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Development of a Simplified Diagnostic Model for Interpretation of Oceanographic Data

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LIST OF SYMBOLS

A, B, C, D variable coefficients, known functions of position

d vertical position of ocean bottom

f Coriolis parameter - vertical component of Earth's rotation vector

fo regional average Coriolis parameter

g acceleration of gravity

H constant used to scale the depth

J Jacobian differential operator $J(a,b) = (\frac{\partial a}{\partial y}\frac{\partial b}{\partial x} - \frac{\partial a}{\partial x}\frac{\partial b}{\partial y})$

K eddy coefficient for diffusion of momentum

$\vec{k}$ unit vector in the positive z direction

L a scale length for horizontal dimensions

N eddy coefficient for diffusion of density

N_1 non-dimensional scaling parameter $\frac{g\varepsilon H}{\rho o f o U L}$

N_2 non-dimensional scaling parameter $\frac{\gamma}{\rho o H}$

N_3 non-dimensional scaling parameter $\frac{\beta L}{fo}$

n, s Locally orthogonal coordinate axes with s along a given curve

P pressure

$P(A)$ atmospheric pressure

$P(d)$ pressure at the bottom of the ocean $z = d$

$P(\zeta)$ pressure on the surface $z = o$

R radius of the Earth

T_n integrated mass transport normal to a given boundary

Tx integrated mass transport in the x direction

Ty integrated mass transport in the y direction

t time-independent variable

U x-component of fluid velocity and horizontal velocity scale factor

$U(d)$ x-component of the geostrophic velocity at the bottom

$\vec{V}$ vector fluid velocity

V y-component of fluid velocity

V_D y-component of the geostrophic velocity at the bottom

x, y, z Coordinate axes - independent variables

α vertical integral of fluid density

$\alpha(d)$ definite integral of fluid density from the surface to the bottom

β gradient of the Coriolis parameter

γ bottom friction coefficient

Δ definite integral of α from the surface to the bottom

ε constant used to scale density variations

∇ del operator $(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j})$

ζ vertical position of the sea surface

ϕ latitude

ρ fluid density

ρo constant representative fluid density

$\vec{\tau}$ horizontal component of wind stress at the surface

τx x-component of surface wind stress

τy y-component of surface wind stress

ψ mass transport stream function

$\vec{\Omega}$ Earth's rotation vector

DEVELOPMENT OF A SIMPLIFIED DIAGNOSTIC MODEL FOR INTERPRETATION OF OCEANOGRAPHIC DATA

J. A. Galt

A steady state numerical model of ocean circulation is formulated to include geostrophic and Ekman dynamic balances as well as the effects of bathymetric variations. The model is diagnostic in that certain segments of the flow are determined from field data. In particular the baroclinic portion of the geostrophic mode is obtained from in situ density measurements and the surface wind driven layer is determined from wind stress data. The model solves for the required barotropic mode and subsequent bottom frictional layer that satisfies continuity and the assumed boundary conditions.

The dependency of the model solution on input data and boundary conditions is discussed. For the case where bottom friction is not included the model reduces to a first order, ordinary differential equation that can be solved along characteristic (f/d) contours. For preliminary model studies this simplified formulation is recommended with a combination of moored current meter data and dynamic height calculations for boundary conditions.

1. INTRODUCTION

In doing oceanographic studies, one has traditionally been faced with the task of estimating circulation. This has proved a formidable undertaking and success has been limited even in restricted areas. Direct measurements have proved particularly difficult. Stable platforms that can survive the ocean environment for extended periods are hard to engineer and recording current meters are expensive. Once the direct measurements are obtained, problems in analysis are common, with typical measurements showing significant energy at higher frequencies. These energetic fluctuations make it difficult to calculate significant mean currents, or obtain coherent flow patterns over even very short distances without averaging quite long records. For many applications it is just this mean flow that is needed and the effects of higher frequency fluctuations could be adequately represented by eddy coefficients.

Since direct current measurements are so difficult to make and apply to regional oceanographic studies, it is reasonable to attempt a theoretical description of currents. To do this, equations representing the flow must be formulated and solved. The Navier-Stokes equation is basically an expression of Newton's second law that can be applied to geophysical fluids. Combining this with equations for continuity and the distribution of density, a closed set of equations is obtained that is theoretically capable of solution. This set of equations may be written in the following form which is appropriate for large-scale oceanic flows:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}.\nabla)\ \vec{v} + 2\vec{\Omega}x\vec{v} = -\frac{1}{\rho}\nabla\rho + \nabla(K\nabla\vec{v}) - g\vec{k} \tag{1}$$

$$\nabla.\vec{v} = o \tag{2}$$

$$\frac{\partial \rho}{\partial t} + (\vec{v}.\nabla)\rho = \nabla(N\nabla\rho) \quad , \tag{3}$$

where

$\vec{v}$ vector velocity

$\vec{\Omega}$ rotation vector of the Earth

ρ density

p pressure

g acceleration of gravity

$\vec{k}$ unit vector in the vertical direction

K eddy coefficients for the diffusion of momentum

N eddy coefficients for the diffusion of density - note: formulating the equations in terms of the diffusion of density eliminates the equation of state.

These equations have been simplified to some degree already; nevertheless they do represent a relatively general formulation of the circulation and solutions can be attempted. Problems arise, however, since this set of equations is extremely rich in solutions. In addition, nonlinear effects make general solutions impossible in analytic form, and even with numerical approximations only limited ranges or scales of flow can be investigated. To get around these mathematical difficulties additional approximations can be introduced. These limited formulations can consistently represent certain classes or parts of the flow that are thought to be dominant. Even in these simplified forms major difficulties still stand in the way of a purely theoretical description of the flow. In particular, the advection of the density field by the currents leads to mathematically complex expressions. If both the fluid velocity and the density are dependent variables then even the simplified forms of the equations are nonlinear. The idea of diagnostic modeling is to get around this particular difficulty.

In a diagnostic model the density is not taken as a dependent variable. It is obtained by direct measurements in the field or region of interest. The direct measurements are then used as coefficients to linearize the theoretical equations. Looking at this in a slightly different light, a diagnostic model solves for the velocity field subject to some observed density distribution and the equations of motion. In this sense it is a combination of direct observations and theoretical solutions. It should be pointed out, however, that the direct observations required are of the density field. (In the ocean this requires temperature and salinity measurements.) These are much easier to obtain on a routine basis than direct measurements of the actual currents. In its simplest form, then, a diagnostic model analyzes the observed temperature and salinity distribution of a region and supplies the current implied by them.

Diagnostic modeling is not new to meteorology or oceanography. The standard geostrophic calculation of currents using the dynamic method (Formin, 1964) is a form of diagnostic modeling. Russian scientists have been particularly active in the development of very general diagnostic models and in applying them to large-scale oceanic circulation. An interesting review and discussion is presented by Sarkisyan and Keondjiyan (1972). Bryan (personal communication) has also investigated large-scale ocean circulation using diagnostic formulations in the initialization of more complex prognostic modeling experiments. Peng and Hsueh (1974) have applied a relatively complete diagnostic model to a coastal zone as part of the CUE study. The initial dynamic processes included in their work are similar to those proposed in this study, i.e., geostrophic plus Ekman flow. This is a quite common point of departure for linear circulation models. In their work Peng and Hsueh carry out relatively detailed scale analysis for the special case where the bottom slope provides the dominant gradient in the potential vorticity. In applying their model a somewhat idealized coastal configuration was used.

