Geostrophic Regimes on a Sphere and a Beta Plane

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ABSTRACT

A general geostrophic equation is derived for a shallow layer of fluid on a sphere. This equation encompasses the planetary, intermediate, and quasi-forms of geostrophy and produces their equations directly when the appropriate parametric ordering relationships are chosen. The three regimes have proven useful for defining and describing oceanic and Jovian eddies and currents on the planetary, intermediate and synoptic scales respectively. The general geostrophic equation may be most useful in describing the interactions among these three different regimes of motion and between motions in high and low latitudes. The accuracy of the β-plane version of these equations is also examined in detail.

1. Introduction

Although most planetary motions are governed by geostrophy, their characteristics can vary widely depending on their scale relative to the deformation radius L_R. In particular, the so-called Planetary-Geostrophic (PG), Intermediate-Geostrophic (IG), and Quasi-Geostrophic (QG) regimes have been identified for the large, intermediate and synoptic scales of motion.

In the PG regime, motions are dominated by nonlinear divergence effects and tend to be steady and forced (Sverdrup, 1947; Burger, 1958). In the QG system, wave dispersion and turbulence generally prevail (Charney (1984). [However, coherent features also may occur under special circumstances (Flierl, 1979).] In the IG regime, wave dispersion and nonlinear divergence act in balance to give long-lived, coherent vortices as the innate forms of motion. Derivations of the IG equation have been given by Flierl (1980), Charney and Flierl (1981), Yamagata (1982), Williams and Yamagata (1984, hereafter WY84).

Connections between the PG and QG regimes are not apparent in the traditional derivations, where separate *ad hoc* scaling arguments are used. However, a more recent derivation (WY84) has shown that the three basic regimes, as well as a number of subregimes, can all be derived in the same systematic manner by assuming ordering relationships of the form $\hat{\epsilon} = E\hat{\beta}^n$, $\hat{s} = S\hat{\beta}^m$ (n = 1–4; m = 0–3) for the Rossby number $\hat{\epsilon}$ and the stratification number $\hat{s}$ in terms of the sphericity parameter $\hat{\beta}$. This derivation also shows that a more general form of PG system exists than the one defined by Burger, that the IG system is unique in parameter space, and that the interaction between the various regimes is described by a term common to all of them—the Jacobian for vorticity advection.

It has been suggested that connections between the regimes can also be deduced by taking less restrictive ordering relationships, ones that do not involve discrete powers of $\hat{\beta}$ (Cushman-Roisin, personal communication, 1984). But it is not clear whether such a procedure is consistent with the β-plane approximation where sphericity is expanded *a priori* in discrete powers of $\hat{\beta}$. Doubts have also been raised as to how accurate the β-plane approximation—which is used in all the derivations—is for the IG system; for, unlike the QG and PG equations which occur at $O(\hat{\beta})$ the main IG equation occurs at $O(\hat{\beta}^2)$.

To address some of these problems, we will show in Section 2 that there exists a general geostrophic (GG) equation that is simpler than the primitive equations but is parametrically general enough to contain all three major regimes as elementary subcases. (Such an equation was first derived, in a different context and in a different form, by Anderson and Killworth, 1979.) The equations can all be derived in spherical coordinates, thus avoiding the β-plane limitations. In addition, in Section 3, we will also examine the accuracy of the β-plane approximation in detail for the IG system because of the novelty and importance of this regime.

The general geostrophic equation should be most useful for studying the interactions between midlatitudes and the tropics and among the various regimes (i.e., *types*[1] of motion), as well as in providing a uniform derivation of their equation sets. Interactions between different types of motion are normally studied

[1] Interactions between different *scales* of motion within a single regime can also be studied by other methods, e.g., closure models for a turbulent regime.

via the primitive equations. However, regime interactions can also be described by the hybrid equations given by multiple-scale expansions (Pedlosky, 1984). The latter method works for the baroclinic equations whereas the general geostrophic approach, although simpler, has so far only been applied to the shallow water equations. The conditions under which IG dynamics occur for continuously stratified fluids have not yet been determined.

Physical interest in the various geostrophic systems stems from their relevance to the multi-scaled oceanic and Jovian eddies. In particular, IG dynamics helps explain the longevity, localization, and anticyclonic bias of Jupiter's Great Red Spot and Ovals (WY84). Single, solitary IG vortices can be produced by barotropically unstable currents and such vortices have been numerically simulated for periods in excess of a century with little change occurring in their shape or strength (Williams and Wilson, 1985). These vortices occur in calculations that eliminate the physical and computational deficiencies in the study of WY84. The spherical coordinate form of the GG equation is useful for analyzing the latitudinal variation in Jovian vortex behavior.

