A Statistical-Dynamical Model of the General Circulation of the Atmosphere[1]

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ABSTRACT

A statistical-dynamical, two-layer model of the atmosphere is constructed for the simulation of the climatic state of the global circulation.

The meteorological variables, velocity, temperature and pressure, are decomposed into their zonal mean parts and eddy parts or deviations. The state of circulation is expressed by the zonal mean parts as well as eddy statistics which are the zonal averages of the product of the deviations. Eddy statistics such as the amount of eddy kinetic energy, and eddy transfer of heat and angular momentum are longitudinally integrated measures of the intensity and structure of individual synoptic-scale disturbances.

The equations for the zonal means of wind, temperature and pressure and that of eddy kinetic energy are obtained from the equations of motion, the thermodynamical equation, and the continuity equation, and include the terms depending on the eddy statistics. The prediction equation for the horizontal eddy transfer of heat, as well as an estimate of the vertical eddy transfer of heat and angular momentum, are derived under the quasi-geostrophic assumption. The horizontal eddy transfer of momentum is estimated by a diagnostic formula similar to the one used by Smagorinsky. The results of theoretical studies of long waves are utilized to determine the pressure interaction term, the characteristic size of eddies, and the phase speed which are involved in certain of the equations.

The model atmosphere expressed by the closed system of equations thus established is controlled by insolation, parameters for radiative heat transfer, static stability, lower boundary conditions for the exchange of momentum and heat, and parameters for horizontal stress and for the lateral diffusion of heat in the free atmosphere due to small-scale eddies. The present model does not include the effect of lateral transfer of latent energy.

A numerical experiment is performed for a fixed annual mean insolation and a given specification of other control factors. The model consists of two layers, each having 48 zonal rings between the north and south poles. Starting from rest and a constant temperature at the middle level, the integration is done for the first 50 days without eddies. A small amount of eddy kinetic energy is superimposed on the axially symmetric flow at 50 days. Then, the primary features of the actual circulation, such as the jet stream, the Ferrel cell in mean meridional circulation, and the poleward eddy transport of heat, evolve, and a quasi-equilibrium state with a mode of fluctuation is attained.

1. Introduction

The response of the atmosphere to the input of energy, i.e., to solar radiation, has been a major subject in the study of the general circulation of the atmosphere. In this paper, we attempt to establish a new kind of time-dependent atmospheric model which should yield the primary features of the dynamic response of the atmosphere.

The straightforward method of dealing with the above problem is to construct a model of the atmosphere based on physical laws and to numerically integrate the system of governing equations through finite difference or spectral methods under appropriate boundary conditions. This kind of numerical experiment has been done in the expectation that the essential macroscopic features of the atmospheric circulation will be obtained in long-term integration. Usually, such a climatic state is represented by a mean field, i.e., an average with respect to space or time or both. The deviation from the mean field is termed the eddy field, which may be standing or transient according to the definition of the mean field. It is known from theory and analysis of observations that: 1) the development and behavior of the eddies depends on the mean field, especially on the baroclinicity, and the eddies are quasi-two-dimensional; 2) the eddies play an active role in the formation, maintainance and variation of the mean field through the process of momentum and heat transfer which is very unlike the classical Austausch process; and 3) the correlation between the mean field and certain statistics of the eddies represents the energetic exchange between mean and eddy fields. In the usual numerical experiments, it is supposed that the eddy conditions as well as the mean field are strongly controlled by the external forcings, and hence the computed eddies are not completely meaningless. Even over periods far beyond the limit of deterministic pre-

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diction, they are climatologically significant. Otherwise, a realistic mean field may not be obtained.

On the other hand, there have been studies aimed at deriving a climatic state from a model in which individual eddies are not traced, but where the collective effect of the eddies on the mean field is somehow represented. Provided that the above-mentioned interrelation between the mean field and the large-scale eddies is reasonably well incorporated in the model, such an approach will be promising. An apparently optimistic view that this could be done by carefully systematizing the results of the theoretical or empirical studies made so far had motivated the present author to design a new model. Before describing this model, we will briefly discuss previous attempts along this line.

Charney (1959) used a finite-amplitude method or second-order perturbation technique to determine the structure and amplitude of the disturbances. He made a study of the zonally averaged field which is in equilibrium in the presence of eddies for fixed insolation and given viscosity parameters. Fjørtoft (1959) discussed the problem of how the magnitudes of kinetic energy of mean flow and eddies are controlled by heat sources and ground friction for an energy balance. Smagorinsky (1964), assuming a mode of momentum balance based on observation and numerical experiment and considering kinematics of baroclinic disturbances, determined the mean meridional circulation, the distribution of surface stress, and the eddy fluxes of heat, momentum and vorticity for the given heating function.

There have been attempts to reconstruct the Austausch approach by taking the behavior of large eddies into consideration. Williams and Davies (1965) related lateral momentum mixing to the baroclinicity of the atmosphere in order to construct a mean motion model. Dolzhanskiy (1969) used the Williams and Davies hypothesis in further studies of mean circulation. Saltzman (1968) derived, for estimating eddy flux of heat, an Austausch coefficient which depended on the character of amplifying baroclinic waves and an energy balance condition. A similar development was given by Green (1969a, b), in which the eddy transfer of all conservative quantities was proportional to a transfer coefficient. The present work does not rely upon an Austausch approach.

In the studies referred to so far, the mean field is assumed to be quasi-stationary. It has been only for barotropic models that the system of prediction equations has been formulated to permit feedback between the mean field and the eddy statistics (Thompson, 1957; Gambo and Arakawa, 1958; Arakawa, 1961).

In the model to be formulated in Sections 3 and 4, the basic climatic feature of the global circulation of the atmosphere are represented by zonal averages of the zonal and meridional components of wind, vertical velocity, temperature, pressure and certain statistical quantities concerning large-scale eddies. The latter quantities express characteristics of synoptic-scale disturbances such as intensity, structure, size and movement, i.e., the evolution of individual eddies is not explicitly treated in this model. Instead, a longitudinally integrated measure of the role of eddies is evaluated together with the zonal mean field as a function of latitude, height and time. The equations for the zonal mean fields are derived from a system of primitive equations. Those for the eddy statistics are formulated by utilizing the results of studies on the dynamics of large-scale eddies. Numerical integrations have been successfully carried out for a fixed annual mean insolation and also for insolation with a seasonal variation. The results of the former case are presented in Section 7, while those of the latter will be reported in a separate paper.

2. Outline of statistical-dynamical model

a. Notation

λ	longitude
θ	latitude
σ	p/p_*
t	time
a	radius of earth
α	$a \cos\theta$
g	acceleration of gravity
f	Coriolis parameter
β	$df/(ad\theta)$
R	gas constant of air
c_p	specific heat of air at constant pressure
ρ	density
u	eastward wind component
v	northward wind component
T	temperature
p	atmospheric pressure
p_*	atmospheric pressure at the lower boundary
$\dot{\sigma}$	$d\sigma/dt$
ω	dp/dt
ϕ	geopotential
K_E	zonal average of eddy kinetic energy
k_E	a quantity defined in Eq. (4.1)
$\bar{a}$	a quantity defined in Eq. (4.1)
K_Z	kinetic energy of zonal mean flow
P_Z	zonal available potential energy
P_E	eddy available potential energy
${}_HF_\lambda$, ${}_HF_\theta$	frictional forces in zonal and meridional directions due to horizontal diffusion of momentum
${}_VF_\lambda$, ${}_VF_\theta$	frictional forces in zonal and meridional directions due to vertical diffusion of momentum
${}_HF_T$	effect of horizontal diffusion of heat
μ_H	horizontal eddy viscosity coefficient
μ_V	vertical exchange coefficient
D	a coefficient defined in Eq. (3.9)
ρ_G	density of air near the earth surface
c_D	drag coefficient

u_G	geostrophic wind on the earth surface
$\kappa, \alpha_G, \kappa_e$	parameters used in Eqs. (3.8) and (3.9)
Γ_2	static stability at level 2
Γ_e	effective static stability
$\dot{q}$	heating rate per unit mass
ζ	vorticity
ψ	stream function
ℓ^2	characteristic horizontal size of eddy
M	characteristic phase speed of eddy given in Eq. (4.11)
N	a quantity defined in Eq. (4.11)
A, B	quantities defined in Eq. (4.15)
$a_0, a_1, \cdots, a_5$	quantities defined in Eq. (4.8)
b_1, b_2, h_1, h_2	coefficients used in Eqs. (4.9) and (4.10)
$\bar{X}$	longitudinal average of a quantity X
X'	$X-\bar{X}$
$(\ )_j$	θ-finite difference index
$(\ )_k$	σ-finite difference index
$\overline{x'y'}_k$	$\overline{x'_k y'_k}$

b. Outline of the model

In order to make clear the framework of the model, we describe schematically how the closed system of equations will be established.

The equations for the zonal mean field are derived from the primitive equations of motion, the first law of thermodynamics, the continuity equation, and the hydrostatic relation. Taking the zonal average of these equations, we obtain the equations for zonal means of wind, temperature and surface pressure, and the formulas for zonal mean vertical velocity and pressure, as shown in the second column of Table 2.1. Surface pressure is assumed to be a function of latitude and time, which renders the formulation very simple. The equations for zonal mean quantities involve correlations between eddy quantities such as shown in the fourth column of the table, where "prime" denotes the deviation of a quantity from its zonal average. These statistical quantities must somehow be evaluated. We also have to specify control factors, which determine basic characteristics of the atmospheric heat engine, namely, the frictional force and the heating function. For simplicity, we assume that $\overline{u'u'}=\overline{v'v'}$, which reduces the number of statistical quantities required for the prediction of the zonal mean field to six.

Table 2.2 shows the physical basis upon which the equations for the required eddy statistics are derived.

The equation for $\overline{v'v'}$ is the equation for eddy kinetic energy K_E. This energy equation yields two new statistical quantities to be evaluated, i.e., $\overline{\phi'\omega'}$ and $\bar{a}$, the latter being defined in the fourth column of the table.

With the use of the linearized vorticity and thermal equations for a quasi-geostrophic, two-level model, we obtain the equations for T', v' and the formula for ω'. In this derivation, we need to know the characteristic size of the eddy. This introduces a new parameter. Multiplying the equations for T', v' with v' and T', respectively, adding the two equations and taking the zonal average, we can derive the prediction equation for the eddy transfer of heat. Multiplying ω' with u', v', T' and ϕ', and taking the zonal average, we obtain formulas for the vertical eddy transfer of these quantities. It is assumed that the formulation mentioned above is at least valid for middle latitudes where the role of large-scale eddies in the energetics is important. For low latitudes, the formulation may not be quite valid. But the role of eddies is presumably relatively small there. However, we apply the derived formulas to all latitudes to avoid an abrupt latitudinal change of a quantity.

As for eddy transfer of momentum, an empirical condition concerning a balance between divergence of eddy flux of relative angular momentum and the computed surface torque for each zonal ring is used. This condition has been successfully used by Smagorinsky (1964). It

Table 2.1. The eddy statistics and control factors required for the prediction of the zonal mean field.

Type of equation	Zonal mean	Corresponding equation in text	Required eddy statistics	Control factor
Equation of motion	$\frac{\partial \bar{p}_*\bar{u}}{\partial t}$	(3.1)	$\overline{u'u'}, \overline{v'v'}, \overline{u'v'}, \overline{u'\omega'}, \overline{v'\omega'}$	Frictional force
	$\frac{\partial \bar{p}_*\bar{v}}{\partial t}$	(3.2)		
Thermodynamical equation	$\frac{\partial \bar{p}_*\bar{T}}{\partial t}$	(3.15)	$\overline{T'v'}, \overline{T'\omega'}$	Heating function
Continuity equation	$\partial \bar{q}/\partial t$	(3.18)		
	$\bar{\dot{\sigma}}$	(3.19)		
	$\bar{\omega}$	(3.20)		
Hydrostatic relation	$\bar{\phi}$	(3.24)		
(Assumption)	$p_*=\bar{p}_*$		$\overline{u'u'}=\overline{v'v'}=K_E$	

TABLE 2.2. The physical basis upon which equations for eddy statistics are derived. Note that two prediction equations and eight diagnostic formulas are derived in the second column. The last column shows the required parameters other than those which appeared in Table 2.1.

Basis for deriving equations in the second column	Equation or formula	Corresponding equation in the text	New statistical quantities
Equation for eddy kinetic energy	$\frac{\partial \bar{p}_* K_E}{\partial t}$	(4.1)	$\overline{\phi'\omega'}$ $\bar{a}=\overline{k_E v'}+\overline{\phi' v'}$
Linearized vorticy and thermal equations under quasi-geostrophic assumption	$\frac{\partial \overline{T'v'}}{\partial t}$	(4.14)	
	$\overline{u'\omega'}$	(4.17)	ℓ^2 (characteristic size of eddy)
	$\overline{v'\omega'}$	(4.18)	
$\left(\frac{\partial v_2'}{\partial t}, \frac{\partial T_2'}{\partial t} \text{ and } \omega_2' \text{ are obtained}\right)$	$\overline{T'\omega'}$	(4.19)	
	$\overline{\phi'\omega'}$	(4.20)	
Momentum balance relationship for zonal ring ($\overline{u'v'}$ is related to surface stress)	$\overline{u'v'}$	(4.21)	
First-order approximation of momentum equation	$\bar{a}$	(4.24)	M (phase speed)
Theoretical study of long waves in baroclinic zonal current. (The size of the neutral eddy, ℓ_n^2, is estimated; then, the ratio of ℓ^2/ℓ_n^2 is appropriately chosen.)	ℓ^2 M	(4.28) (4.29)	

also agrees well with results of numerical experiments with a more sophisticated model.

The quantity $\bar{a}$ which appears in the equation for eddy kinetic energy is the sum of eddy transfer of eddy kinetic energy and the pressure interaction term. This can be estimated from a first-order approximation in a Rossby number expansion of the primitive equations. It requires a knowledge of the characteristic phase speed of the eddies.

In order to evaluate the characteristic size and speed of eddies, we utilize the results of studies on the dynamics of long waves by Charney (1947) and others. These provide an estimate of the size of incipient unstable eddy, ℓ_n^2. Next, we determine the size ratio of unstable eddy to the above eddy, ℓ^2/ℓ_n^2, by the empirical requirement that the steering level for disturbances is approximately the 600-mb level at middle latitudes. This also fixes the characteristic phase speed of eddies.

In the fourth column of Table 2.2, four new quantities are introduced to make total of ten required eddy statistics and parameters. The second column of this table indicates the two prediction equations and eight diagnostic formulas for these quantities. Accordingly, combining Tables 2.1 and 2.2, we have a closed system of equations to establish a statistical-dynamical model. The third column of the tables shows the corresponding complete set of the equation numbers in this paper.

We note here that eddy statistics, i.e., the characteristic intensity and structure which is expressed by the eddy transfer of heat and momentum, as well as the size and phase speed of eddies, are all latitude- and time-dependent variables. The eddy transfer of a quantity is evaluated without using the classical concept of Austausch.