The object of the present work is to formulate a simplified diagnostic model that can readily be used in the interpretation of data from coastal and continental shelf regions. Optional sets of boundary conditions and computational techniques will be presented. A detailed derivation of the model equations will be presented in the following section. Before this, a brief outline will be given of the types of flow included in the formulation.

Scale analysis of the equations of motion for open ocean flows indicate that accelerations are small and that Coriolis and pressure forces nearly balance each other. If this balance is assumed, a geostrophic current is represented. The equation for this is

$$\vec{f} \times \vec{v} = -\frac{1}{\rho} \nabla p \quad , \tag{4}$$

where $\vec{f}$ = vertical component of the Earth's rotation, or Coriolis parameter.

For these same scales of motion the relationship between the pressure and the density is hydrostatic, i.e.,

$$\frac{\partial P}{\partial z} = - \rho g \ , \tag{5}$$

where z = vertical coordinate axis,

g = acceleration of gravity.

To obtain a diagnostic relationship between the density and velocity equation (4) is differentiated with respect to z and equation (5) is substituted for the pressure term. This results in the so-called "thermal wind" equation:

$$f \ x \ \frac{\partial \vec{v}}{\partial z} = \frac{1}{\rho} \nabla P \ . \tag{6}$$

From this equation the geostrophic velocity can be obtained to within a constant of integration providing the horizontal gradients in the density are known.

Near the surface of the ocean a wind-driven layer (Ekman layer) is superimposed on the geostrophic flow. The dynamics of the flow are represented by a balance between the Coriolis force and shear stress. This results in the equation:

$$\vec{f} \ x \ \vec{v} = \frac{\partial}{\partial z}(K \ \frac{\vec{v}}{z}) \ , \tag{7}$$

and boundary conditions

$$K\frac{\partial \vec{v}}{\partial z} \ surface = \vec{\tau} \ , \ and \ \vec{v} \ bottom = 0 \tag{8}$$

where $\vec{\tau}$ = *the surface wind stress*

Equation (7) is also in a diagnostic form in that the wind-driven currents can be determined once the surface stress distribution is known. The simple linear sum of equations (4) and (7) then represents a wind-driven baroclinic regime. Moreover, a recent study by Beardsley and Butman (1974) suggests that this relatively simple formulation can describe a significant portion of the observed flow in some continental shelf regions.

The Ekman plus geostrophic formulation can easily be extended to include bottom stress and a subsequent bottom Ekman layer. Given the density distribution and the wind stress the flow is then completely determined to within the constant of integration that results from equation (6). Traditionally this constant has been evaluated using the assumption that horizontal velocities go to zero at great depth (level of no motion), or equivalently, that the slope of the sea surface is known. While this may be a reasonable assumption in deep water, it is quite clearly not true in the shallower regions over the continental shelf. In this case an alternate method of obtaining the constant of integration must be considered. One straightforward approach is to set continuity constraints on the transport. Typically these will require that the divergence of the total horizontal flow is zero. A classical open ocean model incorporating this type of dynamics was done by Sverdrup (1947). In shallower regions over the continental shelf and slope, the circulation may interact strongly with the sloping bottom. Satisfying an integral condition on the transport has the additional advantage of allowing the flow to couple appropriately with the bathymetry; thus this formulation satisfies potential vorticity constraints and includes the important

Joint Effects of Baroclinicity And bottom Relief (JEBAR) (Sarkisyan and and Ivanov, 1971; Holland, 1973).

From this relatively simple theory a diagnostic model can be developed that includes wind-driven currents, geostrophic flow (barotropic and baroclinic modes), frictionally controlled currents along bottom, and the effects of complex bathymetry. This model will then be used to describe the flow along coastal and continental shelf regions using relatively easy to obtain wind and density data as input.

2. DEVELOPMENT OF MODEL EQUATIONS

We may begin by considering a coordinate system in the northern hemisphere, as shown in Figure 1. The x and y axes are horizontal and at the mean elevation of sea level; x points to the east and y to the north. The z axis is positive up giving a right-handed Cartesian coordinate system; $\zeta(x, y)$ defines the vertical position of the free surface, and $d(x, y)$ gives the vertical position of the bottom.

Making the assumption that the pressure is hydrostatic gives

$$\frac{\partial P}{\partial z} = - \rho g \quad , \tag{9}$$

and, integrating this, gives the pressure at any depth as

$$P\ (z) = P(A) - g \int \rho dz \quad , \tag{10}$$

where $P(A)$ is the local atmospheric pressure at sea level. Assuming that the ocean adjusts to this atmospheric pressure as an inverted barometer, there will be no associated steady-state flow. Without any loss in

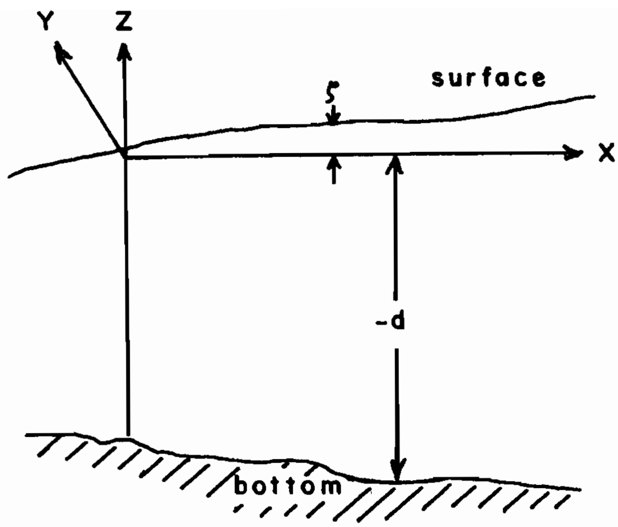

Figure 1. Coordinate system used for model equations.

generality, $P(A)$ will be taken as zero. We may now obtain horizontal components to the pressure gradients by differentiation of (10), i.e.,

$$\frac{\partial p}{\partial x} = - \frac{\partial}{\partial x} (g \int_{\zeta}^{z} \rho dz) = \frac{\partial}{\partial x} (g \int_{0}^{\zeta} \rho dz) + \frac{\partial}{\partial x} (g \int_{z}^{0} \rho dz) \ . \quad (11)$$

Using Leibniz's rule for interchanging the order of differentiation and integration in the first term on the right-hand side gives

$$\frac{\partial p}{\partial x} = (g\rho o) \frac{\partial \zeta}{\partial x} + g \int_{0}^{\zeta} \frac{\partial \rho}{\partial x} dz + \frac{\partial}{\partial x} (g \int_{z}^{0} \rho dz) \quad , \qquad (12)$$

where $\rho(\zeta)$ is replaced by ρo, a characteristic constant density. The second term on the right-hand side is negligible compared with the typical values for the other two terms. It represents the baroclinic contribution to the pressure gradient of something less than the top meter of water. For a homogeneous upper layer this would be identically zero and will be assumed zero in this model. Both of these assumptions will be clearly appropriate when the equations are scaled and nondimensionalized in a following section of this report; thus equation (12) becomes

$$\frac{\partial p}{\partial x} = g\rho o \frac{\partial \zeta}{\partial x} + g \frac{\partial \alpha}{\partial x} \quad , \tag{13}$$

where

$$\alpha(z) = \int_z^o \rho dz \quad . \tag{14}$$

The first term on the right-hand side gives the barotropic contribution to the pressure gradient, i.e., that part caused by the slope of the sea surface. It is obviously independent of depth and with present technology cannot be measured at sea. The second term on the right-hand side gives the baroclinic contribution to the pressure gradient, i.e., that part caused by the internal distribution of mass in the ocean. This can be obtained from standard oceanographic stations using a Nansen bottle or STD data.