Speculations about Jovian vortices have also been made by plasma physicists using equations resembling the general geostrophic equation (Petviashvili, 1980, 1983; Sagdeev *et al.*, 1981; Antipov *et al.*, 1981, 1982; Romanova and Tseitlin, 1984). Their equations are not as complete nor as well defined as ours, nor are they integrated, nor are any regimes nor scales of motion delineated, but they do contain much that is physically appropriate.

The interest of plasma physicists in the geostrophic equation systems originates with the analogy found between plasma drift waves and fluid Rossby waves (Hasegawa *et al.*, 1979), and with the analogous roles these waves play in the turbulent cascades involved in plasma confinement and atmospheric jet formation (Hasegawa and Williams, private communication—see Hasegawa, 1980). The analogy between drift waves and turbulence and Rossby waves and turbulence is now well established (e.g. Meiss and Horton, 1983; Weinstein, 1983).

Given that analogies exist for waves and turbulence, similarities have been sought recently (Petviashvili, 1980) between drift solitons (defined by Todoroki and Sanuki, 1974; Makhankov, 1977) and Rossby solitons of the IG density (not QG shear) type. However, in a planetary atmosphere solitary waves behave like solitons only at the Equator (Boyd, 1980; Williams and Wilson, 1985) while at other latitudes they behave like coalescing vortices (WY84). No coalescing solitary drift waves have been identified in plasma theory or experiment, as yet, so the analogy is incomplete. The solitary Rossby vortices produced in laboratory fluid experiments (Antipov *et al.*, 1981, 1982)—with their anticyclonic bias, etc.—are more akin to the IG vortices than to the Rossby soliton claimed by the authors. For studying Jovian vortices, numerical simulations are far simpler and more quantitative than fluid or plasma analogs.

2. The general geostrophic equation

a. Derivation

The simplest system capable of describing the geostrophic regimes is the shallow water model, whose equations are (in dimensional form):

$$\frac{D\mathbf{v}}{Dt} + f\mathbf{k} \times \mathbf{v} = -\nabla h, \tag{1}$$

$$h_t + \nabla \cdot (h\mathbf{v}) = 0, \tag{2}$$

where h is the geopotential height and incorporates g, the gravity or reduced gravity; where $f = 2\Omega \sin\theta$ is the Coriolis term and D/Dt the total derivative; where $\mathbf{v}$ and ∇ are the horizontal velocity and the horizontal divergence operator, and where $\mathbf{k}$ is the unit vertical vector. Spherical coordinates are assumed.

Equation (1) can be inverted into the form:

$$\mathbf{v} = \frac{1}{f}\mathbf{k} \times \left(\nabla h + \frac{D\mathbf{v}}{Dt}\right). \tag{3}$$

If we now assume that the geostrophic balance dominates, the $D\mathbf{v}/Dt$ terms are all small compared to the others in (3), i.e. they are $O(\xi)$ where ξ is a small number comparable to the Rossby number $\hat{\epsilon}$ or the temporal number $\hat{\tau}$ (see Section 2b for definitions). Then to $O(\xi^2)$, (3) can be written entirely in terms of $\mathbf{v}^g$ or h:

$$\mathbf{v} = \mathbf{v}^g + \frac{1}{f}\mathbf{k} \times \left[\mathbf{v}_t^g + \frac{1}{2}\nabla(\mathbf{v}^g)^2 + (\nabla \times \mathbf{v}^g) \times \mathbf{v}^g\right], \tag{4}$$

where $\mathbf{v}^g = f^{-1}\mathbf{k} \times \nabla h$ is the geostrophic velocity and the second group of terms are of $O(\xi)$. This expression simplifies to the convenient form:

$$\mathbf{v} = \frac{1}{f}\mathbf{k} \times \nabla(h + K) - \frac{1}{f^2}\nabla h_t - \frac{\zeta}{f}\mathbf{v}^g, \tag{5}$$

where K, ζ are the geostrophic components of kinetic energy and vorticity:

$$K = \frac{1}{2}(\mathbf{v}^g)^2 = \frac{1}{2f^2}(\nabla h)^2;$$

$$\zeta = \mathbf{k} \cdot (\nabla \times \mathbf{v}^g) = \nabla\left(\frac{1}{f}\nabla h\right). \tag{6}$$

Substituting (5) into (2) yields a prediction equation for h that involves only h, to $O(\xi^2)$:

$$h_t + \nabla \cdot h\left[\frac{1}{f}(\mathbf{k} \times \nabla h) - \frac{1}{f^2}\nabla h_t + \frac{1}{f}(\mathbf{k} \times \nabla K - \zeta\mathbf{v}^g)\right] = 0. \tag{7}$$