We use a two-layer model with spherical geometry. The vertical coordinate is taken to be $\sigma=p/p_*$. For convenience, the coordinate surfaces corresponding to $\sigma=0$, 0.25, 0.5, 0.75 and 1.0 will be called levels 0, 1, 2, 3 and 4, respectively. A quantity at level k will have the subscript k, if necessary. The variables assigned to level 1 and 3 are $\bar{u}$, $\bar{v}$, $\bar{\phi}$, $\bar{\omega}$, K_E, $\overline{u'v'}$ and $\bar{a}$. Those assigned to level 2 are $\bar{T}$, $\dot{\bar{\sigma}}$, $\overline{T'v'}$, $\overline{u'\omega'}$, $\overline{v'\omega'}$, $\overline{T'\omega'}$ and $\overline{\phi'\omega'}$. The quantities $\bar{p}_*$, ℓ^2 and M are independent of level. We will also derive the formulas to give $\bar{T}$ at levels 1 and 3 and $\overline{v'v'}$ at level 2.

3. Equations for zonal means of linear quantities

a. Horizontal wind

The prediction equations for zonal mean horizontal wind are obtained by taking zonal averages of the equations of motion with the simplification $\bar{p}_*=\overline{p_*(\theta,t)}$:

$$\frac{\partial \bar{p}_*\bar{u}}{\partial t}=-\frac{\partial \bar{p}_*\bar{u}\bar{v}\cos\theta}{\alpha\partial\theta}-\frac{\partial \bar{p}_*\bar{u}\dot{\bar{\sigma}}}{\partial\sigma}+\bar{p}_*\bar{v}\left(f+\frac{\tan\theta}{a}\bar{u}\right)$$
$$-\left(\frac{\partial \bar{p}_*\overline{u'v'}\cos^2\theta}{\alpha\cos\theta\partial\theta}+\frac{\partial \bar{p}_*\overline{u'\dot{\sigma}'}}{\partial\sigma}\right)+{}_H\bar{F}_\lambda+{}_V\bar{F}_\lambda, \quad (3.1)$$

$$\frac{\partial \bar{p}_* \bar{v}}{\partial t}=-\frac{\partial \bar{p}_* \bar{v}\bar{v}\cos\theta}{a\partial\theta}-\frac{\partial \bar{p}_* \bar{v}\bar{\dot{\sigma}}}{\partial\sigma}-\bar{p}_*\bar{u}\left(f+\frac{\tan\theta}{a}\bar{u}\right)$$
$$-\left(\frac{\partial \bar{p}_* \overline{v'v'}\cos\theta}{a\partial\theta}+\frac{\partial \bar{p}_* \overline{v'\dot{\sigma}'}}{\partial\sigma}+\frac{\tan\theta}{a}\bar{p}_*\overline{u'u'}\right)$$
$$-\bar{p}_*\frac{\partial\bar{\phi}_p}{a\partial\theta}+{}_H\bar{F}_\theta+{}_V\bar{F}_\theta. \quad (3.2)$$

The pressure gradient term in (3.2) is evaluated from the gradient of an isobaric surface, $p=\sigma\bar{p}_*$. Such a form is taken based on experience encountered in past numerical experiments (e.g., Kurihara, 1968). All the other terms are written in referring to λ, θ, σ coordinate system. We apply (3.1) and (3.2) to levels 1 and 3. The terms of the type $\partial X/\partial\sigma$ will be evaluated at levels 1 and 3 by using the finite difference of X, i.e., (X_2-X_0) and (X_4-X_2), respectively. Vertical velocity, $\bar{\dot{\sigma}}$ and $\dot{\sigma}'$, is zero at levels 0 and 4. It will be shown by (4.3) that $\overline{p_*u'\dot{\sigma}'}$ and $\overline{p_*v'\dot{\sigma}'}$ at level 2 are approximated by $\overline{u'\omega'}$ and $\overline{v'\omega'}$, respectively. Diffusion of momentum by small-scale eddies, i.e., the effect of Reynolds stress, is expressed for levels 1 and 3 by (3.3) and (3.4):

$${}_H\bar{F}_\lambda=\frac{\partial\mu_H\bar{R}_{\lambda\theta}\cos^2\theta}{a\cos\theta\partial\theta},\quad {}_H\bar{F}_\theta=\frac{\partial\mu_H\bar{R}_{\theta\theta}\cos\theta}{a\partial\theta}, \quad (3.3)$$

where

$$\bar{R}_{\lambda\theta}=\bar{p}_*\cos\theta\frac{\partial}{a\partial\theta}\left(\frac{\bar{u}}{\cos\theta}\right),\quad \bar{R}_{\theta\theta}=\bar{p}_*\frac{\partial\bar{v}}{a\partial\theta},$$

and

$${}_V\bar{F}_\lambda=-g\frac{\partial\bar{\tau}_{\lambda Z}}{\partial\sigma},\qquad {}_V\bar{F}_\theta=-g\frac{\partial\bar{\tau}_{\theta Z}}{\partial\sigma}. \quad (3.4)$$

For the vertical diffusion, the following formulas are used:

$$\bar{\tau}_{\lambda Z}=\bar{\tau}_{\theta Z}=0, \qquad \text{at level 0;} \quad (3.5)$$

$$\left.\begin{aligned}\bar{\tau}_{\lambda Z}&=\rho\mu_V\frac{\partial\bar{u}}{\partial Z}=-g\frac{\sigma}{R\bar{T}}\mu_V\frac{\partial\bar{u}}{\partial\sigma}\\ \bar{\tau}_{\theta Z}&=\rho\mu_V\frac{\partial\bar{v}}{\partial Z}=-g\frac{\sigma}{R\bar{T}}\mu_V\frac{\partial\bar{v}}{\partial\sigma}\end{aligned}\right\}, \qquad \text{at level 2;} \quad (3.6)$$

$$\bar{\tau}_{\lambda Z}=D\bar{u}_4,\quad \bar{\tau}_{\theta Z}=D\bar{v}_4, \qquad \text{at level 4.} \quad (3.7)$$

We will discuss the coefficient μ_H in (3.3) and the coefficient μ_V in (3.6) in Section 5. In (3.7), which represents momentum exchange at the surface due to frictional stress, the mean wind is related to the zonal mean geostrophic wind by using the empirically determined parameters κ and α_G, i.e.,

$$\bar{u}_4=\bar{u}_{G\kappa}\cos\alpha_G,\quad \bar{v}_4=\pm\bar{u}_{G\kappa}\sin\alpha_G, \quad (3.8)$$

where

$$\bar{u}_G=-\frac{1}{f\rho_G}\frac{\partial\bar{p}_*}{a\partial\theta}.$$

The positive sign in (3.8) is to be used for Northern Hemisphere. The coefficient D in (3.7) is written as

$$D=\rho_G c_D\{\bar{u}_4^2+\bar{v}_4^2+2\kappa_e^2K_{E3}\}^{\frac{1}{2}}, \quad (3.9)$$

where κ_e^2 is a prescribed ratio of the eddy kinetic energy at level 4 to that at level 3 [see (A2.7)].

b. Temperature

An attempt to predict the temperature at two levels was rejected since it created difficulties in evaluating the convective transfer of heat and also in partitioning a heating function for two layers. Instead, we predict the temperature at level 2.

Since the temperature at level 2 should be a measure of the total potential energy for the entire column in the crude two-layer model, we will make the equation for $\bar{T}_2$ equivalent to that for total potential energy.

We define the total potential energy for an air column with unit cross section by

$$c_p\tfrac{1}{2}(\bar{T}_1+\bar{T}_3)\frac{\bar{p}_*}{g}. \quad (3.10)$$

For a given static stability, the temperature at levels 1 and 3 is related to $\bar{T}_2$ by the finite difference form

$$\Gamma_2=\left(\frac{\partial\bar{T}}{\partial p}-\frac{R}{c_p p}\bar{T}\right)_2\approx\frac{1}{500}\left(\bar{T}_3-\bar{T}_1-\frac{2}{7}\bar{T}_2\right). \quad (3.11)$$

Furthermore, if a linear relationship on an emagram is assumed for the change of $\bar{T}$ with pressure, we have

$$\bar{T}_2=\left(1-\frac{\ln 2}{\ln 3}\right)\bar{T}_1+\frac{\ln 2}{\ln 3}\bar{T}_3. \quad (3.12)$$

From (3.11) and (3.12), we obtain

$$\left.\begin{aligned}\bar{T}_1&=\frac{5.738}{7}\bar{T}_2-500\times 0.631\Gamma_2\\ \bar{T}_3&=\frac{7.738}{7}\bar{T}_2+500\times 0.369\Gamma_2\end{aligned}\right\}, \quad (3.13)$$

$$\tfrac{1}{2}(\bar{T}_1+\bar{T}_3)=\frac{1}{C_1}\bar{T}_2-500\times 0.131\Gamma_2, \quad (3.14)$$

where

$$C_1=\frac{14}{13.476}.$$

It is seen from (3.10) and (3.14) that, if the product $\bar{p}_*\Gamma_2$ changes little with time, then the time variation of total potential energy is proportional to that of $\bar{p}_*\bar{T}_2$.

The equation for total potential energy includes the terms expressing the energy conversion between total potential and kinetic energy, i.e., $(R/2g)[(\overline{T\omega/\sigma})_1+(\overline{T\omega/\sigma})_3]$, and the generation of total potential energy, $\bar{p}_*/\bar{q}g$. A vertical flux of heat in the free atmosphere does not affect the total potential energy for the air column. Therefore, we write the equation for zonal mean temperature at level 2:

$$\frac{\partial\bar{p}_*\bar{T}_2}{\partial t}=-\frac{\partial\bar{p}_*\bar{T}_2\bar{v}_2\cos\theta}{a\partial\theta}-\frac{\partial\bar{p}_*\overline{T'v_2'}\cos\theta}{a\partial\theta}+C_1\frac{R}{2c_p}\left\{\left(\frac{\bar{T}\bar{\omega}}{\sigma}\right)_1+\left(\frac{\bar{T}\bar{\omega}}{\sigma}\right)_3+\left(\frac{\overline{T'\omega'}}{\sigma}\right)_1+\left(\frac{\overline{T'\omega'}}{\sigma}\right)_3\right\}+C_1\frac{\bar{p}_*}{c_p}\bar{q}+{}_H\bar{F}_T. \quad (3.15)$$

We will see in Section 4a that $[(\overline{T'\omega'}/\sigma)_1+(\overline{T'\omega'}/\sigma)_3]$ is to be replaced by $2\,\overline{T'\omega'}_2$. The effect of horizontal diffusion of heat by small eddies is evaluated by

$${}_H\bar{F}_T=\frac{\partial\mu_H\bar{S}_\theta\cos\theta}{a\partial\theta}, \quad (3.16)$$

where

$$\bar{S}_\theta=\bar{p}_*[\partial\bar{T}_2/(a\partial\theta)].$$

c. Surface pressure and vertical velocity

The zonal average of the continuity equation in a λ, θ, σ coordinate system with the condition $p_*=\bar{p}_*(\theta,t)$ is

$$\frac{\partial\bar{p}_*}{\partial t}=-\frac{\partial\bar{p}_*\bar{v}\cos\theta}{a\partial\theta}-\frac{\partial\bar{p}_*\bar{\dot\sigma}}{\partial\sigma},\quad\text{at levels 1, 3.} \quad (3.17)$$

Since $\bar{\dot\sigma}=0$ at levels 0 and 4, then (3.17) yields

$$\frac{\partial\bar{p}_*}{\partial t}=-\frac{1}{2}\left(\frac{\partial\bar{p}_*\bar{v}_1\cos\theta}{a\partial\theta}+\frac{\partial\bar{p}_*\bar{v}_3\cos\theta}{a\partial\theta}\right), \quad (3.18)$$

$$\bar{\dot\sigma}_2=\frac{1}{4\bar{p}_*}\left(-\frac{\partial\bar{p}_*\bar{v}_1\cos\theta}{a\partial\theta}+\frac{\partial\bar{p}_*\bar{v}_3\cos\theta}{a\partial\theta}\right). \quad (3.19)$$

The product of the pressure gradient term in (3.2) and $\bar{v}$ represents the source of kinetic energy of the zonal mean wind and is related to the so-called pressure work and energy conversion (e.g., Kurihara, 1968); thus,

$$\bar{v}\left(-\bar{p}_*\frac{\partial\bar{\phi}_p}{a\partial\theta}\right)=-\frac{\partial\bar{p}_*\bar{\phi}_\sigma\bar{v}\cos\theta}{a\partial\theta}-\frac{\partial\bar{p}_*\bar{\phi}_\sigma\bar{\dot\sigma}}{\partial\sigma}-\frac{\partial\bar{\phi}_\sigma\sigma}{\partial\sigma}\frac{\partial\bar{p}_*}{\partial t}-R\frac{\bar{T}\bar{\omega}}{\sigma}. \quad (3.20)$$

The diagnostic formula for $\bar{\omega}$ and the hydrostatic relation which gives $\bar{\phi}_\sigma$ must be derived such that (3.20) is satisfied in its finite difference form. Then, $\bar{\omega}$ at levels 1 and 3 is given by

$$\bar{\omega}=\frac{\bar{p}_*}{2}\bar{\dot\sigma}_2+\sigma\left[\frac{\partial\bar{p}_*}{\partial t}+\bar{v}\frac{\bar{p}_*}{R\bar{T}}\left(\frac{\partial\bar{\phi}_p}{a\partial\theta}-\frac{\partial\bar{\phi}_\sigma}{a\partial\theta}\right)\right]. \quad (3.21)$$

In (3.21), $\bar{\phi}_p$ is the geopotential of the isobaric surface, $p=\sigma\bar{p}_*$. It is equal to $\bar{\phi}_\sigma$, i.e., the geopotential of a constant σ surface, just at the point where $\bar{\omega}$ is to be evaluated. The formula for $(\bar{\phi}_p-\bar{\phi}_\sigma)$ will be given in (3.25).

d. Hydrostatic equation

The hydrostatic equation, i.e., $\partial\phi/\partial\sigma=-RT/\sigma$ or $\phi-RT=\partial(\phi\sigma)/\partial\sigma$, takes the following finite difference form, which when combined with (3.21) is consistent with the finite difference version of (3.20):

$$\frac{\bar{\phi}_4-\bar{\phi}_2}{\sigma_4-\sigma_2}=-\frac{R\bar{T}_3}{\frac{1}{2}(\sigma_4+\sigma_2)}, \quad (3.22)$$

$$\bar{\phi}_k-R\bar{T}_k=\frac{(\bar{\phi}\sigma)_{k+1}-(\bar{\phi}\sigma)_{k-1}}{\sigma_{k+1}-\sigma_{k-1}};\quad k=1,3. \quad (3.23)$$

If we adopt the condition $\bar{\phi}_4=0$, we have

$$\left.\begin{aligned}\bar{\phi}_1&=R\,(\bar{T}_1+\tfrac{2}{3}\,\bar{T}_3)\\ \bar{\phi}_2&=R\,\tfrac{2}{3}\,\bar{T}_3\\ \bar{\phi}_3&=R\,\tfrac{1}{3}\,\bar{T}_3\end{aligned}\right\}. \quad (3.24)$$

Temperatures $\bar{T}_1$ and $\bar{T}_3$ are evaluated by (3.13) for given values of $\bar{T}_2$ and Γ_2.

