An Iterative Time Integration Scheme Designed to Preserve a Low-Frequency Wave

YOSHIO KURIHARA AND GREGORY J. TRIPOLI

Geophysical Fluid Dynamics Laboratory, NOAA, Princeton University, Princeton, N. J. 08540

(Manuscript received 17 October 1975, in revised form 17 February 1976)

ABSTRACT

A two-step iterative time integration scheme is formulated, by which the amplitude of a low-frequency wave in a primitive equations model is preserved fairly well for a period of short-range weather prediction while the high-frequency noises are damped. The desired computational characteristics are obtained by separating the terms of the equations at the corrector step. Numerical examples are presented which show the damping property of the proposed scheme. The new scheme does not require more data space in a computer than the amount used in the Euler-backward method.

1. Introduction

Iterative time integration methods have been widely used when the damping of high-frequency modes in the solution of the primitive equations is required. Although iterative methods have been found to damp primarily the high-frequency mode, the use of these methods for long periods may also cause the undesirable damping of waves with relatively low frequency (Kurihara, 1965; Matsuno, 1966).

It is attempted in this paper to formulate a new type of iterative time integration scheme which preserves the low-frequency mode for the time scale of a short-range weather prediction while suppressing the high-frequency mode. This is achieved by separating the terms which contribute to the relatively slow temporal variation from the other terms in the primitive equations and applying an almost neutral marching scheme to the former terms and a damping one to the latter. The strategy of term separation has been used by others in the formulation of the partly implicit or semi-implicit methods (e.g., Kurihara, 1965; Robert, 1969; Kwizak and Robert, 1971) as well as the splitting method (Marchuk, 1965), although the motivation for its use in these methods is different from that of the present work.

In the next section, a new two-step time integration scheme is derived on the basis of some elementary discussion of the damping properties of iterative methods. Numerical examples which demonstrate the behavior of the proposed scheme are presented in Section 3.

2. Formulation of an iterative scheme with term separation

a. Elementary considerations

The equation for a quantity h which propagates in the form of a sinusoidal wave of length L with a phase speed c is written as

$$\frac{\partial h}{\partial t}=F, \tag{2.1}$$

where t is time, $F = -i\nu ch$, $\nu = 2\pi/L$ and $i = \sqrt{-1}$.

The time integration of (2.1) may be performed with the iterative scheme

$$\left.\begin{aligned} h^* &= h^\tau+\Delta t F^\tau \\ h^{\tau+1} &= h^\tau+\Delta t\{(1-w)F^\tau+wF^*\} \end{aligned}\right\}, \tag{2.2}$$

where τ and $\tau+1$ denote the time levels, Δt is the time step, h^* is a temporary value, and w the weight parameter used at a corrector step. In (2.2), F^τ and F^* mean that h^τ and h^* are used respectively to calculate F. In the present paper, discussion will be concerned specifically with the two-time-level, two-step iterative scheme defined by (2.2). However, the principle derived from the considerations which follow is applicable to other iterative schemes. If $w = 1$, the scheme (2.2) gives the Euler-backward method or the Matsuno scheme (Matsuno, 1966; Kurihara, 1965). If $w=0.5$, it becomes equivalent to the modified Euler method.[1] For $w=0$, it is reduced to an unstable explicit forward marching scheme.

Defining b by $\nu c\Delta t$, the relation between $h^{\tau+1}$ and h^τ is expressed by

$$h^{\tau+1}=\lambda h^\tau, \tag{2.3}$$

where

$$\lambda=1-wb^2-ib. \tag{2.4}$$

Accordingly, the amplification rate R is given by

$$R=|\lambda|=\{1+(1-2w)b^2+w^2b^4\}^{\frac{1}{2}}. \tag{2.5}$$

[1] Since $h^{\tau+\frac{1}{2}}$ obtained by a half-step forward differencing is equal to $(h^\tau+h^*)/2$, the scheme (2.2) with $w=0.5$ can be rewritten

$$\left.\begin{aligned} h^{\tau+\frac{1}{2}} &= h^\tau+(\Delta t/2)F^\tau \\ h^{\tau+1} &= h^\tau+\Delta tF^{\tau+\frac{1}{2}} \end{aligned}\right\}.$$