This can be manipulated into its simplest form:

$$h_t - \nabla\cdot\left(\frac{h}{f^2}\nabla h_t\right) + \frac{h}{m}h_x(f^{-1})_y - J\left(h, \frac{h}{f^2}\zeta\right) - J\left(\frac{h}{f}, K\right) = 0, \tag{8}$$

(i) (ii) (iii) (iv) (v)

where $J(h, \alpha) \equiv (\mathbf{k}\times\nabla h)\cdot\nabla\alpha = m^{-1}(h_x\alpha_y - h_y\alpha_x)$ is the advective Jacobian; $(x, y) = a(\lambda, \theta)$, $m = \cos\theta$; a being the planetary radius; (λ, θ) being the longitude and latitude.

Equation (8) describes changes in the height field due to geostrophic motion of any type. We shall refer to (8) as the General Geostrophic (GG) equation for convenience. It involves only the h field and its terms describe (i) time changes, (ii) dispersion, (iii) wave propagation and, at higher order, nonlinear divergence, and (iv), (v) the geostrophic advection of vorticity and kinetic energy. A more complex form of this equation occurs in Anderson and Killworth (1979)—see their Eq. (2.14).

b. Geostrophic regimes

To obtain the various regimes of geostrophic motion that are subsets of (5) and (8), the equations must be nondimensionalized. To do this, we introduce the thickness variable η and the mean thickness H such that $h = g(H + \eta)$ and use the scales $(U, L, T, (LUf_0/g, f_0)$ for the variables $(\mathbf{v}, y, t, \eta, f)$. The associated Rossby, stratification, sphericity and temporal[2] (or frequency) parameters are defined as $\hat{\epsilon} = U/Lf_0$, $\hat{s} = L_R^2/L^2$, $\hat{\beta} = La^{-1}\cot\theta_0$ and $\hat{\tau} = (f_0T)^{-1}$ where $L_R = (gH)^{1/2}/f_0$ is the deformation radius and θ_0 the scaling latitude.

Equations (5) and (8) can then be written, to $O(\xi^2)$, as:

$$\mathbf{v} = \frac{1}{f}\mathbf{k}\times\nabla(\eta + \hat{\epsilon}K) - \frac{\hat{\tau}}{f^2}\nabla\eta_t - \frac{\hat{\epsilon}}{f}\zeta\mathbf{v}^g, \tag{9}$$

$$\frac{\hat{\tau}}{\hat{s}}\left[\eta_t - \hat{s}\nabla\cdot\left(\frac{1}{f^2}\left(1 + \frac{\hat{\epsilon}}{\hat{s}}\eta\right)\nabla\eta_t\right)\right] - \frac{\hat{\beta}}{m_0f^2}\left(1 + \frac{\hat{\epsilon}}{\hat{s}}\eta\right)\eta_x - \hat{\epsilon}J\left(\eta, \left(1 + \frac{\hat{\epsilon}}{\hat{s}}\eta\right)\frac{\zeta}{f^2}\right) - \frac{\hat{\epsilon}^2}{\hat{s}}J\left(\frac{\eta}{f}, K\right) = 0, \tag{10}$$

where all variables are now nondimensional. The three major geostrophic regimes or subsets of (10) are given by selecting the lowest level ordering relationships[3] among the parameters $\hat{\epsilon}$, $\hat{s}$, $\hat{\tau}$, and $\hat{\beta}$:

1) For quasi-geostrophy, we choose $\hat{\tau} = \hat{s}\hat{\epsilon}$ and $\hat{s} \sim 1$, so that (10) reduces, at $O(\hat{\epsilon})$, to

$$\eta_t - \hat{s}\nabla\cdot\left(\frac{1}{f^2}\nabla\eta_t\right) - \frac{\hat{\beta}}{m_0\hat{\epsilon}f^2}\eta_x - J\left(\eta, \frac{\zeta}{f^2}\right) = 0. \tag{11}$$

On the larger QG scales, $\hat{\beta} \sim \hat{\epsilon}$ and motions are characterized by wave propagation and dispersion, while on the smaller QG scales $\hat{\beta} \sim 0$ and motions are similar to those of two-dimensional or f_0 plane fluids.