Consider the constant σ surface and the isobaric surface which intersects it at latitude grid j; $(\bar{\phi}_\sigma-\bar{\phi}_p)$ is of course zero at j. The geopotential height difference of the two surfaces at the adjacent latitude, $j+1$, is obtained by applying hydrostatic equation, $\partial\bar{\phi}/\partial\sigma=-R\bar{T}/\sigma$, i.e.,

$$(\bar{\phi}_\sigma-\bar{\phi}_p)_{j+1}=-\frac{R(\bar{T}_k)_{j+1}}{\frac{1}{2}(\sigma_{k+1}+\sigma_{k-1})}\left[\sigma_k-\frac{\sigma_k(\bar{p}_*)_j}{(\bar{p}_*)_{j+1}}\right],\quad k=1,3. \quad (3.25)$$

Eq. (3.25) is to be used in evaluating $\bar{\omega}$ by (3.21), and also in obtaining $\bar{\phi}_p$ which occurs in (3.2).

4. Equations for eddy statistics

a. Eddy kinetic energy

With the assumption $\overline{u'u'}=\overline{v'v'}$, the zonal average of the equation for eddy kinetic energy is identical to that of $\overline{v'v'}$. Its complete form in a λ, θ, σ coordinate system is as follows:

$$\frac{\partial \bar{p}_* K_E}{\partial t}=-\frac{\partial \bar{p}_* K_E \bar{v}\cos\theta}{a\partial\theta}-\frac{\partial \bar{p}_* K_E \bar{\dot{\sigma}}}{\partial\sigma}-\frac{\partial \bar{p}_* \bar{a}\cos\theta}{a\partial\theta}-\frac{\partial \bar{p}_* \overline{k_E \dot{\sigma}'}}{\partial\sigma}-\frac{\partial \bar{p}_* \overline{\phi'\dot{\sigma}'}}{\partial\sigma}+\left(\frac{\partial \bar{p}_* \overline{u'v'}\cos^2\theta}{a\cos\theta\partial\theta}+\frac{\partial \bar{p}_* \overline{u'\dot{\sigma}'}}{\partial\sigma}\right)\bar{u}+\left(\frac{\partial \bar{p}_* \overline{v'v'}\cos\theta}{a\partial\theta}+\frac{\partial \bar{p}_* \overline{v'\dot{\sigma}'}}{\partial\sigma}+\frac{\tan\theta}{a}\bar{p}_* \overline{u'u'}\right)\bar{v}-R\frac{\overline{T'\omega'}}{\sigma}+(\overline{u'F_\lambda'}+\overline{v'F_\theta'}), \quad (4.1)$$

where

$$\left.\begin{aligned} K_E&=\tfrac{1}{2}(\overline{u'u'}+\overline{v'v'})=\bar{k}_E\\ k_E&=\bar{u}u'+\bar{v}v'+\tfrac{1}{2}(u'u'+v'v')\\ \bar{a}&=\overline{k_E v'}+\overline{\phi' v'}\end{aligned}\right\}.$$

Eq. (4.1) applies for levels 1 and 3. The first and the second terms on the right-hand side represent the transport of K_E by the mean meridional circulation. The next three terms express the eddy transfer of the eddy part of the kinetic energy and the so-called pressure work. The sixth and the seventh terms represent the barotropic exchange of kinetic energy between the zonal and eddy components. The next to last term is the conversion from total potential energy, i.e., the counterpart of the terms in (3.15). The last term is the frictional dissipation.

Terms of the type $\partial X/\partial\sigma$ at levels 1 and 3 are evaluated from the values of X at levels 0, 2 and 4, as was mentioned before in connection with (3.1) and (3.2). The terms $\bar{\dot{\sigma}}$ and $\dot{\sigma}'$ are zero at levels 0 and 4. Since the surface pressure p_* is not a function of longitude in the present model, i.e., $p_*'=0$, then

$$\bar{p}_*\dot{\sigma}'=\omega'-\sigma v'\frac{\partial\bar{p}_*}{a\partial\theta}. \quad (4.2)$$

The order of magnitude of ω', v' and $\partial\bar{p}_*/(a\partial\theta)$ for large-scale motion are 10^{-3} mb sec^{-1}, 10 m sec^{-1} and 10^{-5} mb m^{-1}, respectively. Accordingly, the second term on the right-hand side of (4.2) is smaller than the first term by one order of magnitude. We can approximate the left side of (4.2) by ω', and use the following approximations for the vertical eddy flux of a quantity across level 2 in (3.1), (3.2) and (4.1):

$$\bar{p}_*\,\overline{x'\dot{\sigma}'}\approx\overline{x'\omega'}, \quad x'=u', v', \phi'. \quad (4.3)$$

It is to be noted that ϕ' in (4.1) is a deviation from the zonal mean geopotential of a constant σ surface, and, in the present model, is equal to that of an isobaric surface $p=\sigma\bar{p}_*$, since $p_*'=0$.

Next, we express $-R(\overline{T'\omega'}/\sigma)_{1,3}$ in (4.1) in terms of $(\overline{T'\omega'}/\sigma)$ at level 2, since $\overline{T'\omega'}$ is evaluated only at level 2. In deriving the eddy kinetic energy equation from the quasi-geostrophic vorticity equation, the term $f\partial\omega'/\partial p$ in the vorticity equation results in the terms representing energy conversion and the effect of pressure work in the vertical direction. In case of a two-level model, $f\partial\omega'/\partial p$ at levels 1 and 3 can be approximated by $\pm f\omega'_2/(p_*/2)$ when we take $\omega'=0$ at levels 0 and 4. In this paragraph, when double signs precede a term, the upper sign applies to level 1 and the lower to level 3. In deriving the energy equation corresponding to (4.1) from the vorticity equation, we must multiply $-p_*\phi'_k/f$ (at $k=1,3$) with $\pm f\omega'_2/(p_*/2)$ and take zonal average, yielding

$$-\overline{(\phi_1'-\phi_3')\omega_2'}\mp\overline{(\phi_1'+\phi_3')\omega_2'}.$$

By using a relation, $2\phi'_2=\phi'_1+\phi'_3$, and a hydrostatic relation, $RT'_2=\phi'_1-\phi'_3$, this becomes

$$R\overline{T'\omega_2'}\mp 2\overline{\phi'\omega_2'}. \quad (4.4)$$

The first term of (4.4) is equal to $R\overline{T'\omega'}/(2\sigma)$ at level 2 and should be used for $R\overline{T'\omega'}/\sigma$ at levels 1 and 3. The second term is equivalent to $-\partial\overline{\phi'\omega'}/\partial\sigma$ at levels 1 and 3. From this we see that $[(\overline{T'\omega'}/\sigma)_1+(\overline{T'\omega'}/\sigma)_3]$ in (3.15) can be replaced by $(\overline{T'\omega'}/\sigma)_2=2\overline{T'\omega'}_2$.

The determination of $\bar{a}$ in (4.1) will be described later. We neglect the triple correlations of eddy quantities only in the term $\bar{p}_*\overline{k_E\dot{\sigma}'}$. This term is, therefore, approximated by $(\bar{u}\overline{u'\omega'}+\bar{v}\overline{v'\omega'})$. The form of the frictional term is given in Appendix 2, while the diagnostic relation for $\overline{v'v'}$ at level 2 is derived in Appendix 1.

b. Eddy transfer of heat

The horizontal transfer of heat is one of the important functions of eddies, especially at middle latitudes. We assume that a prediction equation for the behavior of the eddies can be derived, in a first-order approximation, from the vorticity and thermal equations.

The linearized forms of the quasi-geostrophic vorticity equation for two level model are

$$\frac{\partial \zeta_1'}{\partial t} = -\bar{u}_1\frac{\partial \zeta_1'}{a\partial\lambda} - v_1'\frac{\partial\bar{\zeta}_1}{a\partial\theta} - v_1'\beta + f\frac{\omega_2'}{\Delta p},$$

$$\frac{\partial \zeta_3'}{\partial t} = -\bar{u}_3\frac{\partial \zeta_3'}{a\partial\lambda} - v_3'\frac{\partial\bar{\zeta}_3}{a\partial\theta} - v_3'\beta - f\frac{\omega_2'}{\Delta p}.$$

We shall consider the effect of diffusion when it becomes necessary. With the substitution of $\nabla^2\psi'$ for ζ', the above equations are rewritten as

$$\frac{\partial \nabla^2\psi_1'}{\partial t} = -\nabla^2(\bar{u}_1 v_1') - v_1'\left(\beta - 2\frac{\partial^2\bar{u}_1}{a^2\partial\theta^2}\right) + 2\frac{\partial}{a\partial\theta}\left(\frac{\bar{u}_1}{\cos\theta}\right)\frac{\partial v_1'\cos\theta}{a\partial\theta} + f\frac{\omega_2'}{\Delta p},$$

$$\frac{\partial \nabla^2\psi_3'}{\partial t} = -\nabla^2(\bar{u}_3 v_3') - v_3'\left(\beta - 2\frac{\partial^2\bar{u}_3}{a^2\partial\theta^2}\right) + 2\frac{\partial}{a\partial\theta}\left(\frac{\bar{u}_3}{\cos\theta}\right)\frac{\partial v_3'\cos\theta}{a\partial\theta} - f\frac{\omega_2'}{\Delta p}.$$

Let the scale of eddies be characterized by ℓ which has dimensions of length. The determination of ℓ, which may be a function of latitude, will be discussed later. The Laplacian is therefore approximated by

$$\nabla^2\psi' = -\frac{1}{\ell^2}\psi', \quad \nabla^2(\bar{u}v') = -\frac{1}{\ell^2}\bar{u}v'.$$

Since $\psi' = \phi'/f$, we can derive the equation for ϕ'_1 and ϕ'_3, from the vorticity equations above, in the form

$$\frac{\partial \phi_1'}{\partial t} = -f\bar{u}_1 v_1' + f\ell^2 v_1'\left(\beta - 2\frac{\partial^2\bar{u}_1}{a^2\partial\theta^2}\right) - 2f\ell^2\frac{\partial}{a\partial\theta}\left(\frac{\bar{u}_1}{\cos\theta}\right)\frac{\partial v_1'\cos\theta}{a\partial\theta} - \frac{f^2\ell^2}{\Delta p}\omega_2', \quad (4.5)$$

$$\frac{\partial \phi_3'}{\partial t} = -f\bar{u}_3 v_3' + f\ell^2 v_3'\left(\beta - 2\frac{\partial^2\bar{u}_3}{a^2\partial\theta^2}\right) - 2f\ell^2\frac{\partial}{a\partial\theta}\left(\frac{\bar{u}_3}{\cos\theta}\right)\frac{\partial v_3'\cos\theta}{a\partial\theta} + \frac{f^2\ell^2}{\Delta p}\omega_2'. \quad (4.6)$$

The thermal equation for the middle level in linearized form is

$$\frac{\partial T_2'}{\partial t} = -\bar{u}_2\frac{\partial T_2'}{a\partial\lambda} - v_2'\frac{\partial\bar{T}_2}{a\partial\theta} - \Gamma_e\omega_2' + \nabla\cdot(\mu_H\nabla T_2'). \quad (4.7)$$

In (4.7), Γ_e is effective static stability, which includes the effect of condensation implicitly. It is estimated in Section 5 from the climatological mean, partial equivalent potential temperature at levels 1, 2 and 3. The heat diffusion coefficient μ_H is assumed to have the same value as that for momentum diffusion.

We define the following symbols expressing properties of the zonal mean field:

$$\left.\begin{aligned}
a_5 &= \tfrac{1}{2}(\bar{u}_1 - \bar{u}_3) + \ell^2\frac{\partial^2}{a^2\partial\theta^2}(\bar{u}_1 - \bar{u}_3)\\
a_1 &= -2fa_5 + R\frac{\partial\bar{T}_2}{a\partial\theta}\\
a_2 &= -f\ell^2\left(\beta - 2\frac{\partial^2\bar{u}_2}{a^2\partial\theta^2}\right)\\
a_3 &= -2f\ell^2\frac{\partial}{a\partial\theta}\left(\frac{\bar{u}_1}{\cos\theta} - \frac{\bar{u}_3}{\cos\theta}\right)\\
a_4 &= 2f\ell^2\frac{\partial}{a\partial\theta}\left(\frac{\bar{u}_2}{\cos\theta}\right)\\
a_0 &= 2\frac{f^2\ell^2}{\Delta p} - R\Gamma_e
\end{aligned}\right\}, \quad (4.8)$$

where $\bar{u}_2 = \frac{1}{2}(\bar{u}_1 + \bar{u}_3)$.

We shall need to relate the vertical and meridional gradients of eddy velocity to v'_2 and $\partial v_2'/a\partial\lambda$, terms which have a phase difference of 90°. Writing

$$v_1' - v_3' = \tfrac{1}{2}b_1 v_2' + b_2\frac{\partial v_2'}{a\partial\lambda}, \quad (4.9)$$

$$\frac{\partial v_2'\cos\theta}{a\partial\theta} = \tfrac{1}{2}h_1 v_2' - h_2\frac{\partial v_2'}{a\partial\lambda}, \quad (4.10)$$

multiplying each by $v_2' = \frac{1}{2}(v_1' + v_3')$, and taking the zonal average, we get

$$b_1 = \frac{\overline{v'v_1'} - \overline{v'v_3'}}{\overline{v'v_2'}},$$

$$h_1 = \cos\theta\left(\frac{1}{\overline{v'v_2'}}\frac{\partial\overline{v'v_2'}}{a\partial\theta} - 2\frac{\tan\theta}{a}\right).$$

If ϕ'_2 is multiplied into (4.9) and (4.10), and we apply (A1.1), then

$$b_2 = \frac{R}{f}\frac{\overline{T'v_2'}}{\overline{v'v_2'}},$$

$$h_2 = \cos\theta\frac{\overline{u'v_2'}}{\overline{v'v_2'}}.$$

The quantity $v_1'-v_3'$ is related to the tilt of the vertical axis of an eddy. Accordingly, it may be reasonable to expect that the coefficient b_2 would involve $\overline{T'v_2'}$. Likewise, the left-hand side of (4.10) is related to the tilt of the horizontal axis and this is reflected in $\overline{u'v_2'}$ which appears in h_2.

We shall now take the vertical average of (4.5) and (4.6). Using the geostrophic wind relation, $fv'=\partial\phi'/(a\partial\lambda)$, representing $\partial^2/(a^2\partial\lambda^2)$ by $-1/(4\ell^2\cos^2\theta)$ [cf. Section 4f], and neglecting $\partial(v_1'-v_3')\cos\theta/(a\partial\theta)$ which is proportional to $\partial^2T_2'/(a\partial\theta a\partial\lambda)$, we obtain

$$\frac{\partial\phi_2'}{\partial t}=-M\frac{\partial\phi_2'}{a\partial\lambda}+\frac{N}{2}\phi_2', \tag{4.11}$$

where

$$\left.\begin{aligned} M&=\bar{u}_2+\frac{a_2}{f}+\frac{a_5}{4}b_1+\frac{a_4}{2f}h_1\\ N&=\left(\frac{1}{2}a_5b_2-\frac{1}{f}a_4h_2\right)\frac{1}{2\ell^2\cos^2\theta}\end{aligned}\right\}.$$

The coefficient M is the speed of zonal propagation of the ϕ_2' field. Its first two terms give the speed of a Rossby wave with a characteristic size of ℓ^2 and the third and fourth terms are modification factors due to the vertical and horizontal variation of both the zonal mean field and the intensity of the eddies. The coefficient N, on the other hand, represents development of the ϕ_2' field. The factor a_5b_2 is mainly determined by the product of the vertical shear of the zonal mean wind and the horizontal eddy transfer of heat, and hence expresses the supply of energy by the baroclinic process. The supply due to the barotropic process is indicated by a factor a_4h_2 which is proportional to the product of the horizontal shear of the zonal mean wind and the eddy transfer of momentum.