Using the same arguments, the y component of the pressure gradient can be written as

$$\frac{\partial p}{\partial y} = \rho o g \frac{\partial \zeta}{\partial y} + g \frac{\partial \alpha}{\partial y} \quad . \tag{15}$$

We may now consider an ocean region where the currents are the sum of a surface Ekman layer driven by the wind, a geostrophic interior driven by barotropic and baroclinic pressure gradients, and a bottom Ekman layer that matches a zero slip condition along the bottom. The equations to represent this flow are as follows:

$$- f\rho v = - \frac{\partial p}{\partial x} + \frac{\partial}{\partial z} (K\frac{\partial u}{\partial z}) \quad , \tag{16}$$

$$f\rho u = - \frac{\partial p}{\partial y} + \frac{\partial}{\partial z} (K\frac{\partial v}{\partial z}) \quad , \tag{17}$$

where u and v are the x and y components of velocity,respectively. Integrating these equations from d to ζ and using Liebniz's rule gives

$$- fTy = - \frac{\partial}{\partial x} \int_d^\zeta Pdz + P(\zeta)\frac{\partial \zeta}{\partial x} - P(d)\frac{\partial d}{\partial x} - \tau x(d) + \tau x(\zeta) \quad , \tag{18}$$

$$fTx = - \frac{\partial}{\partial y} \int_d^\zeta Pdz + P(\zeta)\frac{\partial \zeta}{\partial y} - P(d)\frac{\partial d}{\partial y} - \tau y(d) + \tau y(\zeta) \quad , \tag{19}$$

where Tx and Ty are the x and y components of the mass transport, $\tau x(d)$ and $\tau y(d)$ are the x and y components of the stress evaluated at the bottom, and $\tau x(\zeta)$ and $\tau y(\zeta)$ are the x and y components of the wind stress acting on the surface of the water.

By cross differentiating equations (18) and (19) and subtracting, the following vorticity equation is obtained: (assuming $f = f_o + \beta y$)

$$-\beta T_y - f\nabla\vec{T} = J(P(\zeta), \zeta) - J(P(d), d) + \vec{k}\cdot\nabla x\vec{\tau}(d) - \vec{k}\cdot\nabla x\vec{\tau}(\zeta) \quad . \tag{20}$$

Setting the divergence of the total transport equal to zero means that the last term on the left-hand side is identically zero. In addition, making the Boussinesq approximation the surface layer is assumed homogeneous and the first term on the right-hand side is zero. This leaves the following vorticity equation:

$$-\beta T_y = - J(P(d), d) + \vec{k}\cdot\nabla x\vec{\tau}(d) - \vec{k}\cdot\nabla x\vec{\tau}(\zeta) \quad . \tag{21}$$

The stress at the bottom can be evaluated using Ekman theory. (Neumann and Pierson, 1966, p. 200). Assuming that the depth of water is greater than the depth of frictional influence, the stress will be linearly related to the geostrophic velocity at the bottom. In Peng and Hseuh's (1974) model this assumption is not made and the more general case where the water depth can be smaller than the Ekman layer thickness $(\pi\sqrt{\frac{K}{\rho f}})$ is included. This leads to more complex coefficients in the final formulation and it is not particularly clear that it extends the usefulness of the model very much. In particular, if the water is shallower than 20-30 meters the linear dynamics become highly questionable. Proceeding with the assumption that the depth is greater than the Ekman depth,

$$\tau x_d = f\gamma(U(d) - V(d)) \ , \qquad (22)$$

$$\tau y_d = f\gamma(U(d) + V(d)) \ , \qquad (23)$$

where $U(d)$ and $V(d)$ are the geostrophic velocity components evaluated at the bottom and γ is a dimensional factor proportional to an eddy coefficient.

The geostrophic velocity can be obtained from the horizontal components of the pressure gradient, i.e.,

$$- f\rho v = - \frac{\partial P}{\partial x} \ ,$$

$$f\rho u = - \frac{\partial P}{\partial y} \ .$$

To simplify this the density on the left-hand side of these equations may be taken as the representative constant value. This is equivalent to making a Boussinesq approximation and gives the following results:

$$\tau x(d) = \frac{\gamma}{\rho_0} \left(- \frac{\partial P(d)}{\partial y} - \frac{\partial P(d)}{\partial x} \right) \ , \qquad (24)$$

$$\tau y(d) = \frac{\gamma}{\rho_0} \left(- \frac{\partial P(d)}{\partial y} + \frac{\partial P(d)}{\partial x} \right) \ . \qquad (25)$$

Cross differentiating these two components gives the curl of the bottom stress as

$$\vec{k} \cdot \nabla x \vec{\tau}(d) = \frac{\gamma}{\rho o} \nabla^2 P(d) \quad . \tag{26}$$

Substituting (26) into (21) yields

$$-\beta T_y = - J(P(d), d) + \frac{\gamma}{\rho o} \nabla^2 P(d) - \vec{k} \cdot \nabla x \vec{\tau}(\zeta) \; .$$

In the absence of significant variations in the Coriolis parameter (i.e., where horizontal length scale is small compared with the radius of the Earth), this equation can be solved for the single dependent variable, bottom pressure. A more useful form of the equation can be obtained by substituting from equations (13) and (15) into the above. Doing this gives the following equation in the two unknowns, surface elevation and y-component of transport:

$$-\beta T_y = - g\rho o \; J(\zeta, d) - gJ(\alpha(d), d) + g\gamma \nabla^2 \zeta + \frac{g\gamma}{\rho o} \nabla^2 \alpha(d) - \vec{k} \cdot \nabla x \vec{\tau}(\zeta) \quad . \tag{27}$$

This equation specifies the complete vorticity balance represented in the model, and it may clarify the physics a bit to identify the significance of each term. The left-hand side of the equation gives the so-called β-effect or planetary tendency associated with variations in the Coriolis parameter. The first two terms on the right-hand side of the equation represent the joint interaction of the flow with the bathymetry. These are the JEBAR terms with the first of the pair giving the contribution from the barotropic mode and the second giving the baroclinic contribution. One can note that for homogeneous water the second term would be identically zero and the first would give the familiar stretching term in the conservation of potential vorticity. On the other extreme, if a complete baroclinic adjustment resulted in no net horizontal pressure gradient at depth, these terms would identically cancel each other. The third term on the right-hand side represents the vorticity contribution by the barotropic mode caused by bottom friction. The fourth term gives the vorticity added through the baroclinic mode by bottom friction. Finally, the last term in equation (27) is the vorticity added to the flow by the wind stress.

