
& P\left(x_{2 k+1}^{2}>c\right)=P\left(x_{2 k-1}^{2}>c\right)+a_{k} \\
& P\left(x_{2 k+2}^{2}>c\right)=P\left(x_{2 k}^{2}>c\right)+b_{k} \tag{1}
\end{align*}
$$

with

$$
\begin{align*}
& P\left(x_{1}^{2}>c\right)=2(1-\Phi(\sqrt{c})), a_{k}=a_{k-1} c /(2 k-1) \quad(k>1) \\
& P\left(x_{2}^{2}>c\right)=\exp (-c / 2), b_{k}=b_{k-1} c / 2 k  \tag{2}\\
& a_{1}=\sqrt{2 c / \pi} \exp (-c / 2), b_{0}=\exp (-c / 2) .
\end{align*}
$$

Using (1) and (2), we do a binary search of the interval <0,f> (successively cut this interval in half) to determine $c$ such that $P\left(\chi_{f}^{2}>c\right)=.5$; this gives us $D_{f}$.

Let

$$
\begin{equation*}
g_{f}(x)=\frac{1}{2^{f / 2} \Gamma(f / 2)} \quad x^{f / 2-1} \exp (-x / 2) \tag{3}
\end{equation*}
$$

the density for $\chi_{f}^{2}$. The value of $E_{f}$ is

$$
\begin{align*}
& \quad \int_{0}^{D_{f}}\left(D_{f}-x\right) g_{f}(x) d x+\int_{D_{f}}^{\infty}\left(x-D_{f}\right) g_{f}(x) d x  \tag{4}\\
& =D_{f}\left[\int_{0}^{D_{f}} g_{f}(x) d x-\int_{D_{f}}^{\infty} g_{f}(x) d x\right]+\int_{D_{f}}^{\infty} x g_{f}(x) d x-\int_{0}^{D_{f}} x g_{f}(x) d x . \tag{5}
\end{align*}
$$

We have

$$
\begin{equation*}
x g_{f}(x)=\frac{2^{f / 2+1} \Gamma(f / 2+1)}{2^{f / 2} \Gamma(f / 2)} g_{f+2}(x)=f g_{f+2}(x) \tag{6}
\end{equation*}
$$

Thus, using the definition of $D_{f}$, we obtain $E_{f}=$

$$
\begin{gather*}
D_{f}(.5=.5)+f\left[P\left(x_{f+2}^{2}>D_{f}\right)-P\left(x_{f+2}^{2}<D_{f}\right)\right]  \tag{7}\\
=f\left[2 P\left(x_{f+2}^{2}>D_{f}\right)-1\right] .  \tag{8}\\
\text { If } f=2 k+1 \quad(k>0) \text {, we have from (1) }
\end{gather*}
$$

$$
\begin{equation*}
E_{f}=f\left\{2\left[P\left(\chi_{f}^{2}>O_{f}\right)+a_{k+1}\right]-1\right\} ; \tag{9}
\end{equation*}
$$

$a_{k+1}$ is obtained from (2) with $D_{f}$ substituted for $c$. By definition of $D_{f}$, again, we are left with $E_{f}=2 f a_{k+1}$. Likewise, if $f=2 k+2(k \geqslant 0)$ we have $E_{f}=2 f b_{k+1}$.

Thus, we have: $D_{1}=0.4549$ and $E_{1}=0.8573, D_{2}=E_{2}=1.386$, $D_{3}=2.366$ and $E_{3}=1.779, D_{4}=3.357$ and $E_{4}=2.103$.

Note correspondences between the formulas for $\mathrm{E}_{\mathrm{f}}$ and the formula (Blyth 1980) for Poisson expected absolute departure from the mean: for x with Poisson mean $\mu, E(|x-\mu|)$ becomes $2 k P(x=k)$ with $k=[\mu]+1$.

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