CORRESPONDENCE

Note on Finite Difference Expressions for the Hydrostatic Relation and Pressure Gradient Force

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1. INTRODUCTION

The computation scheme which was proposed by Kurihara and Holloway [1] was used at the Geophysical Fluid Dynamics Laboratory, ESSA, for the study of the general circulation of the atmosphere incorporating the realistic distribution of orography.

We encountered the following difficulties resulting from the steepness of the mountain slopes at some places:

1) Over the steep slope of mountains, the computation of pressure gradient force had large truncation errors. This error was particularly large in the stratosphere. As a result, fictitious eddies appeared in the stratosphere over the steep slopes, e.g. over the periphery of the Antarctic continent. The zonal mean flow was also affected.

2) The so-called checkerboard pattern appeared in the distribution of surface pressure due to the growth of a computational mode. In order to overcome these difficulties, a modification of the computation scheme was attempted.

The first difficulty is drastically reduced by improving the accuracy in the computation of geopotential height. The second problem is largely overcome by making the estimation of pressure gradient force by a method which was suggested by Smagorinsky and Staff Members [2]. We shall describe the principles for writing the finite difference schemes which are free from the troubles mentioned above and do not yield fictitious sources of energy in the derivation of the difference formula for the budget of the total energy. In this note, the latter condition is called the condition of energy consistency.

The revised numerical scheme is currently used at GFDL in the integration of global model including topography.

2. EXPRESSIONS FOR THE HYDROSTATIC RELATION

When the equations are written in the σ-coordinate system, where σ is pressure P normalized by surface pressure P_*, the conditions at the bottom of the atmosphere can be treated in simple forms since the earth's surface coincides with a coordinate surface (Phillips [3]). This system has been adopted at the general circulation experiments at GFDL. In this case, the height of σ-surface has to be estimated.

Integration of the hydrostatic relation from the pressure P_A to P_B, or from $\sigma_A=P_A/P_*$ to $\sigma_B=P_B/P_*$, yields the geopotential difference $\Delta\phi$ between the two levels. In case of the isothermal atmosphere,

$$\Delta\phi=-RT\ln(\sigma_B/\sigma_A) \tag{2.1}$$

where T is temperature and R is the gas constant.

Assume that the above layer is divided into a finite number of sublayers. If the thickness of each sublayer is computed by a scheme corresponding to $\partial\phi/\partial\sigma=-RT/\sigma$ and accumulated, we have the geopotential difference

$$\Delta\phi=-RT\sum\frac{\Delta\sigma}{\sigma}. \tag{2.2}$$

However, if the above processing is done by a scheme corresponding to

$$\frac{\partial\phi}{\partial\ln\sigma}=-RT \tag{2.3}$$

we obtain

$$\Delta\phi=-RT\sum\Delta(\ln\sigma). \tag{2.4}$$

The result (2.2) does not generally coincide with (2.1), because of truncation error. But, (2.4) is usually equal to (2.1) because of the relation

$$\sum\Delta(\ln\sigma)=\ln(\sigma_B/\sigma_A).$$

The above result suggests that the use of a finite difference form corresponding to (2.3) generally gives a more accurate estimation of geopotential. The improvement in accuracy is significant when the steep high mountain is included in the numerical model. In fact, the modification of the finite difference form of the hydrostatic relation along these lines in our numerical integration could eliminate most of the fictitious motion apparently caused by the above-mentioned truncation error. The description of the detailed form will be made in section 5.

3. ESTIMATION OF PRESSURE GRADIENT FORCE

The pressure gradient force in the equation for $P_*\mathbf{V}$, where $\mathbf{V}$ is the horizontal wind, is written in two equivalent forms

$$\mathrm{PGF}=-P_*\nabla_P\phi \tag{3.1}$$

or

$$\mathrm{PGF}=-P_*\nabla_\sigma\phi-RT\nabla P_* \tag{3.2}$$

where the subscript means the reference surface along which the differentiation of geopotential is done. When we used the finite difference form corresponding to (3.2), the development of a computational mode was noticeable, though it had not been observed in the experiment [1] where the flat lower boundary was assumed. Smagorinsky and Staff Members [2] suggested the computation in the form corresponding to (3.1) in the prediction experiment in which realistic topography was taken into consideration. We adopted their suggestion, and the computational mode which was mentioned above was successfully eliminated.

In the modified version of the estimation of pressure gradient force in a model, it is necessary to obtain the geopotential of a pressure surface from the nearest sigma levels. An interpolation formula for such a purpose can take a form similar to (2.3).

4. CONSIDERATION OF ENERGY CONSISTENCY

In this section, we shall obtain some hints to formulate the finite difference scheme which utilizes the modified versions of the hydrostatic relation and the pressure gradient force but which still has the characteristics of energy consistency.