To solve equation (27) we must come up with an additional relationship between the transport and the surface elevation. This can easily be done by enumerating the components to the total transport. To begin with, the transport in the surface Ekman layer is given by

$$Ty(Ekman) = -\frac{\tau x}{f}(\zeta) \quad , \tag{28}$$

subject to the condition that the water depth is greater than the Ekman layer thickness. Next the barotropic velocity is independent of depth and given by

$$\rho v = \frac{g\rho o}{f}\frac{\partial\zeta}{\partial x} \, ,$$

and, integrating this from d to o, gives

$$Ty\ (barotropic) = -\frac{g\rho od}{f}\frac{\partial\zeta}{\partial x} \, . \tag{29}$$

The baroclinic velocity is given by

$$\rho v = \frac{g}{f}\frac{\partial\alpha}{\partial x} \quad ,$$

and, integrating this from d to o, gives

$$Ty\,(baroclinic) = \frac{g}{f}\int_d^o \frac{\partial\alpha}{\partial x}\,dz \tag{30}$$

$$= \frac{g}{f}\left(\frac{\partial}{\partial x}\int_d^o \alpha dz + \alpha(d)\frac{\partial d}{\partial x}\right)$$

$$= \frac{g}{f}\left(\frac{\partial\Delta}{\partial x} + \alpha(d)\frac{\partial d}{\partial x}\right) ,$$

where Δ is the definite integral

$$\Delta = \int_d^o \alpha dz \quad . \tag{31}$$

The transport in the bottom Ekman layer will be given by:

$$Ty\,(d\text{-}Ekman) = \frac{\tau x(d)}{f} , \tag{32}$$

again subject to the condition that the water depth is greater than the Ekman layer thickness, and substitution into this from (13), (15) and (24) gives

$$Ty\,(d\text{-}Ekman) = -\frac{\gamma g}{f}\left(\frac{\partial\zeta}{\partial x} + \frac{\partial\zeta}{\partial y}\right) - \frac{\gamma g}{\rho_o f}\left(\frac{\partial\alpha(d)}{\partial x} + \frac{\partial\alpha(d)}{\partial y}\right) \quad . \tag{33}$$

The total mass transport in the y direction can now be written as the sum of equations (28), (29), (30) and (33), i.e.,

$$Ty = -\frac{\tau x(\zeta)}{f} - \frac{g\rho_o d}{f}\frac{\partial\zeta}{\partial x} + \frac{g}{f}\left(\frac{\partial\Delta}{\partial x} + \alpha(d)\frac{\partial d}{\partial x}\right) \tag{34}$$

$$-\frac{\gamma g}{f}\left(\frac{\partial\zeta}{\partial x} + \frac{\partial\zeta}{\partial y}\right) - \frac{\gamma g}{\rho_o f}\left(\frac{\partial\alpha(d)}{\partial x} + \frac{\partial\alpha(d)}{\partial y}\right) \quad .$$

For completeness the x-component of the mass transport will be written as

$$T_x = + \frac{\tau_y(\zeta)}{f} + \frac{g\rho_o d}{f}\frac{\partial \zeta}{\partial y} - \frac{g}{f}\left(\frac{\partial \Delta}{\partial y} + \alpha(d)\frac{\partial d}{\partial y}\right) - \frac{\gamma g}{f}\left(\frac{\partial \zeta}{\partial x} - \frac{\partial \zeta}{\partial y}\right) - \frac{\gamma g}{\rho_o f}\left(\frac{\partial \alpha(d)}{\partial x} - \frac{\partial \alpha(d)}{\partial y}\right) . \tag{35}$$

Equations (27), (34) and (35) now give three equations in the three unknowns T_x, T_y and ζ. From these it is possible to obtain a single equation for the elevation of the free surface, or for the transport (expressed in terms of a stream function). Sarkisyan and Keondjiyan (1972) discuss the relative merits of each formulation in some detail. In this work some attention will also be given to the following alternate forms.

2.1 Surface Elevation as Dependent Variable

For the present section the free surface elevation will be chosen as the dependent variable, thus focusing attention on the near surface currents. Substituting equation (34) in (27) and rearranging terms gives

$$\begin{aligned} &+ g\gamma\nabla^2\zeta + \frac{g\gamma}{\rho_o}\nabla^2\alpha(d) - g\rho_o J(\zeta,d) - gJ(\alpha(d), d) - \vec{k}\cdot\nabla\times\vec{\tau}(\zeta) \\ &- \frac{\beta}{f}\tau_x(\zeta) - \frac{g\rho_o d\beta}{f}\frac{\partial \zeta}{\partial x} + \frac{g\beta}{f}\left(\frac{\partial \Delta}{\partial x} + \alpha(d)\frac{\partial d}{\partial x}\right) - \frac{\gamma g\beta}{f}\left(\frac{\partial \zeta}{\partial x} + \frac{\partial \zeta}{\partial y}\right) \\ &- \frac{\gamma g\beta}{\rho_o f}\left(\frac{\partial \alpha(d)}{\partial x} + \frac{\partial \alpha(d)}{\partial y}\right) = 0 . \end{aligned} \tag{36}$$

This, then, is a final model equation. It is elliptic in the single unknown ζ. It represents a diagnostic model in that the variable coefficients are all given in terms of known constants, the depth distribution and the observed density field. Solving this equation is equivalent to finding the barotropic mode that satisfies continuity, subject to a given baroclinic mode, wind stress driven Ekman layer and assumed Ekman dynamics in a bottom Ekman layer.

2.2. Transport Stream Function as a Dependent Variable

In this section a diagnostic equation for the total transport will be derived. This option tends to focus attention on the integrated flow and in some cases leads to more natural, or easier to interpret, boundary conditions. To develop this form of the equation (34) and (35) can be solved for the gradient components of the surface elevation. These values are then differentiated and the results substituted into (27). For the general case the algebra involved is quite tedious and the resulting formulation so unwieldy that it is not of any practical value. One special case is quite useful, however, and this will be developed. In particular, if the frictional drag on the bottom is zero (formally $\gamma = o$), equations (34) and (35) no longer require simultaneous solution, and we may proceed directly as follows:

$$g\rho o \frac{\partial \zeta}{\partial x} = -\frac{fTy}{d} - \frac{\tau x(\zeta)}{d} + \frac{g}{d}\left(\frac{\partial \Delta}{\partial x} + \alpha(d)\,\frac{\partial d}{\partial x}\right), \tag{37}$$

$$g\rho o \frac{\partial \zeta}{\partial y} = \frac{fTx}{d} - \frac{\tau y(\zeta)}{d} + \frac{g}{d}\left(\frac{\partial \Delta}{\partial y} + \alpha(d)\,\frac{\partial d}{\partial y}\right), \tag{38}$$

and substituting these into equation (27), again with the assumption that $\gamma = o$, and introducing a stream function for the total mass transport such that

$$T_y = \frac{\partial \Psi}{\partial x} \qquad T_x = -\frac{\partial \Psi}{\partial y}, \tag{39}$$

the following equation is obtained:

$$J(\Psi,\frac{f}{d}) = -\frac{g}{d}J(\alpha(d),\ d) - \frac{g}{d^2}J(\Delta,d) + \frac{1}{d^2}\vec{k}\cdot\nabla d x \vec{\tau}(\zeta) - \frac{1}{d}\vec{k}\cdot\nabla x \vec{\tau}(\zeta)\ . \tag{40}$$

Here again we have a diagnostic model with the total transport stream function as the dependent variable expressed in terms of the density distribution, bathymetry and the surface wind stress distribution. The solution will yield the transport that satisfies the integrated continuity constraints subject to a measured baroclinic mode, surface wind-driven Ekman layer and bathymetric distribution.

3. BOUNDARY CONDITIONS AND DATA REQUIREMENTS

In the preceding section two diagnostic equations were derived: Equation (36) giving the surface elevation or barotropic mode and (40) giving the total integrated mass transport. To make use of these equations a method of solution must be introduced and appropriate boundary conditions applied. In this section a number of specific cases will be considered and the demands for input data and boundary conditions will be discussed.