The rate R becomes unity for $b=0$ and for $b=(2w-1)^{\frac{1}{2}}/w$, and less than unity for b between these two values. Minimum R or maximum damping is obtained at

$$b=\frac{(2w-1)^{\frac{1}{2}}}{\sqrt{2}w}. \tag{2.6}$$

Specifying certain values for w, the dependency of R on b is shown in Fig. 1. Note that the ordinate scale is very much magnified, and a curve for the Euler-backward scheme ($w=1$) is drawn for comparison. According to its definition, b is determined from the wavelength L, the phase speed of wave c, and the time increment Δt. It is evident that, if b is small, say about 0.2, a stable and almost neutral integration can be made with a choice of 0.505 or slightly larger value for the weight w. In other words, an amplitude of wave will be preserved well if a sufficiently small time step and an appropriate weight are chosen for a given wave frequency. The scheme with $w=0.5$, i.e., the modified Euler method, is unstable for any value of b.

b. Separation of terms

In the primitive equations, the advection term usually yields a low-frequency mode in the solution. Separating the advection term from the other terms, (2.1) may be rewritten in the symbolic form

$$\frac{\partial h}{\partial t}=F_1+F_2, \tag{2.7}$$

where

$$\left.\begin{aligned} F_1&=-i\nu Uh \\ F_2&=-i\nu(c-U)h \end{aligned}\right\}. \tag{2.8}$$

The term F_1 denotes the tendency due to the advection with the wind U and F_2 represents the effect of all other terms. The term F_2 contributes to the appearance of a high-frequency mode.

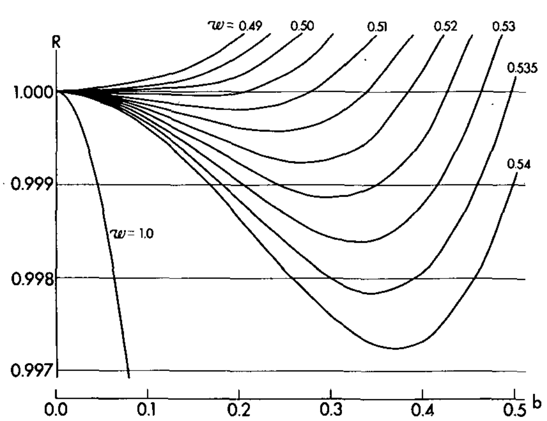

FIG. 1. The amplification rate R vs b [$=\nu c\Delta t$]. Parameter w is a weight factor used in (2.2).

It is intended in the time integration of (2.7) to damp a high-frequency mode while preserving the amplitude of a low-frequency mode. It should be noted that the maximum time step of an iterative method is usually dependent on the frequency of the highest frequency mode. Accordingly, it is sufficiently small to accurately treat the low-frequency mode. Guided by the analysis result obtained in the preceding subsection, the following iterative scheme is proposed:

$$\left.\begin{aligned} h^{*}&=h^{\tau}+\Delta t(F_1{}^{\tau}+F_2{}^{\tau}) \\ h^{\tau+1}&=h^{\tau}+\Delta t\{(1-w_1)F_1{}^{\tau}+w_1F_1{}^{*}\} \\ &\quad+\Delta t\{(1-w_2)F_2{}^{\tau}+w_2F_2{}^{*}\} \end{aligned}\right\}. \tag{2.9}$$

A superscript for F_1 and F_2 indicates the time level of h which is used to estimate F_1 and F_2. In (2.9), the different weights w_1 and w_2 appear at the corrector step. If $w_1=w_2=1$, then (2.9) gives the Euler-backward scheme.