Next, we derive the equation for v_2' by dividing (4.11) by f, taking the zonal derivative, and adding a diffusion term, to obtain

$$\frac{\partial v_2'}{\partial t}=-M\frac{\partial v_2'}{a\partial\lambda}+\frac{N}{2}v_2'+\nabla\cdot(\mu_H\nabla v_2')-\frac{g}{\bar{p}_*}Dv_4'. \tag{4.12}$$

The third term on the right-hand side of (4.12) is the horizontal diffusion of eddy momentum. The form of the vertical diffusion, the fourth term, is consistent with the form (A2.3) which is used in the equation for eddy kinetic energy.

One can obtain the prediction equation for $\overline{T'v_2'}$ by multiplying (4.7) and (4.12) by v_2' and T_2', respectively, adding the two equations and taking a zonal average. The term $v_2'\partial T_2'/(a\partial\lambda)$ which arises can be evaluated from a thermal wind relation and (4.9), i.e.,

$$\overline{v_2'\frac{\partial T_2'}{a\partial\lambda}}=\frac{f}{R}\overline{v_2'(v_1'-v_3')}=\frac{fb_1}{2R}\overline{v'v_2'}. \tag{4.13}$$

If we put $v_4'=\kappa_e v_3'$ for simplicity, and approximate $\overline{\nabla v_2'\cdot\nabla T_2'}$ by $\overline{T'v_2'}/\ell^2$, then we finally have

$$\begin{aligned}\frac{\partial\overline{T'v_2'}}{\partial t}=&(M-\bar{u}_2)\frac{fb_1}{2R}\overline{v'v_2'}-\frac{\partial\bar{T}_2}{a\partial\theta}\overline{v'v_2'}-\Gamma_e\overline{v'\omega_2'}\\ &+\frac{N}{2}\overline{T'v_2'}+\frac{\partial}{a\partial\theta}\left(\mu_H\cos\theta\frac{\partial\overline{T'v_2'}}{a\partial\theta}\right)\\ &-\left(\frac{2\mu_H}{\ell^2}+\frac{g}{\bar{p}_*}D\kappa_e\right)\overline{T'v_2'}.\end{aligned} \tag{4.14}$$

It is of interest to see that the important effects which are associated with baroclinic instability in a two-layer model are included in the right-hand side of (4.14). The initial growth of small perturbations superimposed on a baroclinic zonal current can be discussed in terms of the coefficient for eddy kinetic energy in the equation of the second time-derivative of eddy kinetic energy. When (4.1) is differentiated with respect to time, the time derivative of $-R\overline{T'\omega'}/\sigma$, which represents the baroclinic energy conversion process, appears on the right-hand side of the resulting equation. We will see in (4.19) that $\overline{T'\omega'}$ is approximately correlated with $\overline{T'v'}$, negatively in middle latitudes in the Northern Hemisphere. Accordingly, the coefficients for $\overline{v'v'}$ in the tendency equation of $\overline{T'v'}$, [i.e., (4.14)] govern baroclinic instability. The coefficient of the first term in (4.14) has a factor $(M-\bar{u}_2)$, which is approximately equal to $-\beta\ell^2$. Therefore, for $b_1>0$, the first term expresses stabilization by the β effect. It is larger for the eddies of larger size. The second term represents the destabilizing effect due to baroclinicity. The larger the temperature gradient, the more unstable the eddy. It will be seen later in (4.18) that $\overline{v'\omega'}$ in the third term is proportional to $\overline{v'v'}$. The proportionality coefficient is negative and large for small ℓ^2. Since the effective static stability Γ_e is negative, the third term yields a short-wave limit to instability. The fourth term contributes to instability since the product $N\overline{T'v'}$ usually takes a positive value at middle latitudes. However, an analysis of the results of the numerical integration will show that its role for instability is not primary. The last two terms represent the influence of frictional stress and heat diffusion.

c. Vertical eddy transfer of u, v, T, ϕ

In order to evaluate the vertical eddy transfer of these quantities, we estimate ω_2'. Eliminating the time derivatives from (4.5), (4.6) and (4.7) with the use of $\phi_1'-\phi_3'=RT_2'$, and neglecting $\partial(v_1'-v_3')\cos\theta/(a\partial\theta)$, as we did in deriving (4.11), we obtain the formula for ω_2':

$$\omega_2' = Av_2' + B\frac{\partial v_2'}{\alpha\partial\lambda}, \tag{4.15}$$

where

$$\left.\begin{aligned} A &= \frac{1}{a_0}(a_1 - \tfrac{1}{2}a_2b_1 + \tfrac{1}{2}a_3h_1) \\ B &= \frac{1}{a_0}(-a_2b_2 - a_3h_2) \end{aligned}\right\}.$$

The coefficients A and B determine the amplitude ratio and phase relationship between the ω_2' and v_2' fields. If we either neglect the meridional change or $v_2' \cos\theta$ or put $a_3=0$, (4.15) becomes, by virtue of (4.9), $\omega_2' = [a_1v_2' - a_2(v_1' - v_3')]/a_0$. The main factors in a_1 and a_2 are the meridional gradient of zonal mean temperature and $-f\beta\ell^2$, respectively. Furthermore, a_0 is positive. Consequently, the first term in the above expression of ω_2', namely a_1v_2'/a_0, yields upward motion for southerly flow at middle latitudes in the Northern Hemisphere, while the second term gives downward motion for $v_1' - v_3' > 0$, the magnitude being proportional to ℓ^2. This may be interpreted as a stabilizing influence of β.

The vertical transfer of quantities by large-scale eddies can be now easily estimated. We multiply eddy quantities into (4.15) and take the zonal average; and we make use of (4.13) and the following relationship, which is derived for a condition of non-divergence at level 2:

$$\overline{u_2'\frac{\partial v_2'}{\alpha\partial\lambda}} = -\overline{v_2'\frac{\partial u_2'}{\alpha\partial\lambda}} = \overline{v_2'\frac{\partial v_2' \cos\theta}{\alpha\partial\theta}} = \frac{h_1}{2\cos\theta}\overline{v'v_2'}. \tag{4.16}$$

The resulting formulas follow:

$$\overline{u'\omega_2'} = A\overline{u'v_2'} + B\frac{h_1}{2\cos\theta}\overline{v'v_2'}, \tag{4.17}$$

$$\overline{v'\omega_2'} = A\overline{v'v_2'}, \tag{4.18}$$

$$\overline{T'\omega_2'} = A\overline{T'v_2'} - B\frac{fb_1}{2R}\overline{v'v_2'}, \tag{4.19}$$

$$\overline{\phi'\omega_2'} = -Bf\overline{v'v_2'}. \tag{4.20}$$

In the Northern Hemisphere, A is usually negative and B positive. Accordingly, $\overline{v'\omega'}$ is usually proportional to $-\overline{v'v_2'}$ to make the third term on the right-hand side of (4.14) negative, as discussed before. On the right-hand side of (4.19), the first term is predominant and northward heat transfer is usually associated with upward heat transfer. It is seen from (4.20) that $\overline{\phi'\omega_2'}$ is generally negative. This then implies that the steering level of eddies is below level 2 by the following argument. We consider the thermal equation in linearized quasi-geostrophic form,

$$\frac{\partial T'}{\partial t} = -\bar{u}\frac{\partial T'}{\alpha\partial\lambda} - v_g'\frac{\partial\bar{T}}{a\partial\theta} - \Gamma_e\omega',$$

where v_g' denotes the geostrophic wind. Replacing $\partial/\partial t$ in the above equation by $-M\partial/(\alpha\partial\lambda)$, multiplying by ϕ', and taking zonal averages, we obtain

$$\overline{\phi'\omega'} = \frac{f}{\Gamma_e}(\bar{u} - M)\overline{T'v_g'},$$

where M is the speed of the steering current. The factor $f\overline{T'v_g'}/\Gamma_e$ is usually negative at middle latitudes in both hemispheres. Consequently, $\overline{\phi'\omega'}$ vanishes at the steering level where, by definition, $\bar{u}=M$, and becomes negative above it where $\bar{u}$ increases with height. The formula for M given in (4.11) is consistent with the above discussion, since it yields $\bar{u}_2 - M > 0$, in general.

d. Eddy transfer of momentum

The important parameters for determining the eddy transfer of momentum are the amplitude of the disturbance and the degree of horizontal tilt of the eddy axes. The time change of horizontal tilt is related to the meridional distribution of phase speed of eddies.

It has been shown in a study of the variation of zonal flow with time in barotropic flow (e.g., Arakawa, 1961) that the scales characterizing the zonal mean flow and the eddies are among the factors involved in the expression for the phase speed of eddies. Hence, the relation between the two scales is important for determining the intensification or decay of the jet stream. Following such a view, at one time we attempted to establish a prediction equation for the eddy transfer of momentum. In those trials, the scale of the zonal mean flow was specified somewhat arbitrarily and was included in a formula for the phase speed of eddies. However, there was difficulty in making a proper explicit choice for such a scale.

An alternative approach suggested itself. We first noted that (4.11) was derived by applying a very crude approximation regarding the characteristic eddy scale to the linearized vorticity equation. The scale characteristic of the zonal mean flow is implicitly assumed to be very large. We therefore decided to seek a diagnostic condition for estimating the eddy transfer of momentum.

The formula we will use is similar to the one used in the two-level model by Smagorinsky (1964). His scheme, expressed by Eq. (19) and Fig. 2 in his paper, states that the divergence of the eddy flux of momentum at the upper level of a certain zonal channel is proportional to the stress at the bottom boundary of the channel. Judging from the numerical results of a general circulation experiment using a more sophisticated atmospheric model, this approximation seems to be valid [e.g., Smagorinsky *et al.*, (1965); compare the curves labeled as eddy and surface torque in their Fig. 6A2, and also see their Fig. 6B4]. In the present model, the meridional eddy flux divergence of momentum is related to the stress resulting only from eddies at the surface. In doing so, we preclude a spurious flux divergence of $\overline{u'v'}$ in the case of axially symmetric flow. Smagorinsky assumed $\overline{u'v_3'}\ll\overline{u'v_1'}$, based mainly on observational evidence. We assume the same condition, too, i.e.,

$$-\frac{\partial \overline{p_*}\,\overline{u'v_1'}\cos^2\theta}{\alpha\cos\theta\partial\theta}=-2g\bar{\tau}_e,$$

where $\bar{\tau}_e=(\bar{\tau}_{\lambda z})_4-\rho_G c_D(\bar{u}_4^2+\bar{v}_4^2)^{\frac{1}{2}}\bar{u}_4$ and $\overline{u'v_3'}=0$. Although the above formula requires that the global area integral of $\bar{\tau}_e\cos\theta$ should vanish, $\bar{\tau}_e$, as defined, does not necessarily guarantee it. We therefore take the derivative of the above formula with respect to latitude to obtain

$$\frac{\partial}{a\partial\theta}\frac{\partial\overline{p_*}\,\overline{u'v_1'}\cos^2\theta}{\alpha\cos\theta\partial\theta}=\frac{\partial}{a\partial\theta}(-2g\bar{\tau}_e), \qquad (4.21)$$

and impose an appropriate boundary condition on $\overline{u'v'}$ at the end points of latitudinal grid.

e. Estimation of $\overline{k_E v'}+\overline{\phi'v'}$

The third term in the right-hand side of (4.1) requires a relation for $\overline{k_E v'}+\overline{\phi'v'}$. The equation for the zonal momentum in the first-order Rossby number expansion of the equation for long waves on a β plane is

$$\frac{\partial u_g}{\partial t}+u_g\frac{\partial u_g}{\partial x}+v_g\frac{\partial u_g}{\partial y}-f_0 v_{ag}-\beta_0 y v_g=0,$$

where x is the eastward and y the northward distance from the reference latitude where f_0 and β_0 are defined. We assume that the zero-order field of ϕ is the observed field itself. If the wind $\mathbf{V}=\mathbf{V}_g+\mathbf{V}_{ag}$, then the ratio $|\mathbf{V}_{ag}/\mathbf{V}_g|$ is the order of the Rossby number. Hence, u_g and v_g are given by the zero-order set of equations, $f_0 u_g+\partial\phi/\partial y=0$ and $-f_0 v_g+\partial\phi/\partial x=0$. The above equation is therefore rewritten as

$$\frac{\partial u_g}{\partial t}+v_g\zeta_g+\frac{\partial}{\partial x}\left(\frac{u_g^2+v_g^2}{2}\right)-f_0 v_{ag}-\beta_0 y v_g=0. \qquad (4.22)$$

We multiply (4.22) by ϕ' and take the zonal average. As before we put $\partial u_g/\partial t=-M\,\partial u_g/\partial x$. We also use the approximation $\zeta_g'=-\phi'/(f_0\ell^2)$ to obtain $\overline{\phi'v_g\zeta_g}=0$. Since $\overline{\phi'v_g}=\overline{\phi'v_g'}=0$, $\overline{\phi'v_{ag}}=\overline{\phi'v'}$, and we obtain

$$\overline{[\bar{u}_g u_g'+\tfrac{1}{2}(u_g'u_g'+v_g'v_g')]v_g'}+\overline{\phi'v'}=M\overline{u_g'v_g'}. \qquad (4.23)$$

Assuming that the first term on the left-hand side can be replaced by $\overline{k_E v'}$, and $\overline{u_g'v_g'}$ by $\overline{u'v'}$, respectively, (4.23) becomes

$$\overline{k_E v'}+\overline{\phi'v'}=M\overline{u'v'}. \qquad (4.24)$$

Thus, the eddy transfer of momentum is proportional to a sum of a quantity involving a triple correlation of eddy velocities and the pressure work in the meridional direction.

f. Characteristic size and phase velocity of eddy

It has been assumed so far that the characteristic size and phase speed of eddies can be determined. We will now proceed to establish closing relations for these parameters.

A formula for the phase speed of eddies is given in (4.11). It may be rearranged to take the form

$$M=X-Y\ell^2, \qquad (4.25)$$

where

$$\left.\begin{aligned} X&=\bar{u}_2+\frac{b_1}{8}(\bar{u}_1-\bar{u}_3)\\ Y&=\left(\beta-2\frac{\partial^2\bar{u}_2}{a^2\partial\theta^2}\right)-\frac{b_1}{4}\frac{\partial^2(\bar{u}_1-\bar{u}_3)}{a^2\partial\theta^2}-h_1\frac{\partial}{a\partial\theta}\left(\frac{\bar{u}_2}{\cos\theta}\right)\end{aligned}\right\}.$$

Theoretical studies concerning the behavior of long waves in a baroclinic zonal current show that the speed of propagation of an incipient unstable wave is nearly equal to the surface zonal speed, which is the minimum speed of the zonal current (Charney, 1947; Kuo, 1952; Hirota, 1968). Utilizing this result, if we let ℓ_n^2 be the scale size of such eddies, then the minimum zonal speed is

$$\bar{u}_{\min}=\mathrm{Min}(\bar{u}_1,\bar{u}_3,\bar{u}_4).$$

In particular, we write

$$\bar{u}_{\min}=X-Y\ell_n^2. \qquad (4.26)$$

It is also known, for a given vertical shear of zonal flow, that the scale of a baroclinically unstable eddy is smaller than the critical scale ℓ_n^2. If the scale of those eddies can be related to ℓ_n^2 by an empirical scale ratio $R_1(R_1<1)$ such that

$$\ell^2=R_1\ell_n^2, \qquad (4.27)$$

then it follows from (4.26) that

$$\ell^2=\frac{R_1(X-\bar{u}_{\min})}{Y}. \qquad (4.28)$$

From (4.25) and (4.28), we obtain the phase speed of those eddies

$$M=R_1\bar{u}_{\min}+(1-R_1)X. \qquad (4.29)$$

In the present model, we put $R_1=0.4$. This value is chosen such that the steering level for eddies at middle latitudes is at the 600-mb level.