The formula which relates the energy conversion term $P_*\omega\alpha$ in the equation for c_pP_*T to the work done by the pressure gradient force is

$$P_*\omega\alpha=\underset{(A)}{-\frac{\partial}{\partial\sigma}P_*\phi\bar{\omega}}\underset{(B)}{-\nabla_\sigma\cdot(P_*\phi\mathbf{V})}\underset{(C)}{-\frac{\partial\phi\sigma}{\partial\sigma}\frac{\partial P_*}{\partial t}}\underset{(D)}{-\mathbf{V}\cdot(-P_*\nabla_P\phi)} \quad (E) \tag{4.1}$$

where ω is the vertical P-velocity dP/dt, α is the specific volume, $\bar{\omega}=d\sigma/dt$, c_p is the specific heat at constant pressure. Note that (3.1) is used for expressing the pressure gradient force. For the sake of convenience, each term in (4.1) is designated by (A) through (E).

Next, we will rewrite and expand the term (A) by the use of the hydrostatic relation in the modified form (2.3). The expression for $P_*\alpha$ in a form consistent with (2.3) and the alternate expression for (2.3) are, respectively,

$$P_*\alpha=RT\frac{\partial\ln\sigma}{\partial\sigma} \tag{4.2}$$

and

$$\phi-\frac{\partial\phi\sigma}{\partial\sigma}=\left(\sigma\frac{\partial\ln\sigma}{\partial\sigma}\right)RT. \tag{4.3}$$

When one uses (4.2), (4.3), the continuity equation and the formula for ω

$$\omega=P_*\bar{\omega}+\sigma\left(\frac{\partial P_*}{\partial t}+\mathbf{V}\cdot\nabla P_*\right) \tag{4.4}$$

the term (A) in (4.1) becomes

$$P_*\omega\alpha=R\omega T\frac{\partial\ln\sigma}{\partial\sigma}=\underset{(1)}{P_*\bar{\omega}\frac{1}{\sigma}\left(\phi-\frac{\partial\phi\sigma}{\partial\sigma}\right)}\underset{(2)}{-\phi\frac{\partial P_*\bar{\omega}}{\partial\sigma}}$$
$$\underset{(3)}{-\phi\nabla_\sigma\cdot P_*\mathbf{V}}\underset{(4)}{-\frac{\partial\phi\sigma}{\partial\sigma}\frac{\partial P_*}{\partial t}}$$
$$\underset{(5)}{+RT\sigma\frac{\partial\ln\sigma}{\partial\sigma}\mathbf{V}\cdot\nabla P_*}. \tag{4.5}$$

In order that (4.1) holds in the finite difference version, the finite difference expressions of the right hand side of (4.5) must, after some manipulations, yield the terms corresponding to the right hand side of (4.1). A comparison of (4.5) with (4.1) suggests first that the terms (1) + (2) in (4.5) must be, in the estimation in finite difference form, equal to the term (B) in (4.1). This condition determines the relation between the geopotentials at the interfaces of the layer and at the middle level. Secondly, the terms (3) + (5) in (4.5) must give the quantity corresponding to the terms (C) + (E) in (4.1). This requirement suggests that we do not evaluate P_* in (4.4) directly by differencing of P_* but by the use of the alternate finite difference formula corresponding to

$$P_*\left\{\nabla_P\phi-\nabla_\sigma\phi\right\}\Big/\left(RT\sigma\frac{\partial\ln\sigma}{\partial\sigma}\right).$$

One can easily prove the identity of this term with P_*. Thirdly the finite difference value of $\sigma\partial\ln\sigma/\partial\sigma$, the analytical value of which is unity, should be as close to unity as possible. This condition should be considered in the vertical division of the model atmosphere into a number of layers.

5. AN EXAMPLE OF FINITE DIFFERENCE SCHEME

The formulation of the system of finite difference equations along the lines described in sections 2 to 4 is not unique. It depends on the grid system, the vertical resolution, the position of the levels where the dependent variables are to be assigned and so on.

In this section, as an example, one scheme which is obtained as a modification of the scheme proposed by Kurihara and Holloway [1] is presented. We use the same symbols and the finite difference operators as those used in [1]. When we apply the difference operators similarly on an isobaric surface, we will attach a prime to the operator.

1) Vertical division of the atmosphere:

$\sigma_{1/2}=0$: top of the atmosphere

$\sigma_{K+1/2}=1$: ground surface; bottom of the Kth layer.

The interfaces of the layers $\sigma_{k+1/2}(k=1, \ldots, K-1)$ should be chosen such that (i) $\sigma_k=\sqrt{\sigma_{k-1/2}\sigma_{k+1/2}}$ (for $k=2, \ldots, K$); σ_1 can be any value less than $\sigma_{1\frac{1}{2}}$ but $\sigma_1=\sigma_{1\frac{1}{2}}/e$ ($e=2.71828$) is recommended, (ii) $\frac{1}{2}(\sigma_{k+1/2}+\sigma_{k-1/2})\,\delta_k \ln \sigma/\delta_k\sigma$ should be as close to unity as possible, where $\delta_k \ln \sigma$ for $k=1$ is assumed to be equal to $2(\ln \sigma_{1\frac{1}{2}}-\ln \sigma_1)$.