The first case to be considered will be the one represented in equation (36). This form is relatively general in that it includes both surface and bottom Ekman layers, geostrophic flow (including barotropic and baroclinic modes), the effects of bathymetry, and the variation of the Coriolis parameter with latitude. The equation can be written in the following form:

$$A\ \nabla^2\zeta + B\frac{\partial\zeta}{\partial x} + C\frac{\partial\zeta}{\partial y} + D = O\ , \tag{41}$$

where the non-constant coefficients are known functions of the independent variables, i. e.,

$$A = \gamma g \quad , \tag{42}$$

$$B = g\rho o \left(\frac{\partial d}{\partial y} - \frac{\beta d}{f} - \frac{\beta \gamma}{f \rho o}\right) \quad , \tag{43}$$

$$C = g\rho o \left(\frac{\partial d}{\partial x} - \frac{\beta \gamma}{f \rho o}\right) \quad , \tag{44}$$

$$D = \frac{\gamma g}{\rho o} \nabla^2 \alpha(d) - gJ(\alpha(d), d) - \frac{\gamma g \beta}{f \rho o} \left(\frac{\partial \alpha(d)}{\partial x} + \frac{\partial \alpha(d)}{\partial y}\right) \tag{45}$$

$$\frac{g\beta}{f} \left(\frac{\partial \Delta}{\partial x} + \alpha(z) \frac{\partial d}{\partial x}\right) = \vec{k} \cdot \nabla x \vec{\tau}(\zeta) - \frac{\beta \tau x(\zeta)}{f}$$

The variable part of these coefficients depends on the depth, the density distribution and the wind stress field.

The depth distribution can be obtained with satisfactory accuracy from standard hydrographic charts. In virtually all cases the quantity and quality of the bathymetric data is better than any of the other required input data. It is only necessary to digitize chart data and develop some appropriate method for interpolating and differentiating the results. Once this is done it is a relatively simple task to calculate the depth and gradient of the depth at whatever numerical grid points are desired.

The density information that is required is contained in the values of $\alpha(d)$ and Δ introduced in (14) and (31). These are specified by the density as follows:

$$\alpha(z) = \int_z^0 \rho dz \quad , \tag{46}$$

which is clearly related to the pressure in a hydrostatic fluid. This integral would typically be done numerically after obtaining values for the integrand at discrete depths from Nansen bottle or STD data. Numerically this is quite similar to the type of procedure that is routinely done to calculate specific volume anomaly and dynamic heights. Once this is done the two required definite integrals can also be calculated, i.e.,

$$\alpha_d = \int_d^0 \rho dz \qquad \Delta = \int_d^0 \alpha dz = \int_d^0 \int_z^0 \alpha(z')dz'dz = \int_d^0 (z-d)\rho(z)\, dz \quad .$$

These would also be integrated numerically. A physical interpretation of these terms is perhaps now more apparent. The first obviously relates to the pressure or mean density and the second represents the first moment of the density around the bottom depth. After the field of α_d and Δ is obtained, horizontal derivatives must be calculated at required grid spacings. This would once again require the development of some interpolation scheme. It should be noted that in deep water α_d and Δ may turn out to be very large numbers and that the gradients will be given as relatively small differences between these large numbers. This can lead to the loss of significant figures and subsequent numerical difficulties. For typical oceanic depths (> 4000 m) this can be a serious problem, and even the best quality data may need to be artificially smoothed before reasonable results can be obtained. In shallower water over the continental shelf and slope this accuracy problem tends to be less troublesome for two reasons. First of all the depth is much less, typically several hundred meters; and secondly, the density gradients are often greater than in the open ocean. It appears then that the classical method of calculating currents by dynamic heights and diagnostic models complement each other. In deep water, baroclinic compensation is likely to take place,and the assumption of a stationary reference level is not altogether improbable. This is just the point where inaccuracies in the integrated transport make it difficult to apply continuity constraints, and hence diagnostic modeling techniques are difficult to apply. Conversely, in shallower water baroclinic adjustment is not likely to take place and interactions between the flow and the bathymetry are significant. This then will make a reference level concept unworkable and leaves diagnostic modeling as a strong alternative since the numerical problems associated with the transport constraints will be reduced and this method can include the effects of complex topography.

Another potential source of error in the density field measurements must also be considered. These do not depend on numerical errors, but rather on sampling schemes that are typically used to collect the data. The dynamics of the model assume the flow to be geostrophic, hydrostatic,and steady. To the extent that these conditions are not met and are reflected in the data,the model will introduce erroneous results. Two obvious sources of potential error come quickly to mind. First of all, if there are strong internal tides in the region of interest these may well alias the density field. To test for this eventuality a local study that can resolve internal tides would seem prudent. For example, a few 24-hr series of STD casts at one location should indicate the typical magnitude of density perturbations introduced into the data. Secondly, a problem could develop if the density data is not truly synoptic. Typically a single ship could gather density field data in something like a week. If this is the case, a regional baroclinic current with time scales of a week or less would tend to distort the density field and errors in the calculated gradients may result. The most likely currents to alias the data in this manner are quasi-geostrophic shelf waves. To minimize this difficulty, the regional data should be collected as quickly as possible. In addition, long-term current records can be analyzed to estimate the potential magnitude of these errors.

Both of the problems mentioned above could be minimized by using density fields that represent the average of an ensemble collection of data. This would require the analysis of historical data, and, if enough is available to construct a reasonably smooth mean density field for whatever periods are of interest, would be an attractive option.

The final independent variable input needed for the model is the surface wind stress. For most oceanographic studies, wind data are notoriously poor. Under the best circumstances one might expect to have a few strategically located shore stations and moored buoys that report winds. In cases where this is not available it is possible to estimate the wind stress field using atmospheric pressure data (Aagaard, 1969, 1970). If pressure maps are not available on a synoptic basis it is often possible to use climatological data. In most cases the spatial resolution of the wind field is minimal and only the general characteristics of the wind forcing can be included in the model. Even at this minimal level the wind effect can significantly modify the flow patterns in coastal areas and important additions to the flow dynamics are represented by including them. For example, the barotropic setup along the coastlines caused by Ekman transport is included in the dynamics of the model, and the resulting coastal currents will be simulated.

It now remains to specify the boundary conditions necessary to uniquely solve (41). The form is clearly elliptic, so there are several optional sets of boundary conditions that would close the problem. Because the dependent variable is surface elevation, Dirichlet type boundary conditions will be relatively easy to interpret (Sneddon, 1957), i.e., specification of the surface elevation around the perimeter of the domain will be sufficient to completely determine the solution. Since the sea surface along the boundary gives the normal component of the barotropic mode, this is equivalent to specifying the distribution of barotropic flow into the region. This is represented schematically in Figure 2.

In a mathematical sense the problem is now formally closed, but to make much use of the model a bit more detail on the implications of various boundary value specifications seems in order. In addition, some thought should be given to how the actual boundary values might be obtained in a consistent manner.

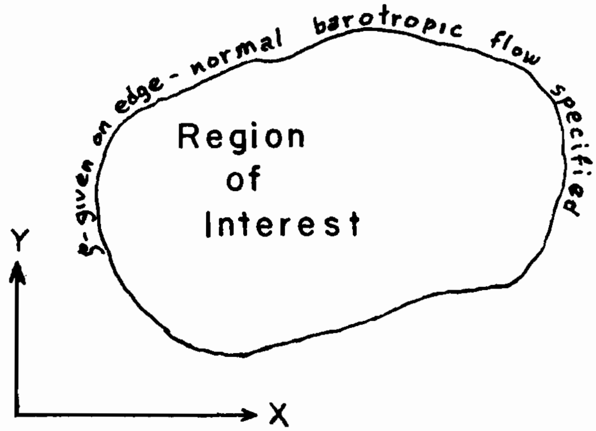

Figure 2. Dirichlet type boundary conditions for region of interest.