The appearance of ℓ^2 in many of the equations and formulas resulted from replacing the Laplacian operator by $-1/\ell^2$ in deriving the basic equations (4.5) and (4.6). Suppose that the field of eddy quantities such as ψ', ϕ' and v' is characterized by a surface spherical harmonic of order m and degree n. The Laplacian for such a field is equivalent to multiplication by $-n(n+1)/a^2$. Accordingly, ℓ can be related to n by

$$\ell=\frac{a}{\sqrt{n(n+1)}}\approx\frac{a}{n}. \tag{4.30}$$

Roughly speaking, characteristic lengths of an eddy in the zonal and meridional directions are approximately the same when $m=n/2=a/(2\ell)$. In this case, the characteristic length in the zonal direction becomes

$$\frac{2\pi a\cos\theta}{m}\frac{1}{2\pi}=2\ell\cos\theta. \tag{4.31}$$

For such a zonal scale, we obtain the approximation

$$\frac{\partial^2 x'}{a^2\partial\lambda^2}\approx-\frac{x'}{4\ell^2\cos^2\theta}. \tag{4.32}$$

5. Control factors

a. Frictional force

In order to estimate frictional effect and lateral heat diffusion in the free atmosphere by (3.3), (3.6), (A2.1), (A2.3) and (3.16), we need to specify the internal viscosity coefficients μ_H and μ_V.

Recently, a theory has been developed for two-dimensional turbulence. It is characterized by a constant rate of cascade of enstrophy to the higher wavenumbers, a non-cascade of energy, and the maintenance of a -3 power energy spectrum. Leith (1968) deduced the eddy viscosity coefficient from such a theory. We shall adopt his form, namely

$$\mu_H=\gamma|\nabla_*\zeta|d^3, \tag{5.1}$$

where γ is a dimensionless constant, $\nabla_*\zeta$ is the finite gradient of vorticity ζ, and d is the mesh interval. In the present model, we have only the latitude grid interval, $a\Delta\theta$. We modify (5.1) by replacing d with $a\Delta\theta$. Furthermore, we approximate $\nabla_*\zeta$ at grid j by

$$|\nabla_*\zeta|_j\approx\frac{1}{a\Delta\theta}|\bar{\zeta}_{j-\frac{1}{2}}-\bar{\zeta}_{j+\frac{1}{2}}|+|\nabla_*\zeta'|.$$

For an estimate of the magnitude of $\nabla_*\zeta'$, we use the approximations:

$$|\nabla_*\zeta'|\approx|\zeta'|/a\Delta\theta,\qquad \zeta'\approx-\psi'/\ell^2,\qquad |\psi'|\approx l|v'|.$$

Then, (5.1) becomes

$$(\mu_H)_j=\gamma\left[|\bar{\zeta}_{j-\frac{1}{2}}-\bar{\zeta}_{j+\frac{1}{2}}|+\left(\frac{|v'|}{\ell}\right)_j\right](a\Delta\theta)^2. \tag{5.2}$$

The approximations we make in evaluating (5.2) are

$$\bar{\zeta}_{j-\frac{1}{2}}\approx-|\bar{u}_{j-1}-\bar{u}_j|/(a\Delta\theta)\quad\text{and}\quad |v'|/\ell\approx(\overline{v'v'}/l^2)^{\frac{1}{2}}.$$

In our experiment with $\Delta\theta=\pi/48$, we choose $\gamma=0.04$. The coefficient μ_H is a function of $\bar{\zeta}$, $\overline{v'v'}$ and ℓ^2. In the experiment to be described in Section 6, the resulting value of μ_H was 1.6×10^5 m^2 sec^{-1} at 45° latitude and 4×10^4 m^2 sec^{-1} at 10°.

The vertical exchange coefficient μ_V assumes the form

$$\mu_V=\rho\ell_v^2\left|\frac{\partial\mathbf{V}}{\partial Z}\right|. \tag{5.3}$$

We put $|\partial\mathbf{V}/\partial Z|\approx(|\bar{u}_1-\bar{u}_3|+|v_1'-v_3'|)/\Delta Z$ at level 2. With $\rho_2=7\times10^{-4}$ gm cm^{-3} and $\Delta Z=7.9$ km, we obtain

$$\mu_V=\ell_v^2(|\bar{u}_1-\bar{u}_3|+|v_1'-v_3'|)\times10^{-3}\ [\text{gm cm}^{-1}\text{sec}^{-1}], \tag{5.4}$$

where $\bar{u}$, v' is in meters per second and ℓ_v in meters. If (4.9) is simplified to yield $v_1'-v_3'\approx b_1v_2'/2$, we can evaluate $|v_1'-v_3'|$ by $|b_1|(\overline{v_2'v_2'})^{\frac{1}{2}}/2$. In the numerical experiment, the value 30 m was chosen for ℓ_v which corresponds to mixing length. If $|\mathbf{V}_1-\mathbf{V}_3|=30$ m sec^{-1}, then $\mu_V=27$ gm cm^{-1} sec^{-1}.

As for the surface friction, the forms are given by (3.7) and (A2.3). To determine the surface wind by (3.8) and (A2.7), the parameters $\kappa=|\mathbf{V}_4|/|\mathbf{V}_G|$, $\kappa_e=|\mathbf{V}_4'|/|\mathbf{V}_3'|$ and the angle α_G have to be fixed. The values we use are $\kappa=0.6$, $\kappa_e=0.7$, and $\alpha_G=22.5°$. The surface wind thus determined is related to the surface stress by the coefficient D, which is expressed by (3.9) and involves the density of the air ρ_G and drag coefficient c_D. We put $\rho_G=1.25\times10^{-3}$ gm cm^{-3} and $c_D=0.0025$. D is therefore a function of the variables $\mathbf{V}_4$ and K_{E3}.

b. Heating function

In the present model, the mean rate of heating for an entire air column is evaluated from the given insolation and the temperature at level 2. We assume a heat balance at the surface and the hydrologic cycle is not considered. The static stability is a function of latitude but does not change with time.

Smagorinsky (1963) formulated a parameterization for the diabatic heating of the vertically integrated atmosphere. His scheme is used in the following. The notation we use is as follows:

S_0 solar radiation at top of atmosphere
A_0 planetary albedo

c_0 ratio of solar radiation absorbed at the earth's surface to that absorbed by both atmosphere and earth
$\nu_\uparrow$ normalization parameter for upward long-wave atmospheric radiation $L_\uparrow$
$\nu_\downarrow$ normalization parameter for downward long-wave atmospheric radiation $L_\downarrow$
B_0 long-wave absorptivity of the atmosphere
e ratio of emission of long-wave radiation from the surface to the long-wave back radiation
s Stephan-Boltzmann constant
Γ_2 static stability
Γ_e effective static stability

The amount of short-wave radiation absorbed directly by the air column is, by definition, $S_0(1-A_0)(1-c_0)$. The air column loses heat by long-wave atmospheric radiation by the amount $L_\uparrow+L_\downarrow$, where $L_\uparrow=\nu_\uparrow s\bar{T}_2^4$ and $L_\downarrow=\nu_\downarrow s\bar{T}_2^4$. On the other hand, it is warmed by the absorption of long-wave radiation emitted from the surface in the amount $B_0 s\bar{T}_*^4$, where $\bar{T}_*$ is the surface temperature. The parameter e defined above is $s\bar{T}_*^4/L_\downarrow$. This is done in order to avoid the explicit determination of $\bar{T}_*$. The short-wave radiation absorbed at the surface and the net input of long-wave radiation at the surface are $S_0(1-A_0)c_0$ and $L_\downarrow-s\bar{T}_*^4=(1-e)L_\downarrow$, respectively. Since the heat capacity of the earth is assumed to be zero, the sum of these two quantities must be transferred to the atmosphere through some heat exchange process which we need not specify.

Integrating all the effects above-mentioned, we obtain a formula for the mean rate of heating per unit mass, i.e.,

$$\bar{q}=ES_0+F\bar{T}_2^4+(GS_0+H\bar{T}_2^4), \tag{5.5}$$

where

$$\left.\begin{aligned} E&=\frac{g}{\bar{p}_*}(1-A_0)(1-c_0), & F&=\frac{g}{\bar{p}_*}[(B_0e-1)\nu_\downarrow-\nu_\uparrow]s \\ G&=\frac{g}{\bar{p}_*}(1-A_0)c_0, & H&=\frac{g}{\bar{p}_*}(1-e)\nu_\downarrow s \end{aligned}\right\}.$$

The three terms in the right side of (5.5) represent direct heating due to insolation, that due to long-wave radiation, and that passing through the bottom of air column, respectively.

Table 5.1 gives the values of the parameters used in the experiment. The data source for the parameters A_0 through e is the paper by Smagorinsky (1963). Reference is made to the table and row number in his paper. It is seen that the variation of c_0, $\nu_\uparrow$, $\nu_\downarrow$, and especially B_0 and e, with latitude, is rather small. Insolation S_0 is the annual mean value and is derived from the 1951 edition of the *Smithsonian Meteorological Tables* (p. 418).

Static stability is also listed in Table 5.1; its definition was given in (3.11). The required data are taken partly from Fig. B4 in the paper by Smagorinsky (1963) and partly from *Geophysical Memoirs* No. 101 (Meteorological Office, England, 1958).

However, the effective static stability Γ_e enters in (4.7), and is estimated in a similar way to Γ_2, by using the partial equivalent temperature derived from Fig. B4 of Smagorinsky's paper. Comparing Γ_e thus estimated with Γ_2, we find that Γ_e can be closely related to Γ_2 by the empirical formula

$$\Gamma_e=\Gamma_2 \sin\theta. \tag{5.6}$$

6. Design of the numerical experiment for annual mean insolation

The present model atmosphere covers the entire surface of the earth and consists of two layers separated by $\sigma=0.5$. Each layer is divided into zonal rings with the same latitudinal width $\Delta\theta$. The number of rings between the north and south poles is 48; hence, $\Delta\theta$ is $\pi/48$. The meridional grid index J increases southward. The northernmost latitude grid, $J=1$, is at the north latitude, $\pi/2-\Delta\theta/2$.

For the finite difference scheme, the box method is applied to each zonal ring. In particular, version I of the schemes formulated by Kurihara and Holloway (1967) is used. The numerical integration is performed with a time increment of 10 min; this is set by gravity waves. In general, centered differencing is used for the

TABLE 5.1. Values of the parameters used in the experiment. Table and row number in the last column indicates those pertaining to Smagorinsky (1963), from which the quantities are derived.

Parameter	0	10	20	30	40	50	60	70	80	90	Data source
	Latitude (deg)										
A_0	0.326	0.310	0.283	0.284	0.335	0.389	0.443	0.527	0.602	0.669	Table A3, row 4
c_0	0.665	0.683	0.711	0.733	0.725	0.716	0.709	0.698	0.646	0.564	{Table A3, row 1; Table A3, row 7
$\nu_\uparrow$	0.751	0.773	0.808	0.810	0.812	0.836	0.849	0.852	0.856	0.861	Table A2, row 12
$\nu_\downarrow$	1.303	1.338	1.349	1.325	1.304	1.295	1.257	1.205	1.145	1.050	Table A2, row 13
B_0	0.961	0.959	0.948	0.945	0.946	0.951	0.953	0.955	0.953	0.947	Table A2, row 10
e	1.170	1.176	1.205	1.210	1.196	1.180	1.171	1.167	1.198	1.237	{Table A1, row 6; Table A1, row 7
S_0(ly day^{-1})	854	840	804	746	668	576	478	396	356	341	See text
$-\Gamma_2$(10^{-4} K mb^{-1})	425	425	429	443	487	548	610	672	708	725	See text

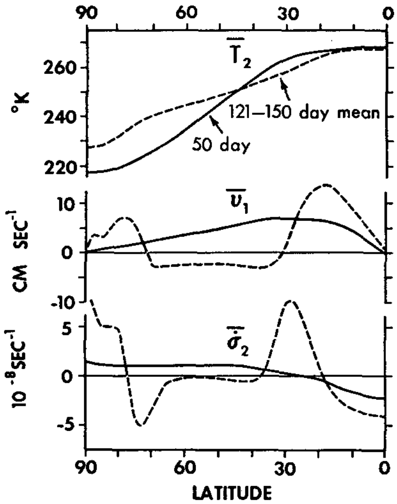

FIG. 1. Latitudinal distribution of the zonal mean temperature ($\overline{T}_2$), the mean meridional flow at level 1 ($\overline{v}_1$), and the mean vertical velocity at level 2 ($\overline{\dot{\sigma}}_2$). Solid lines show the state at 50 days; dashed lines show average for 121–150 day period.

terms of the prediction equations, except for the terms of diffusion or dissipation tppe for which forward or backward differencing is applied. We occasionally use Euler-backward differencing (Kurihara, 1965) to damp high-frequency external gravity waves. In case of centered differencing, the development of a computational mode is suppressed by applying the modified Euler method every 72 time steps. (See Appendix 3 for some additional remarks on the numerical scheme.)

The numerical integration is performed for a fixed annual mean insolation, starting from rest and with a constant temperature of 250K at level 2. The parameters which specify the frictional force and the heating function were given in Section 5. The integration for the first 50 days is done without eddies, i.e., $K_E=0$. At 50 days, a small amount of eddy kinetic energy, 0.01 m^2 sec^{-2}, which is less than 0.01% of the final mean value of K_E, is introduced for all latitude grids at level 1 and 3. The computation is then continued to 150 days, at which time a quasi-equilibrium is attained.

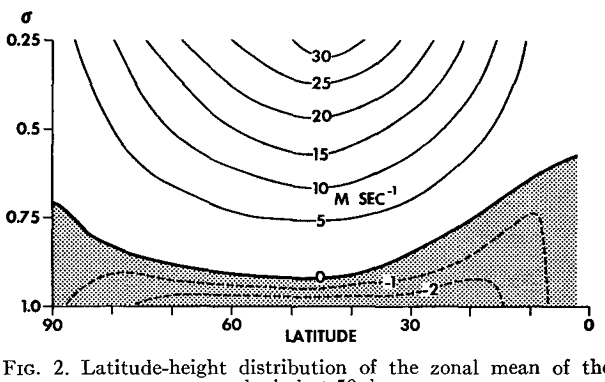

FIG. 2. Latitude-height distribution of the zonal mean of the zonal wind at 50 days.

The computation time required for a one-day forecast, i.e., 144 steps in the present experiment with centered differencing, is 42 sec on the UNIVAC 1108 computer. This includes the time to make fairly comprehensive diagnostic integral calculations during the course of integration.

7. Results and analyses

a. Build-up of axially symmetric flow

The latitudinal variation of the annual mean insolation imposed upon the atmosphere which was at rest at day 0 causes upward motion at low latitudes and downward at high latitudes. Simultaneously, westerly flow in most of the atmosphere and weak easterly flow near the earth's surface are formed by Coriolis force and surface friction. Such a field of flow, later being modified by advection due to the meridional circulation and diffusion, develops during the initial period of integration without eddies.