2) Hydrostatic relation (revision of (3.8A) and (3.9) in [1]):

$$\delta_k\phi=-RT_k\delta_k \ln \sigma \tag{5.1}$$

or

$$\phi_k-\frac{\delta_k(\phi\sigma)}{\delta_k\sigma}=RT_k\left(\frac{\sigma_{k+1/2}+\sigma_{k-1/2}}{2}\,\frac{\delta_k \ln \sigma}{\delta_k\sigma}\right)\equiv R\tilde{T}_k \tag{5.2}$$

Here, ϕ_k is related to $\phi_{k\pm1/2}$ by

$$\phi_{k\pm1/2}=\phi_k\mp RT_k(\delta_k \ln \sigma)/2$$

which insures that $\phi_k=\frac{1}{2}(\phi_{k+1/2}+\phi_{k-1/2})$.

3) Pressure gradient force: The revised forms of this term for version I in [1] are $-L'_\lambda(P_*, \phi_p)$, and $-L'_\theta(P_*, \phi_p)$, where ϕ_p is the geopotential of a pressure surface P. The value ϕ_p is obtained from the heights of the nearest sigma surfaces by using an interpolation formula which is consistent with (5.1). The modified forms for version II in [1] are $-P_*G'_\lambda(\phi_p)$ and $-P_*G'_\theta(\phi_p)$.

4) Thermodynamics equation and formula for ω: Modified form of (3.3A) in [1]:

$$\frac{\partial}{\partial t}\left(P_{*0}T_0\right)=-D\left(\frac{T_l+T_0}{2}\right)+\frac{R}{c_p}T_0\omega_0\frac{\delta_k \ln \sigma}{\delta_k\sigma}+\frac{P_{*0}}{c_p}\hat{q}+\left(F_T\right)_0.$$

Modified form of (3.7A) in [1]:

$$\omega_{0k}=P_{*0}\frac{\bar{\omega}_{k+1/2}+\bar{\omega}_{k-1/2}}{2}+\frac{\sigma_{k+1/2}+\sigma_{k-1/2}}{2}\left[\frac{\partial P_{*0}}{\partial t}\right.$$
$$+u_0\left\{L'_\lambda\left(P_*, \phi_p\right)-L_\lambda\left(P_*, \phi_\sigma\right)\right\}\Big/R\tilde{T}_0$$
$$\left.+v_0\left\{L'_\theta\left(P_*, \phi_p\right)-L_\theta\left(P_*, \phi_\sigma\right)\right\}\Big/R\tilde{T}_0\right]$$

for version I.

$$\omega_{0k}=P_{*0}\frac{\bar{\omega}_{k+1/2}+\bar{\omega}_{k-1/2}}{2}+\frac{\sigma_{k+1/2}+\sigma_{k-1/2}}{2}\left[\frac{\partial P_{*0}}{\partial t}\right.$$
$$+u_0P_{*0}\left\{G'_\lambda\left(\phi_p\right)-G_\lambda\left(\phi_\sigma\right)\right\}\Big/R\tilde{T}_0$$
$$\left.+v_0P_{*0}\left\{G'_\theta\left(\phi_p\right)-G_\theta\left(\phi_\sigma\right)\right\}\Big/R\tilde{T}_0\right]$$

for version II. Here, ϕ_p and ϕ_σ are the geopotentials of a pressure surface and a sigma surface, respectively, and $R\tilde{T}$ is a quantity defined in (5.2).

5) Finally, in regard to the grid system used, we recommend the use of a system which has no grid points at the Poles. Otherwise, the surface pressure at the Pole tends to be inconsistent with the meridional pressure gradient in surrounding latitudes due to a variation in the weights involved in the estimation of pressure gradient force by the box method. Moreover, removing the polar boxes makes the programming simpler. In this case, the numerical schemes for the northernmost or southernmost boxes take forms similar to the ones for other boxes by considering that the areas of the poleward interfaces of these boxes are zero.

Note Added in Proof—The recent results suggest that the present scheme still tends to cause small-scale irregularity of the flow pattern at the highest level over the steep slopes of mountains. Further improvement of the computation scheme is desired.

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Comments on "A Comparison of the Climate of the Eastern United States During the 1830's With the Current Normals"

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In his recent article, Wahl [1] indicated that: "A comparison of climatic data for the eastern United States from the 1830's and 1840's with the currently valid climatic normals indicates a distinctly cooler, and in some areas, wetter climate in the first half of the last century."