To begin with, we may note that the barotropic transport across a unit length of boundary (where s is distance measured along a prescribed line) is given by:

$$T_n = -\frac{g\rho o d}{f}\frac{\partial \zeta}{\partial s},$$

and across any section of the boundary:

$$\int T_n = -\frac{g\rho o d}{f}\int \frac{\partial \zeta}{\partial s}\, ds .$$

If the depth is constant along a closed path of integration (i.e., the boundary), then the net barotropic transport into the region is identically zero (unless the region is large enough so that variations in f are significant). The same will be true for the total geostrophic transport, and thus for the baroclinic mode and barotropic mode, individually. This can also easily be seen from (30), assuming d and f are constants, meaning that for flat bottom regions or closed bathymetric contours the geostrophic flow cannot directly contribute to the net advection of either mass or vorticity into the domain. For this particularly simple case any wind-driven convergence in the surface Ekman layer must be balanced by divergence in the bottom layer. Assuming Ekman dynamics for the lower layer, any divergence will be proportional to the negative curl of the geostrophic flow. From this, the dependence of the solution on the forcing and boundary conditions can easily be seen, i.e., the rotational component of the flow is determined by the wind stress distribution. In addition, an incompressible component is added to satisfy the given boundary values. It is also clear that the problem will become degenerate as the frictional coefficient for the bottom layer approaches zero (equation 41). Physically the geostrophic vorticity will approach infinity and mathematically the higher order terms drop out and the equation goes from an elliptic form to a first order partial differential equation. This important special case will be considered in more detail later.

We may now consider a more general situation where there is a variable depth region to be investigated. Once again,if the surface elevation is given around the perimeter,the inflow (outflow) of the barotropic mode is determined. In this case it is possible, in fact likely, that the barotropic mode will give some net advection of mass or vorticity into the domain. In general, the only case where the barotropic mode does not add any net contribution to either the mass or vorticity is where water enters and exits the model region on the same bathymetric contour. This of course satisfies the conditions for conservation of potential vorticity in a homogeneous geostrophic fluid (a limited subset of the physics included in the model formulation).

One of the major difficulties in formulating the model boundary conditions should now be clear. The divergence in the surface Ekman layer and the baroclinic mode are given by the diagnostic input data (i.e., wind stress and density field). In addition, the boundary conditions specify the net divergence of the barotropic mode. If these do not sum to zero, the only alternative available to the model is to satisfy continuity with the secondary flow in the bottom frictional layer. If the imposed boundary conditions are accurate then there is no difficulty. If, on the other hand, the balance is not correct, extraneous circulation in the barotropic mode around the boundary will result. In particular, a clockwise boundary circulation will give divergence in the bottom layer and a counterclockwise boundary current will give a net convergence in the bottom layer (fig. 3). In addition, we can see that the secondary flow required for the continuity balance is coupled to the geostrophic currents with the bottom frictional coefficient. If this is reduced, the couple becomes weaker, and much stronger boundary currents are required. Much the same kinds of arguments can be made about the vorticity balance within the model. This type of behavior is not uncommon in the solutions to differential equations where the highest order terms are multiplied by a small parameter, and boundary layers can be anticipated (Cole, 1968). This leads to the troubling conclusion that for small values of bottom friction coefficient the model may be very sensitive to the boundary conditions.

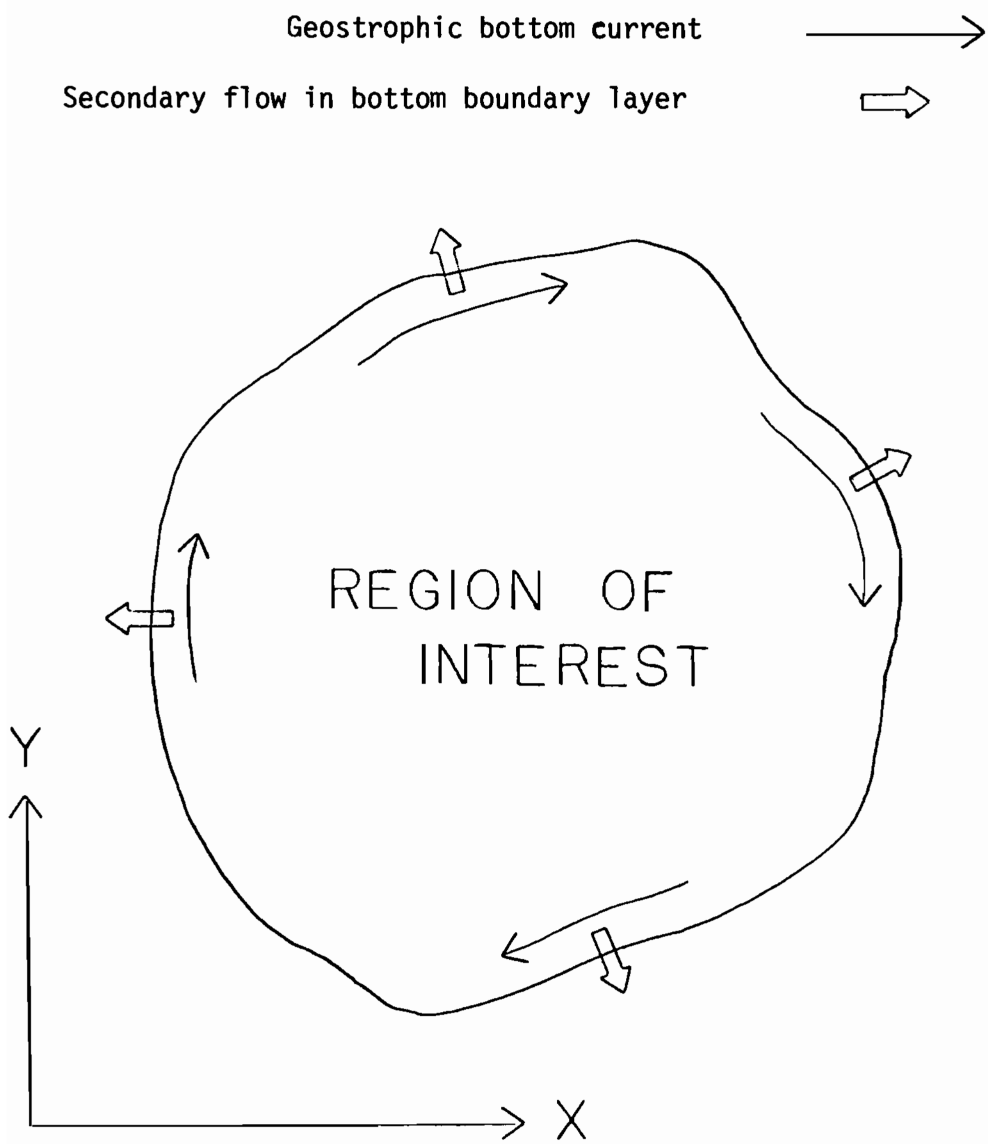

Figure 3. Net divergence within the area due to transport in secondary bottom boundary currents.

It should perhaps be pointed out that many partial basin models, both prognostic and time dependent, allow a steady geostrophic solution and are subject to these same potential difficulties. In some cases, this does not seem to be clearly understood, and insufficient care in the specification of boundary conditions has made model results difficult to interpret.

In other model studies, the effect of boundary currents is reduced by making the solution area larger than the actual area of interest. In this way it is hoped that errors near the model boundaries will not seriously affect the solution in the area of interest. This appears to be fairly successful in some cases but adds considerable complexity to the problem and is not consistent with the present goal of coming up with an operational easy-to-apply regional model.

To approach the problem of boundary conditions from a different and slightly more optimistic point of view, we can consider the equations without bottom friction and look at the implications of how the flow will be modified by introducing the small secondary flow associated with weak bottom friction.