2) For planetary geostrophy, we choose $\hat{\tau} = \hat{s}\hat{\epsilon}$ and $\hat{s} \sim \hat{\epsilon}$, so that (10) reduces at $O(\hat{\epsilon})$, to

$$\eta_t - \frac{\hat{\beta}}{m_0\hat{\epsilon}f^2}\left(1 + \frac{\hat{\epsilon}}{\hat{s}}\eta\right)\eta_x - J\left[\eta, \left(1 + \frac{\hat{\epsilon}}{\hat{s}}\eta\right)\frac{\zeta}{f^2}\right] - \frac{\hat{\epsilon}}{\hat{s}}J\left(\frac{\eta}{f}, K\right) = 0. \tag{12}$$

The most interesting form of PG motion occurs when $\hat{\beta} \sim \hat{\epsilon}$, with wave propagation and steepening by nonlinear divergence determining the flow form.

3) For intermediate-geostrophy, we choose $\hat{\tau} = \hat{s}\hat{\beta}$, $\hat{\epsilon} \sim \hat{s}^2$, $\hat{\beta} \sim \hat{s}$ and retain higher order terms (in $\hat{s}$) so that (10) reduces to

$$\eta_t - \hat{s}\nabla\cdot\left(\frac{1}{f^2}\nabla\eta_t\right) - \frac{1}{m_0f^2}\left(1 + \frac{\hat{\epsilon}}{\hat{s}}\eta\right)\eta_x - \frac{\hat{\epsilon}}{\hat{\beta}}J\left(\eta, \frac{\zeta}{f^2}\right) = 0. \tag{13}$$

The first order terms (η_t, η_x) describe long wave propagation, while the $O(\hat{s})$ terms give the slow changes. For the wave propagation to dominate, i.e., for scale separation to occur, $\hat{\beta}$ must be relatively small $(\sim\hat{s})$. Thus the f variations in (13) may be of little consequence. Alternatively, we could regard (10) as providing an approximate description of IG flows when only the ordering relationship $\hat{\epsilon} \sim \hat{s}^2$ is specified.

The advection term $J(\eta, \zeta/f^2)$ occurs in all 3 regimes and provides a process through which the regimes can interact, as well as describing scale interactions within each regime. Subregimes of (11) and (12) can be defined by choosing higher level parameter relationships (see WY84, Table 1). The IG system in (13), however, is unique and physically irreducible.

[2] In WY84, we chose $T = L/|c_\beta|$ as the most suitable time scale, where $c_\beta = -\beta L_R^2$ is the long-wave speed. Here we use a more general formulation so that the f_0 plane cases may also be considered.

[3] The only constraint on $\hat{s}$, that $(1 + \hat{\epsilon}\eta/\hat{s}) > 0$, is important only for strong cyclonic motion.

c. Midlatitudinal beta-plane forms

To derive the β-plane form of the GG equation (10) and its subcases, it is useful to eliminate the f^{-1} factors from inside the operators. Thus:

$$\frac{\hat\tau}{\hat s}\left\{\eta_t - \frac{\hat s}{f^2}\nabla\cdot\left[\left(1+\frac{\hat\epsilon}{\hat s}\eta\right)\nabla\eta_t\right]\right\} - \frac{\hat\beta}{m_0 f^2}\left(1+\frac{\hat\epsilon}{\hat s}\eta\right)\eta_x - \frac{\hat\epsilon}{f^3}\left(1+\frac{\hat\epsilon}{\hat s}\eta\right)J(\eta,\nabla^2\eta) - \frac{\hat\epsilon^2}{\hat s f^3}J\left(\eta,\frac{1}{2}(\nabla\eta)^2\right) - \hat\beta R = 0, \quad (14)$$

where the residual variations due to f_y can be written:

$$R \equiv \frac{1}{f^4}\left[\hat\tau f R_1 + \hat\epsilon\left(1+\frac{\hat\epsilon}{\hat s}\eta\right)R_2 + \frac{\hat\epsilon^2}{\hat s}R_3\right],$$

$$R_1 = -2\frac{m}{m_0}\left(1+\frac{\hat\epsilon}{\hat s}\eta\right)\eta_{yt},$$

$$R_2 = 3D_1(\nabla^2\eta) - J\left(\eta, \frac{m}{m_0}\eta_y\right) - 4\frac{\hat\beta}{f}D_1\left(\frac{m}{m_0}\eta_y\right),$$

$$R_3 = (D_2 + 2D_1)\left(\frac{1}{2}(\nabla\eta)^2\right),$$

and where $D_1(\alpha) \equiv -(\eta_x/m_0)\alpha$, $D_2(\alpha) \equiv (\eta/m_0)\alpha_x$ are simple operators. The form of R is of little importance and is given only for completeness. It can be neglected only when $\hat\beta$ is small, as in the $\hat\beta$-plane approximation, or in defining an alternative IG equation:

$$\eta_t - \frac{\hat s}{f^2}\nabla^2\eta_t - \frac{1}{m_0 f^2}\left(1+\frac{\hat\epsilon}{\hat s}\eta\right)\eta_x - \frac{\hat\epsilon}{\hat\beta f^3}J(\eta,\nabla^2\eta) = 0. \quad (15)$$