We also observe a poleward shift of mass which was uniformly distributed at the beginning. This movement of mass is almost completed by about 30 days. After that, the rate of change of $\bar{p}_*$ is very slow. Starting from the uniform value of 1013.25 mb, $\bar{p}_*$ at the grid nearest the pole rises to 1049.45 mb by 30 days and is 1051.43 mb at 50 days. Meanwhile, $\bar{p}_*$ at the grid nearest to the equator lowers to 1000.88 mb by 30 days and is 1001.43 mb at 50 days.

The development of a single Hadley cell is rather rapid in the early period. Later, its intensity changes only slightly. Fig. 1 shows the northward flow at level 1 and vertical velocity $\overline{\dot{\sigma}}_2$ at 50 days. The center of the Hadley cell is at 30° latitude. Since all fields are either symmetric or antisymmetric with respect to the equator, only the distribution for the Northern Hemisphere is presented. The stabilizing effect of the Hadley circulation is of inadequate intensity to counterbalance the heating gradient. The cooling rate near the pole at about 50 days is 0.44C day^{-1} and the warming rate near the equator is 0.19C day^{-1}. Thus, the meridional gradient of $\overline{T}_2$ keeps increasing. In Fig. 1, the $\overline{T}_2$ field at 50 days is shown. The gradient is enough for the baroclinic waves to grow.

The wind at the surface is easterly everywhere, as seen in Fig. 2. This implies that the total absolute angular momentum of the atmosphere continues to increase. Although the surface torque supplies westerly momentum to the lower layer, the actual change of momentum is largely governed by the meridional circulation. The Hadley cell yields convergence of the momentum due to the earth's rotation at the upper level and divergence at the lower level. As a result, the increasing rate of westerly momentum at the upper level is about three times that at the lower level at 50 days. The change of vertical wind shear reflects that of meridional gradient of temperature. The zonal mean flow at 50 days, shown in Fig. 2, is in almost perfect geostrophic balance at all latitudes including the tropics.

b. Evolution of a climatic state

After adding a small amount of eddy kinetic energy, 0.01 J cm^{-2}, which corresponds to wind speed of 14 cm sec^{-1}, at 50 days at each point, we continue the integration until 150 days.

Fig. 3 shows how the zonal mean surface pressure changes with time. It is seen that mass is redistributed in a rather short period. At about 60–70 days, the kinetic energy of eddies increases at middle latitudes, a large poleward transport of heat by eddies takes place, and the Ferrel cell circulation is formed. At the same time, as shown in Fig. 3, the high pressure belt in the subtropics and low pressure at higher latitudes appear and the surface wind at the middle latitudes becomes westerly. The pressure near the equator rises from about 1001 mb at 50 days to 1012 mb at 80 days, remaining so thereafter. It appears that the intensity of the subtropical high belt and the low pressure belt

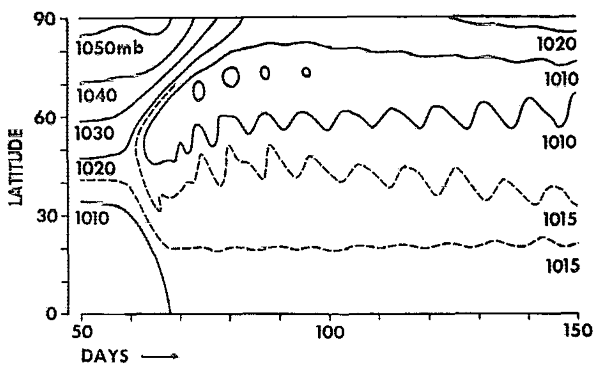

FIG. 3. Time variation of the latitudinal distribution of zonal mean surface pressure during the 50–150 day period.

at ~70° are correlated with each other as they pulsate with a period of about 10 days. Zonal mean temperature and zonal mean wind attain a state of quasi-equilibrium by 100 days. In fact, beyond this time most characteristics exhibit a very small time variation. We choose the 30-day period between 121 and 150 days for the analysis of the climatic state obtained by the present model.

c. Analyses of the climatic state

The latitudinal distribution of zonal mean temperature $\bar{T}_2$ is given in Fig. 1 by the dashed line. Comparing the line with the curve for 50 days, we see that heat has been removed from the subtropical latitudes and transfered to high latitudes.

The field of $\bar{T}_2$ is maintained through a balance between the non-adiabatic heating and the dynamics. Fig. 4 shows the rate of temperature change due to the heating function, the mean meridional circulation, the

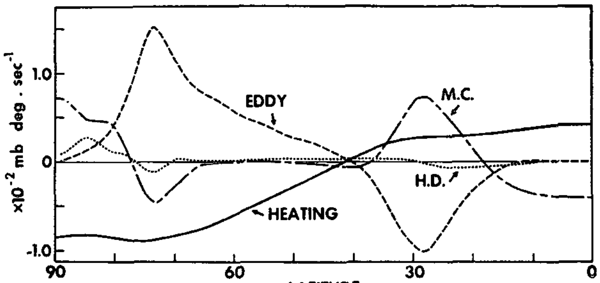

FIG. 4. Latitudinal distribution of change of $p_*\bar{T}_2$ due to non-adiabatic heating (HEATING), the meridional circulation (M.C.), the large-scale eddies (EDDY), and the horizontal diffusion (H. D.).

large-scale eddy flux and the small-scale diffusion of heat. This distribution is obtained by taking a 30-day average of daily instantaneous states. It is seen that the non-adiabatic heating in the tropics is counterbalanced by the mean meridional circulation. In the subtropics, the warming is accentuated by the meridional circulation and the eddies make the balance. Similar results are obtained in the numerical experiment by Smagorinsky *et al.* (1965, Fig. 5B3). Generally speaking, the cooling by the heating function north of 40° is balanced by the eddy motion. A detailed analysis shows that the temperature change due to the meridional circulation is mainly due to adiabatic expansion associated with mean vertical motion, yielding a conversion of total potential energy to kinetic energy of mean zonal flow. In the case of the eddies, the cooling due to vertical motion is overbalanced by the relatively large horizontal heat transfer north of 40°, giving a net warming.

Fig. 5 gives the distribution of the components of the heating function. The present model includes neither the hydrologic cycle nor the effect of energy transfer by oceans. Accordingly, the curve of net heating rate is monotonic. If those effects were taken into consideration, the net heating would have shown a secondary maximum at middle latitudes (e.g., Smagorinsky, 1963, Fig. A5; Manabe *et al.*, 1965, Fig. 12B6).

Next, we examine the mean meridional circulation which influences the heat budget of the atmosphere as

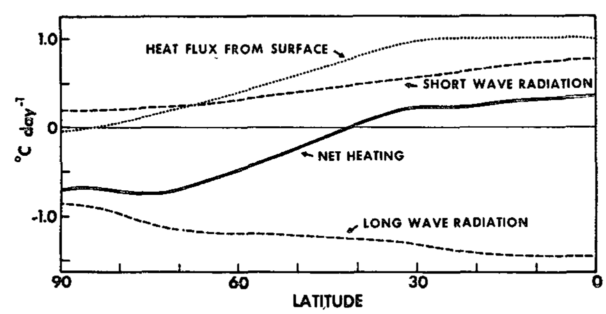

FIG. 5. Latitudinal distribution of mean heating rate for the total air column due to short-wave radiation, long-wave radiation, upward heat flux at the earth's surface, and the net effect.

TABLE 7.1. Average of heat and momentum flux and various parameters over the time interval of 121–150 days.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Grid j	Latitude (deg)	$\overline{T'v_2'}$ (°K m sec^{-1})	$\overline{T'\omega_2'}$ (10^{-3} K mb sec^{-1})	Percentage geostrophic departure* of $\bar{u}_1$	Deviation** of $\bar{u}_1$ (10^{-1} m mb sec^{-2})	Quantity defined by Eq. (7.3) (10^{-1} m mb sec^{-2})	$\overline{u'v_1'}$ (m^2 sec^{-2})	Change of angular momentum by surface torque (10^7 dyn cm^{-1})	Characteristic scale l^2 (10^{11} m^2)	Representative zonal wavenumber	Characteristic phase speed (m sec^{-1})	Horizontal diffusion coefficient μ_H (10^5 m^2 sec^{-1})	Vertical exchange coefficient μ_v (gm cm^{-1} sec^{-1})	$\overline{T'T_2'}$ (°K)^2
1	88.12	0.03	−0.00	1.4	0.14	0.53	−0.01	0.1	1.31	8.8	1.9	0.74	6.1	0.0
2	84.38	0.16	−0.02	1.1	0.26	0.49	−0.04	1.4	1.21	9.2	5.5	0.83	13.1	0.0
3	80.62	0.73	−0.10	1.7	0.53	0.78	−0.44	4.1	1.20	9.2	7.2	1.04	17.7	0.1
4	76.87	2.09	−0.23	3.3	1.01	1.36	−1.04	5.7	1.46	8.1	7.6	1.28	18.7	0.7
5	73.12	6.62	−0.37	6.2	1.40	1.93	−1.56	2.5	2.39	6.5	7.2	0.93	15.8	2.8
6	69.37	12.03	−0.82	6.9	1.13	1.72	−1.06	−3.7	3.78	5.2	6.5	1.34	12.6	5.8
7	65.63	12.05	−0.70	3.8	0.57	1.17	0.10	−5.5	3.64	5.3	6.2	1.15	12.3	6.4
8	61.87	14.80	−0.77	3.3	0.48	1.02	1.04	−5.8	3.21	5.6	6.2	1.24	12.5	8.3
9	58.12	14.37	−0.86	5.0	0.72	1.20	1.76	−5.6	3.02	5.8	6.4	1.31	13.5	8.4
10	54.37	15.84	−1.02	6.8	1.01	1.45	2.33	−5.9	3.00	5.8	6.8	1.42	14.9	9.9
11	50.62	15.20	−1.22	8.0	1.24	1.64	2.78	−5.9	3.07	5.8	7.4	1.49	16.6	11.3
12	46.88	15.76	−1.44	8.5	1.39	1.73	3.18	−6.2	3.20	5.6	8.1	1.58	18.6	13.4
13	43.12	15.16	−1.63	8.8	1.51	1.77	3.61	−6.9	3.38	5.5	9.1	1.64	20.9	15.4
14	39.37	14.94	−1.88	7.6	1.39	1.53	4.10	−7.7	3.61	5.2	10.2	1.71	23.3	17.2
15	35.62	13.87	−2.22	2.3	0.46	.39	4.66	−9.5	3.73	5.2	11.5	1.76	25.6	16.7
16	31.87	11.71	−2.40	− 4.8	−1.08	−1.39	5.05	−5.7	3.15	5.7	12.8	1.89	27.6	11.6
17	28.13	7.09	−1.85	− 7.9	−1.86	−2.29	4.78	1.9	2.58	6.3	13.5	1.96	29.1	6.0
18	24.37	3.23	− .99	− 8.1	−1.73	−2.09	3.62	10.3	2.51	6.4	13.4	1.68	29.6	3.2
19	20.62	1.07	− .38	− 8.6	−1.48	−1.70	2.17	14.0	2.81	6.0	12.5	1.26	28.2	2.0
20	16.87	0.24	− .09	− 9.9	−1.22	−1.32	1.12	13.8	3.25	5.6	10.8	0.88	24.5	1.1
21	13.13	−0.10	0.01	−10.7	−0.77	−0.80	0.63	10.9	3.00	5.8	8.3	0.50	18.5	0.4
22	9.38	−0.02	0.01	−10.2	−0.31	−0.32	0.42	7.0	2.05	7.0	5.1	0.42	11.6	0.1
23	5.63	−0.00	0.00	− 8.4	−0.07	−0.07	0.25	3.1	1.12	9.5	2.4	0.40	6.1	0.0
24	1.88	−0.00	0.00	− 6.9	−0.01	−0.01	0.08	1.0	0.59	13.2	1.0	0.32	3.0	0.0

* See Eq. (7.1).
** See Eq. (7.2).

seen in Fig. 4. The 30-day average of mean meridional circulation is given by the distribution of $\bar{v}_1$ and $\bar{\dot{\sigma}}_2$ in Fig. 1. It is characterized by two direct cells, at low and high latitudes, respectively, and a weak indirect cell between them. It is seen that $\bar{\dot{\sigma}}_2$ is highly correlated with the rate of change of temperature due to the meridional circulation shown in Fig. 4. The pattern of meridional circulation, however, is not quasi-stationary. Fig. 6 shows the variation of the latitudinal distribution of $\bar{v}_1$ with time for the period from 100 to 150 days. Ferrel cells are formed one after another with a period of about 10 days at 70°. Each cell propagates equatorward down to ~30°. Although the life of each cell is ~2 weeks, a new evolution begins at high latitudes every 10 days. In consequence, we sometimes obtain a 5-cell circulation, for example, at 130 days. Matsumoto (1962) pointed out that 5 cells in the meridional circulation are sometimes observed in the actual atmosphere. The intensity of the tropical Hadley cell increases when the 3-cell circulation is formed, for example, at 135 days. This may be one of the indications of interaction between the flows in middle and low latitudes.

The role of large-scale eddies in the heat balance of the atmosphere is determined by the horizontal and vertical transport of heat. Latitudinal distribution of these quantities is tabulated in columns 3 and 4 of table 7.1. Large values of $\overline{T'v_2'}$ are seen at 45–55°. A countergradient meridional heat flux occurs equatorward of about 15°. Since the horizontal heat flux is governed by (4.14), it may be of interest to see the balance of the terms on the right-hand side of the equation. The magnitude of the first three terms is much larger than the others so that balance is almost completely determined by these three terms. Fig. 7 shows the latitudinal distribution of these terms. As mentioned in Section 4, these quantities represent the role of baroclinicity, of the β effect, and of the effective static stability upon the development of baroclinic waves. Baroclinicity is of course favorable for the growth of eddies and most effective at 30–35°. On the other hand, the β effect and particularly the static stability act to suppress development.

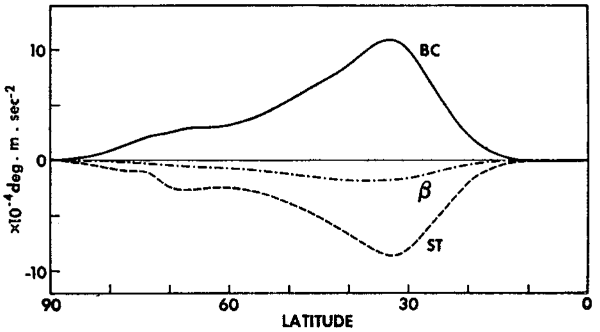

FIG. 7. Latitudinal distribution of the change of $\overline{T'v'}$ due to the first term in the right-hand side of Eq. (4.14) (β effect: β), the second term (baroclinicity: BC), and the third term (stability: ST).