In many respects this is in line with intuitive ideas of ocean currents. In general, it seems that bottom friction is not a dominant factor in the dynamics even in shallow water currents, and in deeper water the effects of bottom friction are essentially negligible.

Proceeding along this line, we may rewrite (36) assuming that γ is zero:

$$
\begin{aligned}
&- g\rho o J(\zeta,d) - gJ(\alpha(d),\, d) - \frac{g\rho o d\beta}{f}\frac{\partial\zeta}{\partial x} + \frac{g\beta}{f}\left(\frac{\partial\Delta}{\partial x} + \alpha(d)\frac{\partial d}{\partial x}\right) \\
&- \vec{k}\cdot\nabla x\vec{\tau}(\zeta) - \frac{\beta}{f}\tau x(\zeta) = 0 \qquad\qquad . \qquad (47)
\end{aligned}
$$

Multiplying this equation by f/d^2 and collecting terms gives

$$g\rho o J(\zeta, \frac{f}{d}) = \frac{gf}{d2}J(\alpha(d), d) - \frac{g\beta}{d2}(\frac{\partial\Delta}{\partial x} + \alpha(d)\frac{\partial d}{\partial x}) + \frac{f}{d2}\vec{k}\cdot\nabla x\vec{\tau}(\zeta) + \frac{\beta}{d2}\vec{\tau}x(\zeta) \quad . \tag{48}$$

This is now a first-order partial differential equation, and the appropriate boundary conditions are quite different from those required for (41). To get some indication of what these are and how they will determine the solution, we may consider the simplified case of a barotropic fluid with no surface wind. Under these assumptions the density is a function of z only, i.e.,

$$\rho = \rho\ (z) \ .$$

This results in the first two terms on the right-hand side of (48) being zero. With no stress the last two terms on the right-hand side are also zero, leaving:

$$J(\zeta, \frac{f}{d}) = o \quad . \tag{49}$$

Physically the problem has been reduced to the familiar conservation of potential vorticity where the flow follows f/d contours. In a mathematical sense the most general solution to equation (49) is

$$\zeta = w\ (\frac{f}{d}) \tag{50}$$

where $w(\frac{f}{d})$ is an arbitrary function (Courant and Hilbert, 1962). Clearly, then, the solution for the entire domain is known once w is determined. To do this, the value of ζ must be given along any line that runs monotonically from the lowest value of f/d in the region to the highest (fig. 4). Even in the baroclinic case, the f/d contours will represent characteristics for the differential equation, which means that along any of the f/d contours the partial differential equation can be written as an ordinary differential equation. To clarify this a bit, we may rewrite the Jacobian in a local orthogonal right-handed coordinate system where the n-axis is normal to f/d contour and the s-axis is along the contour (fig. 5). In this form, (48) becomes

$$\frac{\partial \zeta}{\partial n} \frac{\partial}{\partial s} \left(\frac{f}{d}\right) - \frac{\partial \zeta}{\partial s} \frac{\partial}{\partial n} \left(\frac{f}{d}\right) = RHS \tag{51}$$

where RHS just represents the right-hand side of (48) divided by $g\rho_0$. Next we see that the choice of coordinate systems makes the first term on the left-hand side equal to zero, and we are left with an ordinary differential equation:

$$\frac{d\zeta}{ds} = - \frac{RHS}{\frac{\partial}{\partial n}\left[\frac{f}{d}\right]} \tag{52}$$

From a numerical point of view this is now a straightforward problem. The right-hand side of (52) is given in terms of known quantities. Starting from the known point (boundary condition) on each f/d contour the equation can be integrated along the contour in either direction. In most cases a simple desk-top computer could handle the problem after some initial data analysis. Qualitatively the model reduces to about the same level of difficulty as the problem of calculating geostrophic currents using dynamic heights.

From the form of (52) we can also clearly see the physical significance of the right-hand side terms in (48). They represent the components of cross contour flow associated with the baroclinic (first two

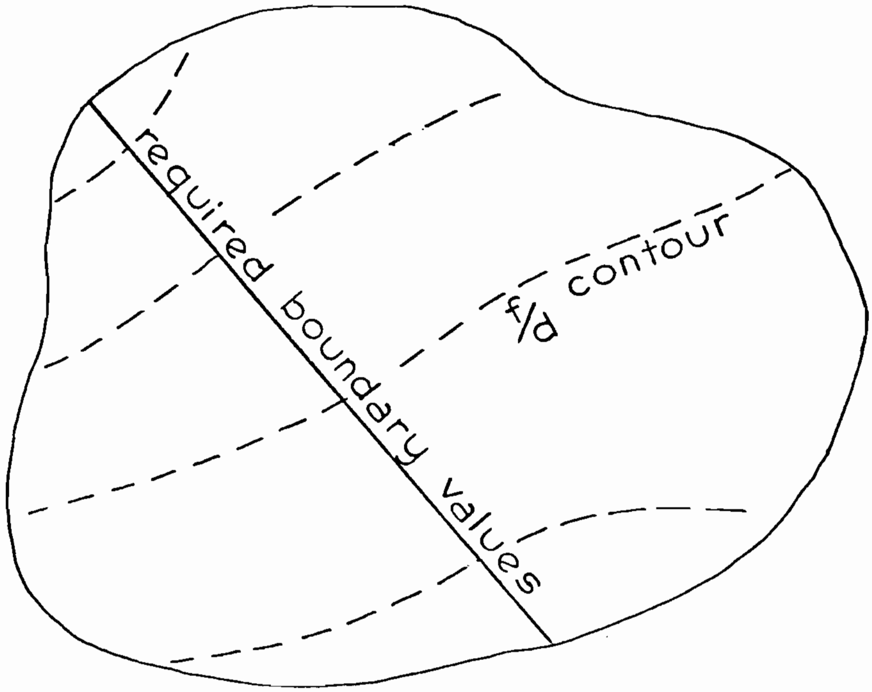

Figure 4. Required boundary conditions and characteristics for formulation without bottom friction.

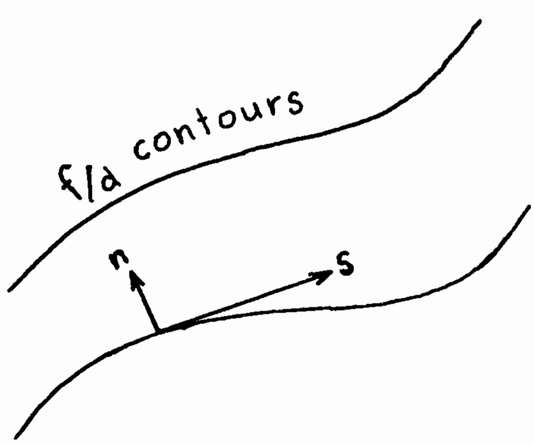

Figure 5. Orthogonal coordinate system related to f/d contours.

terms) and Ekman wind-driven modes (last two terms). The change of ζ along the contour physically represents the barotropic component of cross-contour flow, and to conserve the potential vorticity in the water column this barotropic stretching effect must just balance the contributions from the baroclinic and Ekman flow. In this context it is also obvious what happens in a region where the f/d gradient vanishes. The required stretching from the barotropic mode is impossible and a vorticity balance cannot be obtained. As was mentioned before, in a uniform depth region the geostrophic modes are non-divergent and a non-geostrophic component to the flow is required for solution, unless the curl of the wind stress is zero and it has no meridional component. In the context of the present model this non-geostrophic component would be the bottom frictional layer.