Introducing sphericity to $O(\hat\beta)$ into (14), via the expansions $f = 1 + \hat\beta y$, $m_0/m = 1 + \hat\beta\gamma y$, and retaining all second order terms gives:[4]

$$\eta_t - \hat s(1-2\hat\beta y)\left[\nabla'\cdot\left(1+\frac{\hat\epsilon}{\hat s}\eta\right)\nabla'\eta_t - \hat\beta\gamma\left(1+\frac{\hat\epsilon}{\hat s}\eta\right)\eta_{yt}\right] - \frac{\hat s}{\hat\tau}\hat\beta(1-2\hat\beta y)\left(1+\frac{\hat\epsilon}{\hat s}\eta\right)\eta_{x'} - \frac{\hat s}{\hat\tau}\hat\epsilon(1-3\hat\beta y)(1+\hat\beta\gamma y)\left(1+\frac{\hat\epsilon}{\hat s}\eta\right) \times J'(\eta, \nabla'^2\eta - \hat\beta\gamma\eta_y) - \frac{\hat\epsilon^2}{\hat\tau}(1-3\hat\beta y)(1+\hat\beta\gamma y) \times J'\left(\eta, \frac{1}{2}(\nabla'\eta)^2 - \hat\beta\gamma y\eta_{x'}^2\right) - \frac{\hat s}{\hat\tau}\hat\beta R = 0, \quad (16)$$

where $x' = m_0 x$, $\gamma = \tan^2\theta_0$, $J'(\eta, \alpha) = \eta_{x'}\alpha_y - \eta_y\alpha_{x'}$, $\nabla'\alpha = \mathbf{i}\alpha_{x'} + \mathbf{j}\alpha_y$. The prime superscript is dropped hereafter for convenience. If we retain only those terms that are significant over the three major parametric regimes, (14) reduces to the β-plane (i.e. $\gamma \equiv 0$) form of the general geostrophic equation without having to specify γ:

$$\eta_t - \hat s\nabla^2\eta_t - \frac{\hat s}{\hat\tau}\hat\beta\left(1 - 2\hat\beta y + \frac{\hat\epsilon}{\hat s}\eta\right)\eta_x - \frac{\hat s}{\hat\tau}\hat\epsilon\left(1+\frac{\hat\epsilon}{\hat s}\eta\right) \times J(\eta,\nabla^2\eta) - \frac{\hat\epsilon^2}{\hat\tau}J\left(\eta,\frac{1}{2}(\nabla\eta)^2\right) = 0. \quad (17)$$

All essential processes are represented in this form of the GG equation, which thus provides the simplest model for analytical studies of geostrophic regime interactions. A similar equation has been derived by Cushman-Roisin (personal communication, 1984). For numerical studies, the basic spherical equations are easily, rapidly, and accurately solvable (Williams and Wilson, 1985).

For the QG, PG and IG ordering relationships used in (11)–(13), (17) yields the three major regime equations as given in WY84:

$$\eta_t - \hat s\nabla^2\eta_t - \frac{\hat\beta}{\hat\epsilon}\eta_x - J(\eta,\nabla^2\eta) = 0, \quad (18)$$

$$\eta_t - \frac{\hat\beta}{\hat\epsilon}\left(1+\frac{\hat\epsilon}{\hat s}\eta\right)\eta_x - \left(1+\frac{\hat\epsilon}{\hat s}\eta\right)J(\eta,\nabla^2\eta) - \frac{\hat\epsilon}{\hat s}J\left(\eta,\frac{1}{2}(\nabla\eta)^2\right) = 0, \quad (19)$$

$$[\eta_t - \eta_x] - \left[\hat s\nabla^2\eta_t + \frac{\hat\epsilon}{\hat s}\eta\eta_x - 2\hat\beta y\eta_x + \frac{\hat\epsilon}{\hat\beta}J(\eta,\nabla^2\eta)\right] = 0. \quad (20)$$

Equations (18) and (19) represent the most general β-plane forms of QG and PG balances, while reduced (subregime) β-plane forms can be obtained by using higher level ordering relationships (see WY84, Table 1, for details[5]).

3. Accuracy of the beta plane approximation for the IG system

We now take a closer look at *all* the variables involved in the IG system—momentum, vorticity and potential vorticity—and examine the accuracy of the β-plane approximation for their prediction equations. To achieve this the $O(\hat\beta^2)$ terms are included

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[4] Higher order sphericity expansions are considered in Section 3. Equivalent forms of (16) are obtained if (14) is first multiplied throughout by f^3.