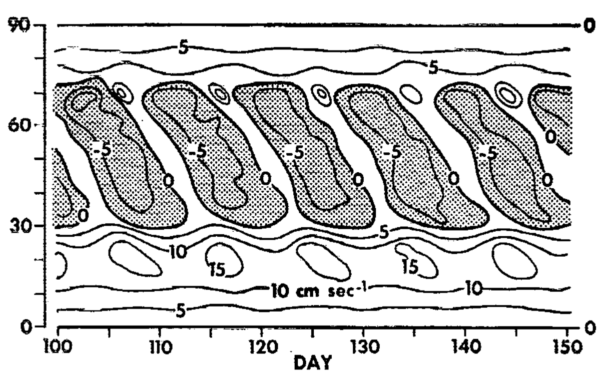

FIG. 6. Time variation of the latitudinal distribution of mean meridional flow at level 1 for the 100–150 day period. Northerly flow is indicated by shading.

We look again at the distribution of $\bar{T}_2$ in Fig. 1. The gradient is relatively large at the 20–30° and 75–85° latitude belts. The field of mean zonal flow, given in Fig. 8, shows the double jet streams located at the same latitudes. We shall now examine the degree of geostrophic balance of the mean zonal wind at level 1. Smagorinsky (1963) made such an analysis and pointed out that the geostrophic departures are highly correlated with the gradient of the variance of the meridional wind component. In our case, the problem is to investigate the balance among the terms in (3.2). Table 7.1, column 5, shows the percentage geostrophic departure defined by

$$\left[\bar{u}\left(f+\frac{\tan\theta}{a}\bar{u}\right)+\frac{\partial\bar{\phi}_p}{a\partial\theta}\bigg/\frac{\partial\bar{\phi}_p}{a\partial\theta}\right]\times 100. \qquad (7.1)$$

The geostrophic departure of the mean zonal flow is at most 11%. The geostrophic balance of the mean zonal wind at equatorial latitudes is very good and comparable to that at the middle latitudes. Column 5 and also column 6, which gives the deviation

$$\bar{p}_*\bar{u}\left(f+\frac{\tan\theta}{a}\bar{u}\right)+\bar{p}_*\frac{\partial\bar{\phi}_p}{a\partial\theta}, \qquad (7.2)$$

show that mean zonal flow is super-geostrophic north of 35° and sub-geostrophic to the south. The third largest term in (3.2) is

$$-\frac{\partial\bar{p}_*\overline{v'v'}\cos\theta}{a\partial\theta}, \qquad (7.3)$$

which is tabulated in column 7 of the table. A high correlation between (7.2) and (7.3) can be seen. All the above results agree well with those obtained by Smagorinsky (1963). The magnitude of (7.3) is generally

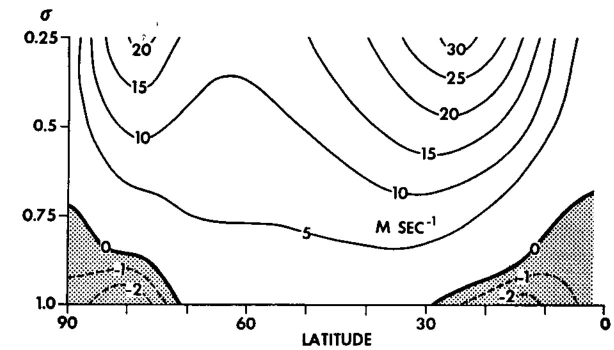

FIG. 8. Latitude-height distribution of the zonal mean of zonal wind averaged for the 121–150 day period.

larger than that of (7.2). If two more terms, i.e.,

$$-\partial \overline{p_* \overline{v'\dot{\sigma}'}}/\partial\sigma \quad \text{and} \quad -\overline{p_* \overline{u'u'}} \tan\theta/a,$$

are added to (7.3), the sum yields almost 100% of (7.2).

An analysis of the budget of absolute angular momentum gives the result that the transfer of the angular momentum due to earth's rotation by the mean meridional circulation is well balanced by the convergence of eddy transfer of relative angular momentum at the upper layer, while at the lower layer the former is largely cancelled by the gain due to surface torque. The distribution of $\overline{u'v_1'}$ and surface torque are shown in columns 8 and 9 of Table 7.1, respectively. The result seems qualitatively reasonable. However, their magnitude is small by factor of about 4 as compared with the estimates made from the actual atmosphere. This is also reflected in the rather weak mean meridional circulation, the small gradient in surface pressure, and hence the weak surface wind in the present model.

It should be noted that the role of eddies in transporting the angular momentum in the present model is passive. This is due to the use of a diagnostic means for estimating the horizontal eddy transfer of momentum, and is certainly a weakness of the model. With such a parameterization, the eddy transfer of momentum cannot be a causal factor for the evolution of the system of surface wind, although it favors the maintenance of

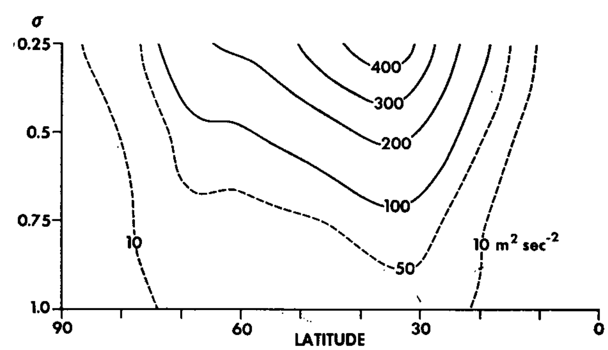

FIG. 9. Latitude-height distribution of eddy kinetic energy averaged for the 121–150 day period.

TABLE 7.2. Budget of eddy kinetic energy (10^3 ergs sec^{-1} cm^{-2}) based on an average for a period of 121–150 days.

	Conversion from total potential energy	Gain due to pressure work at level 2	Gain due to work done by Reynolds stress at level 2	Gain from flux of eddy kinetic energy across level 2	Total	Dissipation due to horizontal diffusion of momentum	Dissipation due to vertical diffusion of momentum	Dissipation at earth's surface	Total	Conversion from kinetic energy of zonal mean flow
Upper layer	1.28	0.43	−0.05	0.03	1.69	−1.59	−0.06		−1.65	0.01
Lower layer	1.28	−0.43	0.05	−0.03	0.87	−0.09	−0.01	−0.83	−0.93	0.00
Total column	2.56				2.56	−1.68	−0.07	−0.83	−2.57	0.01

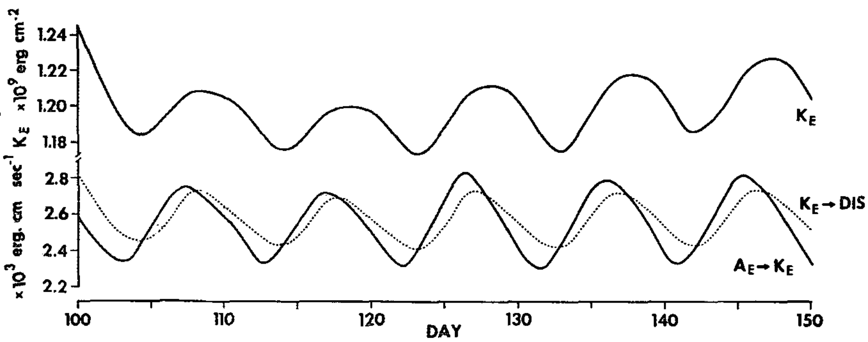

FIG. 10. Time variation of eddy kinetic energy (K_E), conversion of eddy available potential energy to eddy kinetic energy ($A_E \rightarrow K_E$), and dissipation of eddy kinetic energy ($K_E \rightarrow DIS$) for the 100–150 day period.

the wind system which resulted primarily from the other causes.

Next, we will discuss some statistics concerning eddies. Fig. 9 gives the distribution of eddy kinetic energy. A maximum is at level 1 and at ~36°. The characteristic size, estimates of the zonal wavenumber which corresponds to the zonal scale given by (4.31), and the velocity of propagation of the eddies are listed in columns 10–12 of Table 7.1, respectively. Except for tropical and very high latitudes, the estimated characteristic zonal wavenumber is between 5.2 and 6.5. The diffusion coefficient in the present model depends on the intensity and scale of the eddies which have evolved [see (5.2) and (5.4)]. The latitudinal distribution of the resulting horizontal and vertical diffusion coefficients at level 2 are given in columns 13 and 14. The coefficient for horizontal diffusion takes on a relatively small value at equatorial latitudes, as does the vertical diffusion coefficient.

The budget of eddy kinetic energy is investigated by taking global averages of the contributing factors for the upper and lower layers separately. The result is shown in Table 7.2. The pressure interaction at the middle level is most effective in exchanging energy between the two layers. Dissipation of eddy kinetic energy at the upper layer is mainly by horizontal diffusion of momentum. On the other hand, at the lower layer, it is due to the work done against the frictional stress at the surface. Time series of total integral of eddy kinetic energy, conversion from eddy available potential energy, i.e., the integral of $-R(\overline{T'\omega'}/\sigma)/g$ with respect to σ, and total dissipation of eddy kinetic energy for the period from 100 to 150 days are presented in Fig. 10. Fluctuation of total eddy kinetic energy with an approximate period of 10 days and with amplitude of 2% of the total amount is seen. Variation of total dissipation follows that of conversion with a time lag of about one day.

Finally, four conventional forms of energy and their transformations are given in Table 7.3. Eddy available potential energy is evaluated by the area average of $\gamma_2(\overline{T'T_2'}/2)\bar{p}_*/g$, where the factor γ_2 is the area average of $-R/(p_2\Gamma_2)$. The quantity $\overline{T'T_2'}$ is obtained by (A1.10) and listed in column 15 of Table 7.1. An estimate of the decrease of P_E by horizontal heat diffusion is made with the area average of $\gamma_2(\mu_H/\ell^2)\overline{T'T_2'}\bar{p}_*/g$. Estimates of other transformations are made by applying the formulas usually used for budget analysis of energy (e.g., Smagorinsky *et al.*, 1965, Section 7). Results of similar analyses made by Smagorinsky *et al.* (1965), Smagorinsky (1963) and Phillips (1956) for their numerical experiments are also shown in Table 7.3, together with estimates for the actual atmosphere for the Northern Hemisphere (Oort, 1964). As compared with other models, the amount of P_Z and K_Z of the present model is smaller and closer to observation. Note, in particular, that K_E of our model is comparable to K_Z. The smallness of P_E is common to all models. The energy transformation and the conversion due to the baroclinic process, i.e., $\{P_Z, P_E\}$ and $\{P_E, K_E\}$, are within the range of observation. Dissipation of K_E is somewhat larger than in other models. However, its ratio to K_E is smaller than in the models by Smagorinsky *et al.* and by Smagorinsky by more than a factor of two. The conversion $\{K_E, K_Z\}$ in the present model is weak. This is due to the smallness of eddy transfer of momentum. One should also note that dissipation of K_Z in the present model is very small.

8. Summary and remarks

A statistical-dynamical, two-layer model has been constructed; the equations consist of those for zonal averages of the meteorological variables and those for the eddy condition. This approach is certainly one way to study the complicated system of circulation. The concept may be utilized for other problems.

A numerical experiment has been performed with a given specification of diffusion parameters for an annual mean insolation. Starting from rest, axially symmetric flow is formed in a 50-day integration from which eddies are excluded. At that point a small amount of eddy kinetic energy was added. After a period of transi-

TABLE 7.3. Energy (J cm^{-2}) and its transformation (10^{-3} J cm^{-2} day^{-1} mb^{-1}). The transformation notation is the same as that defined by Phillips (1956).

	Present model	Smagorinsky *et al.* (1965)	Smagorinsky (1963)	Phillips (1965)	Estimates by Oort (1964)
Available potential energy					
P_Z	582.3	736.6	940.4	1266	400±100
P_E	37.9	34.9	17.6	26.2	150±50
Kinetic energy					
K_Z	109.6	144.0	273	290	80±30
K_E	120.1	32.0	35.2	72.4	70±30
{heating, P_Z}	38.3	44.9	29.8	22.4	26.8±8.6
{P_Z, horizontal diffusion}	2.6	7.5	10.7	2.8	
{P_Z, P_E}	31.2	32.1	24.8	32.4	25.9±8.6
{heating, P_E}		−1.9	−0.7		−6.9±8.6
{P_E, horizontal diffusion}	3.2	6.2	1.7	1.0	
{P_E, K_E}	21.8	25.9	21.1	32.1	19.0±8.6
{K_E, horizontal diffusion}	14.3	13.4	8.6	4.5	{15.6±8.6} (horizontal diffusion, vertical diffusion and surface friction combined)
{K_E, vertical diffusion}	0.6	{5.5} (vertical diffusion and surface friction combined)	0.6	0.0	
{K_E, surface friction}	7.1		3.8	3.7	
{K_E, K_Z}	−0.1	6.3	10.4	13.7	3.5±1.7
{K_Z, horizontal diffusion}	0.4	2.4	4.0	2.8	{4.3±1.7} (horizontal diffusion, vertical diffusion and surface friction combined)
{K_Z, vertical diffusion}	0.7	{3.1} (vertical diffusion and surface friction combined)	3.8	0.0	
{K_Z, surface friction}	0.2		3.0	6.0	
{K_Z, P_Z}	−1.7	1.3	0.9	3.6	−0.9±1.7

tion, when eddies grow, most zonal mean quantities attain a state of quasi-equilibrium.

The mean zonal flow which evolved possesses a double jet-stream structure. The horizontal eddy flux of sensible heat is large and poleward at 45–55° latitude. It is very small and countergradient at low latitudes. The eddy transfer of relative angular momentum is rather small, though it agrees qualitatively with estimates for the actual atmosphere. The eddies have maximum intensity at about 36° in the upper layer. The scale of eddies at most latitudes corresponds to zonal wavenumbers between 5.2 and 6.5. An analysis of the time variation, such as presented in Fig. 3, 6 and 10, indicates that the quasi-equilibrium is not stationary but has a mode of fluctuation. Its period is ~10 days in the present case. Furthermore, inter-latitudinal coupling is suggested by Figs. 3 and 6.

Following are some remarks regarding the formulation and control factors of the present model:

1) This model is characterized by dealing with zonal averages. It is impossible to include explicitly the effects of the geographical distribution of land, sea and mountains. Separation of eddies into quasi-standing and transient components is also not possible. The details of deviations from the zonal average can be discussed only by using an orthodox three-dimensional model.

2) Another feature of the present model is its two-layer formulation. It is due to this constraint that a heating function is defined only for the total air column and ω' can be determined only at one middle level. On the other hand, actual observation suggests the necessity for finer resolution in the vertical to simulate a realistic state of certain variables. For example, the mean meridional circulation in the actual atmosphere has maximum intensity near the surface and compensating return flow at the tropopause level (Oort and Rasmusson, 1970). If one increases the number of layers, however, new problems are introduced, such as how to partition the heating function into layers. It is the author's opinion that a multi-layer formulation of a model of this kind cannot be strongly recommended. Therefore, the inherent limitations of a two-layer formulation imposes a limit of capability to the statistical-dynamical approach.

3) Little is known about mechanism of energy dissipation in the actual atmosphere. Although analysis of energy dissipation in the present model yields the results as shown in Table 7.2, it must not be taken too literally. It may be possible that a seemingly realistic climatic state can be obtained with a specification of a diffusive parameterization which is quite different from that used here.

4) As for the heating function, a change of some parameters may be useful in understanding climatic response. An experiment with insolation having a seasonal variation has been carried out. The hydrologic cycle may be taken into consideration by parameterizing the moisture content of the atmosphere. The effect of heat transfer in the oceans can be included empirically in specifying the heating function.