From (2.8) and (2.9), the following relation expressing the computational characteristics of the proposed scheme is derived:

$$h^{\tau+1}=\lambda h^{\tau}, \tag{2.10}$$

where

$$\left.\begin{aligned} \lambda&=1-wb^2-ib \\ b&=\nu c\Delta t \\ w&=\frac{U}{c}w_1+\frac{c-U}{c}w_2 \end{aligned}\right\}. \tag{2.11}$$

Note that (2.11) is formally the same as (2.4). The amplification rate is given by (2.5) with the parameter w replaced by the one defined above. Therefore, the scheme is computationally stable if the condition $b\leqslant(2w-1)^{\frac{1}{2}}/w$ is satisfied. This condition, together with the damping characteristics for (2.11), which should be the same as those obtained previously with respect to (2.4), has to be considered in determining the weights w_1 and w_2 and the time step Δt.

Suppose that the solutions of the primitive equations for wavenumber ν contain a meaningful wave being advected by the wind U as well as the high-frequency waves represented by the maximum phase speed $c_{\max}$. It will be assumed that $c_{\max}\gg U$. Let w_2 be fixed to unity in the present instance. Then w for the high-frequency mode becomes approximately unity unless w_1 is very large. It follows from (2.6) that the most efficient damping of the high-frequency mode is achieved with

$$\Delta t=\frac{1}{\sqrt{2}\nu c_{\max}}.$$

The above Δt satisfies the stability condition for the case of $w=1$, i.e., $\nu c_{\max}\Delta t \leqslant 1$. Using this time step and setting $c=U$, the parameter b for a slowly moving meteorological wave becomes $U/(\sqrt{2}c_{\max})$. The amplitude of this wave would be preserved if the parameter w is specified so that a corresponding curve in Fig. 1 intersects the line for $R=1$ at the value of b given above. The value w thus obtained is an optimum value for w_1 since $w=w_1$ for this wave.

The above-mentioned steps of procedure for determining w_1, w_2, and Δt are applicable to the cases when U is very large and also when the phase speed of the low-frequency mode is different from U. When a wave system with many wavenumbers is treated, the weights and time step have to be chosen so that the computational stability condition is satisfied for all wave components.

3. Numerical examples

A system of equations used by Kurihara (1965) in investigating the feature of a time integration scheme is adopted as a test of the proposed scheme. The equations written in rectangular coordinates are

$$\left.\begin{aligned}\frac{\partial u}{\partial t}&=-U\frac{\partial u}{\partial x}+fv-\frac{\partial \phi}{\partial x}\\ \frac{\partial v}{\partial t}&=-U\frac{\partial v}{\partial x}-fu\\ \frac{\partial \phi}{\partial t}&=-U\frac{\partial \phi}{\partial x}+fUv-gH\frac{\partial u}{\partial x}\end{aligned}\right\}, \tag{3.1}$$

where u and v are x and y components of the perturbation wind, U is a constant zonal wind in the x direction, f is the Coriolis parameter, ϕ is the perturbation of the geopotential, g is the acceleration of gravity, and H is the mean height of the atmosphere. For a given wavenumber $\nu=2\pi/L$, where L is the wavelength, the solutions of (3.1) describe waves with the following phase speeds:

$$\left.\begin{aligned}c_1&=U+2(-\tfrac{1}{3}a)^{\frac{1}{2}}\cos\left(\frac{\epsilon}{3}+\frac{4}{3}\pi\right)\approx U\\ c_2&=U+2(-\tfrac{1}{3}a)^{\frac{1}{2}}\cos\frac{\epsilon}{3}\approx U+(gH)^{\frac{1}{2}}\\ c_3&=U-2(-\tfrac{1}{3}a)^{\frac{1}{2}}\cos\left(\frac{\epsilon}{3}+\frac{2}{3}\pi\right)\approx U-(gH)^{\frac{1}{2}}\end{aligned}\right\}, \tag{3.2}$$

where $\epsilon=\tan^{-1}[(-4a^3/27b^2)-1]^{\frac{1}{2}}$, $a=-(f^2/\nu^2)-gH$, $b=-f^2U/\nu^2$. The speeds c_2 and c_3 represent those of the inertia-gravitational waves. Specifically, the solutions of (3.1) are expressed by

$$\left.\begin{aligned}u&=\sum_{l=1}^{3}u_l, & u_l&=\phi_l\frac{\nu^2(U-c_l)}{f^2-(U-c_l)^2\nu^2}\\ v&=\sum_{l=1}^{3}v_l, & v_l&=\phi_l\frac{i\nu f}{f^2-(U-c_l)^2\nu^2}\\ \phi&=\sum_{l=1}^{3}\phi_l, & \phi_l&=S_l\exp[i\nu(x-c_lt)]\end{aligned}\right\}, \tag{3.3}$$

where $S_l(l=1, 2, 3)$ are amplitudes of three waves.