[5] Where the convention $\hat\epsilon\hat\beta^{-1}J(\eta, \zeta) \equiv -J'$ and $\hat\epsilon^2\hat s^{-1}J(\eta, K) = -J''$ holds, with the right-hand side expressions being those defined in WY84. For convenience, the meteorological malpractice of writing $K = \frac{1}{2}v^2 = 0.5f^{-2}(\nabla h)^2$ rather than $K = \frac{1}{2}\mathbf{v}\cdot\mathbf{v} = 0.5 f^{-2}(\nabla h)^2$ has been adhered to.

in the sphericity expansions. This analysis is quite separate from that of Section 2.

a. Sphericity expansion of the primitive equations

We begin by writing the basic equations in the nondimensional form:

$$\hat{\beta}\hat{s}\frac{D\mathbf{v}}{Dt} + f\mathbf{k}\times\mathbf{v} = -\nabla\eta, \qquad (21a,b)$$

$$\hat{\beta}\hat{s}\frac{D\zeta_a}{Dt} + \hat{\epsilon}\zeta_a\nabla\cdot\mathbf{v} = 0, \qquad (21c)$$

$$\hat{\beta}\hat{s}\frac{D\eta}{Dt} + (\hat{s} + \hat{\epsilon}\eta)\nabla\cdot\mathbf{v} = 0, \qquad (21d)$$

$$\frac{D\Pi}{Dt} = 0, \qquad (21e)$$

where $D(\ )/Dt = (\ )_t + (\hat{\epsilon}/\hat{\beta}\hat{s})\mathbf{v}\cdot\nabla(\ )$ is the total derivative, $\zeta = \mathbf{k}\cdot\nabla\times\mathbf{v}$ the relative vorticity, $\zeta_a = f + \hat{\epsilon}\zeta$ the absolute vorticity, and $\Pi = \zeta_a/(1 + (\hat{\epsilon}/\hat{s})\eta)$ the potential vorticity.

The sphericity is introduced into (21) by expanding the spherical factors about values at the representative latitude and longitude, θ_0 and λ_0:

$$\frac{\sin\theta}{\sin\theta_0} = 1 + \hat{\beta}y - \hat{\beta}^2\frac{\gamma}{2}y^2 + O(\hat{\beta}^3),$$

$$\frac{\cos\theta_0}{\cos\theta} = 1 + \hat{\beta}\gamma y + \hat{\beta}^2\delta y^2 + O(\hat{\beta}^3),$$

$$\frac{\tan\theta}{\tan\theta_0} = 1 + \hat{\beta}y(1+\gamma) + O(\hat{\beta}^2), \qquad (22)$$

where $a(\theta - \theta_0) = Ly$, $a\cos\theta_0(\lambda - \lambda_0) = Lx$, $\gamma = \tan^2\theta_0$ and $\delta = (\gamma/2)(1+\gamma)$. Only those terms needed to attain $O(\hat{\beta}^2)$ accuracy in the basic equations have been retained. These expansions are valid when L/a and $\hat{\beta}$ are less than unity. Because manipulation of the sphericity expansions is limited and can lead to errors (e.g., the identity $\cos^{-1}\theta(\sin\theta)_\theta = 1$ does not hold), it is necessary to expand the ζ equation in parallel with those for u and v as a checking device.

Introducing these spherical expansions and the IG ordering relationships ($\hat{\epsilon} = E\hat{\beta}^2$, $\hat{s} = S\hat{\beta}$, $T = \hat{\beta}t$, where T now represents the slow time scale) gives equations accurate to $O(\hat{\beta}^2)$:

$$S\hat{\beta}^2u_t - v\zeta_a = -(1 + \hat{\beta}\gamma y + \hat{\beta}^2\delta y^2)\left(\eta + \frac{E}{2}\hat{\beta}^2\mathbf{v}^2\right)_x, \qquad (23a)$$

$$S\hat{\beta}^2v_t + u\zeta_a = -\left(\eta + \frac{E}{2}\hat{\beta}^2\mathbf{v}^2\right)_y, \qquad (23b)$$

$$\frac{D\zeta_a}{Dt} + \tilde{E}\zeta_a\nabla\cdot\mathbf{v} = 0, \qquad (23c)$$

$$\hat{\beta}\frac{D\eta}{Dt} + (1 + \tilde{E}\hat{\beta}\eta)\nabla\cdot\mathbf{v} = 0, \qquad (23d)$$