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APPENDIX 1

Calculation of $\overline{v'v'}$ and $\overline{T'T'}$ at Level 2 and the Interpolation of Certain Eddy Statistics

In the derivation of the formulas which give $\overline{v'v'}$ and $\overline{T'T'}$ at level 2, we use the relations

$$\left.\begin{array}{l} \phi_1'+\phi_3'=2\phi_2', \quad \phi_1'-\phi_3'=RT_2' \\ fv'=\dfrac{\partial\phi'}{a\partial\lambda}, \quad \overline{\left(\dfrac{\partial x'}{a\partial\lambda}\right)^2}=\dfrac{\overline{x'x'}}{4\ell^2\cos^2\theta} \end{array}\right\}. \qquad \text{(A1.1)}$$

The second and the third relations are the hydrostatic and geostrophic wind relations, respectively. The last relation results from (4.32). From (A1.1), we obtain

$$2v_2'=v_1'+v_3'.$$

Accordingly,

$$\overline{4v'v_2'}=\overline{v'v_1'}+\overline{v'v_3'}+\overline{2v_1'v_3'}. \qquad \text{(A1.2)}$$

In order to estimate $\overline{v_1'v_3'}$ in (A1.2), we put

$$v_1'=av_3'+b\frac{\partial v_3'}{a\partial\lambda}, \qquad \text{(A1.3)}$$

where a and b are to be determined below. Taking the square of (A1.3), we then have, using (A1.1),

$$\overline{v'v_1'}=a^2\overline{v'v_3'}+b^2\frac{1}{4l^2\cos^2\theta}\overline{v'v_3'}. \qquad \text{(A1.4)}$$

On the other hand, multiplying (A1.3) by ϕ_3', taking the zonal average, and using (A1.1), we obtain after some manipulation

$$\overline{v_1'\phi_3'}=-R\overline{T'v_2'}=-bf\overline{v'v_3'}. \qquad \text{(A1.5)}$$

Therefore, (A1.4) becomes

$$a^2(\overline{v'v_3'})^2=\overline{v'v_1'}\cdot\overline{v'v_3'}-\frac{1}{4\ell^2\cos^2\theta}\left(\frac{R}{f}\right)^2(\overline{T'v_2'})^2. \qquad \text{(A1.6)}$$

It is easily seen, by multiplying (A1.3) with v_3' and taking the zonal average, that the left-hand side of (A1.6) is equal to the square of $\overline{v_1'v_3'}$. If the condition

$$(\overline{T'v_2'})^2\leqslant 4\ell^2\cos^2\theta\left(\frac{f}{R}\right)^2\overline{v'v_1'}\cdot\overline{v'v_3'} \qquad \text{(A1.7)}$$

is satisfied, then $(\overline{v_1'v_3'})^2\leqslant\overline{v_1'v_1'}\cdot\overline{v_3'v_3'}$, i.e., the Schwarz inequality holds. We assume furthermore that $\overline{v_1'v_3'}$ is positive. Then, by (A1.7), we obtain from (A1.2) and (A1.6)

$$4\overline{v'v_2'}=\overline{v'v_1'}+\overline{v'v_3'}+2\left[\overline{v'v_1'}\cdot\overline{v'v_3'}-\frac{1}{4\ell^2\cos^2\theta}\left(\frac{R}{f}\right)^2(\overline{T'v_2'})^2\right]^{\frac{1}{2}}. \qquad \text{(A1.8)}$$

Next, we derive the formula for $\overline{T'T_2'}$. Using the first formula of (A1.1), we have

$$\overline{(\phi_1'-\phi_3')^2}=2(\overline{\phi'\phi_1'}+\overline{\phi'\phi_3'})-4\overline{\phi'\phi_2'}. \qquad \text{(A1.9)}$$

We can rewrite (A1.9) by making use of (A1.1) to obtain

$$\overline{T'T_2'}=8l^2\cos^2\theta\left(\frac{f}{R}\right)^2(\overline{v'v_1'}+\overline{v'v_3'}-2\overline{v'v_2'}), \qquad \text{(A1.10)}$$

which is always positive or zero for $\overline{v'v_2'}$ given by (A1.8).

We estimate $\overline{u'v_2'}$ by taking the average of $\overline{u'v'}$ at level 1 and 3. The quantity $\overline{v'v'}$ at the mid-point between two latitude grids is obtained as the square of the average of the square root of $\overline{v'v'}$ at the two grids. The mid-point value of $\overline{u'v'}$ and $\overline{T'v'}$ is the average of their respective values at two latitude grids.

APPENDIX 2

Estimate of Frictional Effects in the Equation for Eddy Kinetic Energy

In Eq. (4.1) the effect of the frictional force is given by the terms $\overline{u'F_\lambda'}+\overline{v'F_\theta'}$.

We assume that horizontal component of frictional force is written as

$$\left.\begin{array}{l} {}_HF_\lambda'=\dfrac{\partial\mu_H R_{\lambda\lambda}'}{a\partial\lambda}+\dfrac{\partial\mu_H R_{\lambda\theta}'\cos^2\theta}{a\cos\theta\partial\theta} \\ {}_HF_\theta'=\dfrac{\partial\mu_H R_{\theta\lambda}'}{a\partial\lambda}+\dfrac{\partial\mu_H R_{\theta\theta}'\cos\theta}{a\partial\theta}+\dfrac{\tan\theta}{a}\mu_H R_{\lambda\lambda}' \end{array}\right\}, \qquad \text{(A2.1)}$$

where

$$\left.\begin{array}{l} R_{\lambda\lambda}'=\bar{p}_*\left(\dfrac{\partial u'}{a\partial\lambda}-\dfrac{\tan\theta}{a}v'\right) \\ F_{\lambda\theta}'=\bar{p}_*\cos\theta\dfrac{\partial}{a\partial\theta}\dfrac{u'}{\cos\theta} \\ R_{\theta\lambda}'=\bar{p}_*\dfrac{\partial v'}{a\partial\lambda} \\ R_{\theta\theta}'=\bar{p}_*\dfrac{\partial v'}{a\partial\theta} \end{array}\right\}.$$

We then have

$$\overline{u'_H F_\lambda'}+\overline{v'_H F_\theta'}=\frac{\partial}{a\partial\theta}\left[\mu_H \bar{p}_* \cos\theta\left(\frac{\partial K_E}{a\partial\theta}+\frac{\tan\theta}{a}\overline{u'u'}\right)\right]$$
$$-\frac{\mu_H}{\bar{p}_*}\left[\overline{(R_{\lambda\lambda}')^2}+\overline{(R_{\lambda\theta}')^2}+\overline{(R_{\theta\lambda}')^2}+\overline{(R_{\theta\theta}')^2}\right].$$

Neglecting the factor $\overline{u'u'}\tan\theta/a$ in the first term on the right-hand side of the above formula, and using the approximation

$$\overline{(R_{\lambda\lambda}')^2}+\overline{(R_{\lambda\theta}')^2}+\overline{(R_{\theta\lambda}')^2}+\overline{(R_{\theta\theta}')^2}=\bar{p}_*^2\frac{\overline{u'u'}+\overline{v'v'}}{\ell^2}=\bar{p}_*^2\frac{2K_E}{\ell^2},$$

where ℓ^2 is a characteristic size of eddy and defined in the text, we can obtain

$$\overline{u'_H F_\lambda'}+\overline{v'_H F_\theta'}=\frac{\partial}{a\partial\theta}\left(\mu_H \bar{p}_* \cos\theta\frac{\partial K_E}{a\partial\theta}\right)-\mu_H\bar{p}_*\frac{2K_E}{\ell^2}. \quad \text{(A2.2)}$$

On the right-hand side of (A2.2), the first term is the work done by the frictional stress and the second the dissipation of eddy kinetic energy.

For vertical diffusion we use forms similar to (3.4)–(3.7) in the text, namely,

$$_V F_\lambda'=-g\frac{\partial\tau_{\lambda z}'}{\partial\sigma},\quad {}_V F_\theta'=-g\frac{\partial\tau_{\theta z}'}{\partial\sigma}, \quad \text{(A2.3)}$$

where

$$\left.\begin{array}{l}\tau_{\lambda z}'=\tau_{\theta z}'=0, \quad \text{at level 0}\\ \left.\begin{array}{l}\tau_{\lambda z}'=-g\dfrac{\sigma}{R\bar{T}}\mu_V\dfrac{\partial u'}{\partial\sigma}=\dfrac{g}{R\bar{T}_2}\mu_V(u_1'-u_3')\\ \tau_{\theta z}'=-g\dfrac{\sigma}{R\bar{T}}\mu_V\dfrac{\partial v'}{\partial\sigma}=\dfrac{g}{R\bar{T}_2}\mu_V(v_1'-v_3')\end{array}\right\}, \text{ at level 2}\\ \tau_{\lambda z}'=Du_4', \quad \tau_{\theta z}=Dv_4', \quad \text{at level 4}\end{array}\right\}.$$

After some manipulation we obtain

$$(\overline{u'_V F_\lambda'}+\overline{v'_V F_\theta'})_1=-\frac{2g^2}{R\bar{T}_2}\mu_V(K_{E1}-K_{E3})-\frac{g^2}{R\bar{T}_2}\mu_V[\overline{(u_1'-u_3')^2}+\overline{(v_1'-v_3')^2}], \quad \text{(A2.4)}$$

$$(\overline{u'_V F_\lambda'}+\overline{v'_V F_\theta'})_3=\frac{2g^2}{R\bar{T}_2}\mu_V(K_{E1}-K_{E3})-\frac{g^2}{R\bar{T}_2}\mu_V[\overline{(u_1'-u_3')^2}+\overline{(v_1'-v_3')^2}]-2gD(\overline{u_3'u_4'}+\overline{v_3'v_4'}). \quad \text{(A2.5)}$$

The first terms on the right-hand sides of (A2.4) and (A2.5) represent the effect of work done by stress at level 2, and the second terms the dissipation of eddy kinetic energy. The third term in (A2.5) shows the work done by atmosphere at the lower boundary, i.e., frictional dissipation at the earth's surface. In evaluating the second term, we apply the thermal wind relation, the approximation characterizing the temperature field in terms of the characteric eddy size, and (A1.10), yielding

$$\overline{(u_1'-u_3')^2}+\overline{(v_1'-v_3')^2}=\left(\frac{R}{f}\right)^2\left[\overline{\left(\frac{\partial T_2'}{a\partial\theta}\right)^2}+\overline{\left(\frac{\partial T_2'}{a\partial\lambda}\right)^2}\right]=\left(\frac{R}{f}\right)^2\frac{\overline{T'T_2'}}{l^2}$$
$$=8\cos^2\theta(\overline{v'v_1'}+\overline{v'v_3'}-2\overline{v'v_2'}). \quad \text{(A2.6)}$$

We have to estimate $\overline{u_3'u_4'}+\overline{v_3'v_4'}$ in (A2.5). We assume that the surface eddy wind is related to the eddy wind at level 3 by the speed ratio κ_e and the phase difference α_G corresponding to an Ekman spiral, i.e.,

$$\left.\begin{array}{l}u_4'=\kappa_e(u_3'\cos\alpha_G\mp v_3'\sin\alpha_G)\\ v_4'=\kappa_e(\pm u_3'\sin\alpha_G+v_3'\cos\alpha_G)\end{array}\right\}, \quad \text{(A2.7)}$$

where upper sign is used for the Northern Hemisphere and lower sign for the Southern. We then get

$$\overline{u_3'u_4'}+\overline{v_3'v_4'}=2\kappa_e\cos\alpha K_{E3}. \quad \text{(A2.8)}$$

Adding (A2.4) or (A2.5) to (A2.2), we have an estimate for whole effect of the frictional force on the change of K_{E1} or K_{E3}.

APPENDIX 3

Remarks on the Numerical Scheme

As mentioned in Section 6, the finite difference scheme used in general is Version I of the box method described by Kurihara and Holloway (1967). An exception is the computation of the first and second terms in the right-hand side of (4.1). In this case, the meridional flux of a quantity χ at the mid-point, $J=j+\frac{1}{2}$, where the advective velocity is $v_{j+\frac{1}{2}}=\frac{1}{2}(v_j+v_{j+1})$, is given by

$$\tfrac{1}{2}[(v_{j+\frac{1}{2}}+|v_{j+\frac{1}{2}}|)\chi_{j+1}+(v_{j+\frac{1}{2}}-|v_{j+\frac{1}{2}}|)\chi_j],$$

instead of $\frac{1}{2}v_{j+\frac{1}{2}}(\chi_j+\chi_{j+1})$. A similar formula is used for the vertical flux. Such a scheme preserves conservation of the volume integral of χ.

In many of the terms in prediction equations and diagnostic formulas, latitudinally smoothed values, weighted 1/6, 4/6, 1/6, are used rather than a grid value for a non-differentiated quantity, provided that a desirable condition, such as energy consistency, is not violated by doing so. The weighted value can be taken as a representative one for a small latitude span. The weighting formula is equivalent to the numerical integral by Simpson's first rule which has third-order accuracy. This accuracy is comparable to that of the three-point formula for the finite difference approximation for a first differential, i.e., second-order accuracy. On the other hand, the accuracy is first order when the grid value is used for a non-differential.

The coefficients b_1, b_2, h_1 and h_2 in (4.9) and (4.10) have the denominator $\overline{v'v_2'}$. In actual computation we replace it by the average of $\overline{v'v'}$ at levels 1, 2, 3 with weights 1/4, 2/4, 1/4 to obtain a smoother numerical result.

We use the method similar to the one described by Richtmyer (1957, pp. 101–104) for solving (4.21) to obtain $\bar{p}_* \overline{u'v'} \cos^2\theta$. In this case we impose the condition

$$-(\bar{p}_* \overline{u'v'} \cos\theta)_{j=1\frac{1}{2}} w_{sj=1} = -2g(\bar{\tau}_e)_{j=1} \cos\theta_{j=1},$$

$$\overline{u'v'}_{j=1\frac{1}{2}} = \tfrac{2}{3}\overline{u'v'}_{j=2},$$

where w_S is a weight factor appearing in the box method. The above relation expresses a momentum balance at the northern polar cap and gives $\overline{u'v'}$ at $j=2$. A similar relation for the southern polar cap fixes $\overline{u'v'}$ at $j=47$, i.e., another end-point value needed for solving the equation. We assume that $\overline{u'v'}$ at $j=1$ and 48 is one-third of that at $j=2$ and 47, respectively.

The dissipation of eddy kinetic energy due to vertical mixing of momentum is expressed by the second term on the right-hand side of (A2.4) and (A2.5). This effect is first estimated for the total air column. It is then partitioned to two layers in proportion to the eddy kinetic energy present at levels 1 and 3.

In computing ℓ^2 by (4.28), we put $\ell^2=\ell_c^2$, where $\ell_c^2=6\times10^9\text{m}^2$, if the numerator happens to be negative. When the denominator is less than β, it is replaced by β, and, if the resulting ℓ^2 is smaller than ℓ_c^2, we take ℓ_c^2 for ℓ^2.

When the numerical integration is done with eddies included, a check of K_E is made at each time step at each grid. If it is observed to be smaller than the criterion, 0.01 m^2 sec^{-2} in the present case, it is set to this value. A check is also made of $\overline{T'v'}$ so that (A1.7) is satisfied. We also examine $\overline{u'v'}$ and may adjust it so that $|\overline{u'v'}_k| \leqslant \overline{u'v'}_k$ is satisfied. During the integration it was found that such adjustments are rarely if ever needed.

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