The proposed scheme (2.9) was applied to the time integration of (3.1). The right hand side of (3.1) was estimated for a grid network by using a centered finite differencing method. As mentioned before, F_1 in (2.9) denotes the term for horizontal advection and F_2 all the other terms. It is noted here that there may be cases where it is appropriate to include the Coriolis term in F_1. The constants used in the test are $U=50$ m s^{-1}, f taken at 45° latitude and $gH=8\times10^4$ m^2 s^{-2}. The grid distance Δx was set to 60 km, and the two cases, $L=10\ \Delta x$ and $L=70\ \Delta x$, were treated. In deciding the parameters w_1 and w_2 in the time integration scheme and a time step Δt, the computational phase speeds of the waves were considered rather than the analytical values given by (3.2). Computational modification of phase speed or frequency depends on a choice of the spatial differencing method and also on the wavelength. For example, in the case of centered finite differencing, the wave, either gravitational or meteorological, with wavelength $4\Delta x$ yields maximum frequency (see Appendix 1 in Kurihara, 1965). In the present test the weights w_1 and w_2 were fixed to 0.506 and 1, respectively, and $\Delta t=60$ s was used. These values were chosen so that damping of the present meteorological waves does not exceed 10% after 24 h.

The time integrations were performed at first for the case with gravity waves only. The initial fields of u, v, and ϕ were given by (3.3) with $S_1=0$, $S_2=S_3=50$ m^2 s^{-2}. Since the fields after one marching step included the computationally excited noise, the first time level was redefined as the initial time. During the integration for 48 h, the energy defined by

$$E=\left[\frac{u^2+v^2}{2}+\frac{1}{gH}\frac{\phi^2}{2}\right],$$

where the brackets indicate the numerical sum over one wavelength, was monitored. In Fig. 2, the energy normalized by its initial value is plotted with dotted lines against time. It is evident that medium-scale gravity waves ($L=600$ km) are almost completely suppressed within two hours after $t=0$. The damping of long gravity waves ($L=4200$ km) is relatively slow because the frequency is low for long waves. The integration for the case of gravity waves alone was also

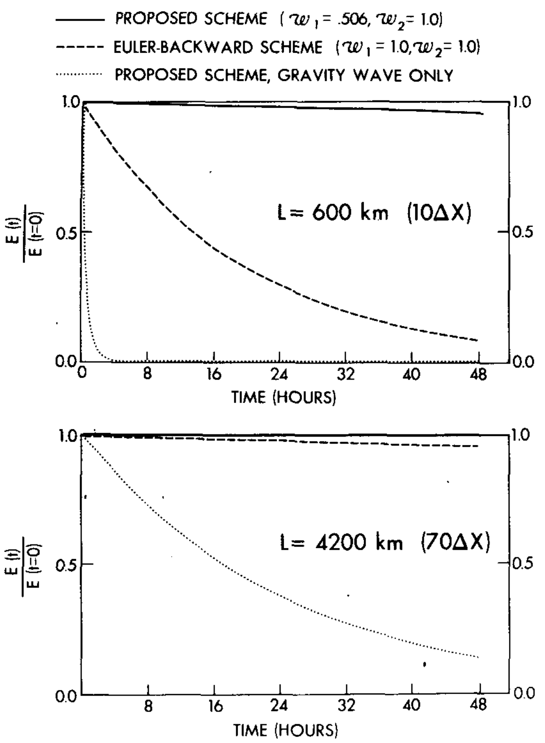

FIG. 2. Variation of energy E normalized by its initial value with time. The upper figure is for the wave with 600 km wavelength and the lower one for 4200 km wavelength. The results from the integration with the proposed scheme (solid line) and with Euler-backward scheme (dashed line) are shown for the case when the initial field contains both meteorological and gravity waves. The result from the integration with the proposed scheme (dotted line) for the case of gravity waves only is also shown.

carried out with the Euler-backward scheme. The resulting lines showing the temporal variation of normalized energy (not plotted in Fig. 2) were indistinguishable from those of the proposed scheme.