where

$$\frac{D(\ )}{Dt} = (\ )_t + \hat{\beta}(\ )_T + \tilde{E}[(1 + \hat{\beta}\gamma y + \hat{\beta}^2\delta y^2)u(\ )_x + v(\ )_y],$$

$$\nabla\cdot\mathbf{v} = (1 + \hat{\beta}\gamma y + \hat{\beta}^2\delta y^2)u_x + v_y - v\hat{\beta}(\gamma + 2\hat{\beta}\delta y),$$

$$\zeta = (1 + \hat{\beta}\gamma y)v_x - u_y + u\hat{\beta}\gamma,$$

$$\zeta_a = \left(1 + \hat{\beta}y - \hat{\beta}^2\frac{\gamma}{2}y^2\right) + E\hat{\beta}^2\zeta,$$

$$\Pi = 1 + \hat{\beta}(y - \tilde{E}\eta) + \hat{\beta}^2\left(E\zeta - \tilde{E}y\eta + \tilde{E}^2\eta^2 - \frac{\gamma}{2}y^2\right), \qquad (24)$$

and where $\tilde{E} = ES^{-1}$. These equations can then be expanded by writing the dependent variables in series form, e.g.,

$$v = \sum_{n=0}^{\infty}\hat{\beta}^n v^{(n)}.$$

b. Lower-order equations

The $O(1)$ and $O(\hat{\beta})$ equations given by (23) and (24) are:

$$v^{(0)} = \eta_x^{(0)}, \qquad (25a)$$

$$u^{(0)} = -\eta_y^{(0)}, \qquad (25b)$$

$$u_x^{(0)} + v_y^{(0)} = 0, \qquad (25c)$$

and

$$v^{(1)} + (1-\gamma)yv^{(0)} = \eta_x^{(1)}, \qquad (26a)$$

$$u^{(1)} + yu^{(0)} = -\eta_y^{(1)}, \qquad (26b)$$

$$u_x^{(1)} + v_y^{(1)} + v^{(0)} - \gamma(yv^{(0)})_y = 0, \qquad (26c)$$

$$\eta_t^{(0)} + u_x^{(1)} + v_y^{(1)} - \gamma(yv^{(0)})_y = 0, \qquad (26d)$$

$$\eta_t^{(0)} - \eta_x^{(0)} = 0. \qquad (26e)$$

[All sets of equations are written in the same sequence as (21) to identify their origin.] Cross differentiation of the momentum equations (25a,b) and (26a,b) leads to expressions identical to those in the vorticity equations (25c) and (26c). Similarly, subtraction of the height equation (26d) from the vorticity equation (26c) gives the same expression as the potential vorticity equation (26e). Thus, at these levels of $\hat{\beta}$, all the equations are mutually consistent.

c. Higher-order equations

The $O(\hat{\beta}^2)$ equations can be written, after some algebra, as:

$$Su_t^{(0)} - \left[v^{(2)} + yv^{(1)} + E\zeta^{(0)}v^{(0)} - \frac{\gamma}{2}y^2v^{(0)}\right] = -\left[\eta^{(2)} + \frac{E}{2}(\mathbf{v}^{(0)})^2\right]_x - \gamma y\eta_x^{(1)} - \delta y\eta_x^{(0)}, \quad (27a)$$

$$Sv_t^{(0)} + \left[u^{(2)} + yu^{(1)} + E\zeta^{(0)}u^{(0)} - \frac{\gamma}{2}y^2u^{(0)}\right] = -\left[\eta^{(2)} + \frac{E}{2}(\mathbf{v}^{(0)})^2\right]_y, \quad (27b)$$

$$[S\nabla^2\eta_x^{(0)} + EJ(\eta^{(0)}, \nabla^2\eta^{(0)}) + u_x^{(2)} + v_y^{(2)} + \eta_x^{(1)} - 2y\eta_x^{(0)}] + [\gamma(yu_x^{(1)} - v^{(1)}) - \delta(y^2v^{(0)})_y] = 0, \quad (27c)$$

$$[\eta_t^{(1)} + \eta_T^{(0)} + u_x^{(2)} + v_y^{(2)} - \tilde{E}\eta^{(0)}\eta_x^{(0)}] + [\gamma(yu_x^{(1)} - v^{(1)}) - \delta(y^2v^{(0)})_y] = 0, \quad (27d)$$

$$\eta_T^{(0)} - \tilde{E}\eta^{(0)}\eta_x^{(0)} - S\nabla^2\eta_x^{(0)} + 2y\eta_x^{(0)} - EJ(\eta^{(0)}, \nabla^2\eta^{(0)}) = -[\eta_t^{(1)} - \eta_x^{(1)}]. \quad (27e)$$

In this set, the potential vorticity equation (27e)—the so called IG equation—is consistent with the vorticity and height equations and equals their difference. In contrast, cross differentiation of the momentum equations yields a vorticity equation that differs from (27c) by a term $\gamma^2yv^{(0)}$ originating in the term $\gamma y\eta_x^{(1)}$ in (27a). The γ^2 term is only significant when $\theta_0 \geqslant 45°$. This inconsistency illustrates the danger involved in multiple differentiation of the approximated sphericity factors, particularly of the f^{-1} forms prevalent in the equations.