In the next test experiment, a meteorological wave with large amplitude was included in the initial perturbation field; i.e., $S_1 = 1000\ \mathrm{m^2\ s^{-2}}$, $S_2 = S_3 = 50\ \mathrm{m^2\ s^{-2}}$. Accordingly, the energy E is mostly that of the meteorological wave. The energy component due to the gravity waves should decrease as examined in the previous experiment, especially when L is 600 km. Therefore, the ratio $E(t)/E(t=0)$ essentially reflects the damping of the meteorological wave. As shown by the solid lines in Fig. 2, the energy of this wave was preserved well in the integration with the proposed scheme. For the wave with $L = 4200$ km, the decrease of the energy in 48 h is negligible. For the medium-scale wave ($L = 600$ km), E at 48 h is lower than the initial value, but only by about 5.5% of the initial. On the other hand, when the Euler-backward scheme was used in the time integration, the medium-scale meteorological wave was damped significantly, leaving only 8% of the initial value after 48 h. The dashed lines in Fig. 2 show the energy variation in these integrations.

4. Remarks

By separating the terms of the primitive equations at the corrector step of the iterative time integration method, preservation of the amplitude of low-frequency waves can be significantly improved. A scheme such as proposed in the present paper would be useful in treating medium-scale waves when the continuous damping of high-frequency noise in the model is desired.

If the diffusion effect is explicitly included in the equation, it also yields damping of waves. Usually, explicit diffusion terms can discriminate the different space scales among the waves with the same frequency while the damping scheme through time differencing cannot. On the other hand, the latter scheme can distinguish the modes with different frequencies when their wavelengths are the same, while the former method cannot. In a complex system, dependency of diffusion effect on frequency and wavelength is not simple because of the interrelation between the wavelength and frequency. If both of the above-mentioned schemes are suitably formulated and used, then small-scale high-frequency waves may be suppressed most efficiently.

It is noted here that the proposed scheme does not require more data space in a computer than the amount used in the Euler-backward scheme. This is true because the data at the current level, i.e., h^τ, may be replaced by $h_p = h^\tau + \Delta t\{(1-w_1)F_1{}^\tau + (1-w_2)F_2{}^\tau\}$ at the end of the predictor step. To obtain $h^{\tau+1}$ at the corrector step, h_p and h^* are then used. In the case of the Euler-backward method, h^τ and h^* are needed at the corrector step. This is considered a special case of the proposed scheme since h_p becomes identical with h^τ in the Euler-backward method.

Acknowledgments. The authors would like to express their thanks to Drs. C. T. Gordon, J. D. Mahlman, and B. B. Ross for the valuable comments on the original manuscript. Thanks are also due to Mrs. E. Thompson and Messrs. J. N. Conner and P. G. Tunison for preparing the paper.

REFERENCES

Kurihara, Y., 1965: On the use of implicit and iterative methods for the time integration of the wave equation. *Mon. Wea. Rev.*, **93**, 33–46.

Kwizak, M., and A. J. Robert, 1971: A semi-implicit scheme for grid point atmospheric models of the primitive equations. *Mon. Wea. Rev.*, **99**, 32–36.

Marchuk, G. I., 1965: A new approach to the numerical solution of differential equations of atmospheric processes. WMO Tech. Note 66, Geneva.

Matsuno, T., 1966: Numerical integrations of the primitive equations by a simulated backward difference method. *J. Meteor. Soc. Japan*, **44**, 76–84.

Robert, A. J., 1969: The integration of a spectral model of the atmosphere by the implicit method. *Proc. WMO/IUGG Symposium on Numerical Weather Prediction*, Tokyo, 1968, Japan Meteor. Agency, VII, 19–24.