The derivation of the vorticity equation (27c) is not completely independent of the momentum equations as it involves using (26a). However, as no differentiation of spherical factors is involved and as all the $O(\hat{\beta})$ equations are mutually consistent, (27c) is thus the correct version of the vorticity equation. To avoid errors, it appears to be advisable that the vorticity equation always be expanded in parallel with the momentum equations, even though this could involve some redundancy at the lower orders of expansion. The potential vorticity equation (27c) is identical with the form (17) derived directly from the spherical GG equation.

d. Simpler derivation of Π *equation*

The basic IG equations (26e) and (27e) can be derived simply and directly from the potential vorticity equation, (21e), by following the procedure of Section 2 in which we first define the geostrophic wind components $v^g = \eta_x$, $u^g = -\eta_y$ and then expand them as $\mathbf{v}^g = \mathbf{v}^{(0)} + \hat{\beta}\mathbf{v}^{(1)} + O(\hat{\beta}^2)$. Substitution in the momentum equations (23a, b) reveals that the total velocity can be written, to $O(\hat{\beta}^2)$, as:

$$v = [1 - \hat{\beta}(1 - \gamma)y]v^{(0)} + \hat{\beta}v^{(1)},$$
$$u = [1 - \hat{\beta}y]u^{(0)} + \hat{\beta}u^{(1)}. \quad (28)$$

Then the total derivative can be defined, to $O(\hat{\beta})$, in a simple Jacobian form:

$$\frac{DA}{Dt} = A_t + \hat{\beta}A_T + \{[1 - \hat{\beta}(1 - \gamma)y]\tilde{J}(\eta^{(0)}, A) + \hat{\beta}\tilde{J}(\eta^{(1)}, A)\}, \quad (29)$$

where $\tilde{J} = \tilde{E}J$. Applying this operator to the $O(\hat{\beta}^2)$ potential vorticity

$$\Pi = 1 + \hat{\beta}[y - \tilde{E}\eta^{(0)}] + \hat{\beta}^2\left[\tilde{E}(S\zeta^{(0)} - \eta^{(1)} - y\eta^{(0)} + \tilde{E}\eta^{(0)2}) - \frac{\gamma}{2}y^2\right], \quad (30)$$

yields (27e), after two terms of the form $\pm\gamma yv^{(0)}$ have been cancelled out. Although this derivation is simpler than that in Section 3c, it is less revealing of the limitations and inconsistencies.

e. Beta-plane accuracy

From the above, Sections (b)–(d), we see that γ factors occur in all of the $\mathbf{v}$, ζ, η equations at $O(\hat{\beta})$ and $O(\hat{\beta}^2)$ but that they do not occur in the Π equation (27e). Thus the β-plane approximation, given by setting $\gamma \equiv 0$ is valid for the potential vorticity at $O(\hat{\beta}^2)$ but for the other variables it attains only an $O(\gamma)$ accuracy. In the QG system, a similar dependence on γ occurs (Lipps, 1964; Pedlosky 1979, p. 322) but no inconsistency of the $O(\gamma^2)$ type arises because $O(\hat{\beta}^2)$ terms are not necessary and multiple differentiation of sphericity factors is not involved.

4. Discussion

We have shown that a general geostrophic equation for the potential vorticity can be derived in either spherical (8) or midlatitudinal β-plane (17) form. The equation contains the three major geostrophic regimes as subsets and provides a uniform derivation of their governing equations. The equation also describes the interactions among the various regimes and between the various regions. The GG equation (8) also applies to geostrophic motion in low latitudes where it can be approximated by an equatorial β-plane and can then describe, for example, the Rossby solitons discussed by Boyd (1980)—see Williams and Wilson (1985) for details.

For the shallow water system, it does not appear to be necessary to resort to multiple-scale methods for dealing with regime interactions. That method may, however, be essential for dealing with continuously stratified, baroclinic fluids (Pedlosky, 1984). Whether baroclinic versions of the GG and IG equations exist remains an interesting problem for future research.

The β-plane approximation is as valid for the IG regime as it is for the QG regime, i.e., only the equation for potential vorticity is fully accurate.

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