We are now in a position to work back to the full model including the effects of weak bottom friction. Starting with wind and density data, for some region, the surface elevation along any line running from the shallow to deep extremes is required. These data can be obtained in a variety of ways. In deeper regions a level of no motion assumption and dynamic height calculations might provide reasonable estimates. In shallower segments a line of moored current meter arrays across the f/d gradient should yield the most useful data, particularly if simultaneous measurements of bottom pressure are also obtained. If all else fails, a judicious guess could provide a point of departure for the careful numerical exploration of the implications of a given wind and density field.

Once the input data is accumulated the first phase of the model would solve for the simplified physics represented in equation (52).

From this the values of ζ around the entire boundary could be obtained, and the second phase of the model would use these with the full information given in equation (41). Obviously the potential for problems associated with the boundary conditions still exists, but for small values of bottom friction this method should prove a useful point of departure. It is likely that careful testing and qualitative iterations on the boundary values will give useful insights into model results.

As a final point, the stream function representation of the model given in equation (40) has the same mathematical form as equation (48) and can be solved in much the same way. These two equations represent identical physical processes and the choice of which one to use is essentially a matter of emphasis.

The boundary conditions required along a closed boundary are obvious when the transport stream function is the dependent variable. A no-flux condition requires a constant value for the stream function along the boundary. With the surface elevation as the dependent variable, the formulation is not quite so straightforward. For that case the change in surface elevation along the boundary must give a barotropic current that just balances the contribution across the boundary from the baroclinic and Ekman modes. These components to the flow must be calculated from the local values of input data for the model, and once again, if bottom friction is included boundary currents may be expected if net transport conditions are not met.

4. NON-DIMENSIONAL FORMULATION OF THE MODEL EQUATIONS

The actual solution of the model equations and their application to a geophysical situation will be greatly simplified if they are written in a consistently scaled non-dimensional form. This will also make it possible to come up with at least order-of-magnitude estimates for the relative significance of various terms and in some cases to suggest alternate formulations.

As a first step the pressure distribution and its gradient will be non-dimensionalized. This can be done by defining the following non-dimensionalized (primed) variables and constant dimensional scaling factors:

$$X = L(x') \ , \tag{53}$$

$$P = \rho o g z + \rho o f o U L \ \ (P') \ , \tag{54}$$

$$\rho = \rho o + \varepsilon(\rho') \ , \tag{55}$$

where clearly the pressure is divided into a component which is hydrostatic with respect to the typical density and a variable part that depends somehow on the density variations. It is likewise assumed that the density is made up of an average part and a fluctuating part. Also considering typical ocean situations, $\varepsilon \ O(10^{-3} g/cm^3) \ll \rho o \ O(1 g/cm^3)$, it should be obvious that when considering variations in α and Δ, only the $\varepsilon(\rho')$ component will contribute and that this partitioning will reduce the loss of significant figures that could result if the entire density field were scaled with a single constant. The depth variations will also be scaled in two parts, reflecting the basic partitioning of the pressure gradient terms into barotropic and baroclinic parts. For elevation of the sea surface the following non-dimensional variable is used:

$$\zeta = \frac{foUL}{g}(\zeta') \quad , \tag{56}$$

and for depths within the water column,

$$z = H(z') \quad . \tag{57}$$

These definitions may now be substituted into (11) and, applying Liebniz's rule, yield

$$\rho o f o U(\frac{\partial P'}{\partial x'}) = [\rho o + \varepsilon(\rho')] f o U(\frac{\partial \zeta'}{\partial x'}) + \varepsilon f o U \int_{0}^{\zeta'} \frac{\partial \rho'}{\partial x'} dz'$$
$$+ \frac{g\varepsilon H}{L} \int_{z'}^{0} \rho' dz' \quad ,$$

and dividing through by $\rho o f o U$ gives

$$\frac{\partial P'}{\partial x'} = [1 + \frac{\varepsilon}{\rho o}(\rho')] \frac{\partial \zeta'}{\partial x'} + (\frac{\varepsilon}{\rho o}) \int_{0}^{\zeta'} \frac{\partial \rho'}{\partial x'} dz'$$
$$+ (\frac{g\varepsilon H}{\rho o f o U L}) \frac{\partial}{\partial x'} \int_{z'}^{0} \rho' dz' \tag{58}$$

Since all of the non-dimensional terms are now scaled to be $o(1)$ and $\varepsilon/\rho o$ $o(10^{-3})$, it is clear that the $\varepsilon/\rho o$ terms can be neglected; thus

$$\frac{\partial P'}{\partial x'} = \frac{\partial \zeta'}{\partial x'} + (\frac{g\varepsilon H}{\rho o f o U L}) \frac{\partial \alpha'}{\partial x'} \quad , \tag{59}$$

where (60)

$$\alpha'(z') = \int_{z'}^{0} \rho' dz' \quad .$$

Equation (59) is the non-dimensional form of (13), and the assumptions that were made can now clearly be identified as dropping terms that are typically three orders of magnitude smaller than the dominant terms. The non-dimensional constant coefficient (which from now on will simply be referred to as N_1) obviously represents the ràtio of the baroclinic component of the pressure gradient to the barotropic component of the pressure gradient. For typical continental shelf situations we can estimate the magnitude of this coefficient as follows:

$g = 10^{-3}\ cm/sec^{2}$ $\qquad$ $L = 10^{7}\ cm$

$\varepsilon = 10^{-3}\ g/cm^{3}$ $\qquad$ $H = 2 \times 10^{4}\ cm$

$\rho o = 1\ g/cm^{3}$ $\qquad$ $U = 20\ cm/sec$

$fo = 10^{-4} sec^{-1}$

which gives

$$N_1 = \frac{g\varepsilon H}{\rho o\, foUL} \approx 1 \quad . \tag{61}$$

Equation (15) may now also be written in non-dimensional form as:

$$\frac{\partial P'}{\partial y'} + \frac{\partial \zeta'}{\partial y'} + N_1 \frac{\partial \alpha'}{\partial y'} \tag{62}$$

To non-dimensionalize the vorticity balance, the following additional non-dimensional variables are defined:

$$f - fo + \beta L\ (y') \ , \tag{63}$$

$$\tau = \rho o foUH(\tau') \ , \tag{64}$$

$$T = \rho oUH(T') \ . \tag{65}$$

It should perhaps be pointed out that the non-dimensionalization used for the stress (eq. 64) results in a non-dimensional numerical value that is typically not order one, but somewhat smaller. An alternate approach would be to scale the stress with the Ekman depth instead of the total depth. This would bring the numerical value back in line, but leaves an Ekman number dependence in the final form of the vorticity equation that is misleading. The curl of the wind stress term (eq. 36) physically represents the divergence or convergence in the surface Ekman layer and this of course is independent of Ekman number, providing only that it is not zero. To have this term multiplied by some power of the Ekman number implies a dependence on the eddy coefficient that simply is not there. To avoid this potential difficulty it is preferable to deal with stress values that are numerically smaller.

Substituting these and the previously defined non-dimensional variables into (27) and then dividing by $\rho o f o \; UH/L$ yields

$$- N_3 T y' = - J(\zeta', d') - N_1 J(\alpha'(d'), d') + N_2 \nabla^2 \zeta' + N_1 N_2 \nabla^2 \alpha'(d') - \vec{k} \cdot \nabla x \tau'(\zeta') , \qquad (66)$$

where two more non-dimensional coefficients have been defined as

$$N_2 = \frac{\gamma}{\rho o H} \text{ and } N_3 = \frac{\beta L}{fo} . \qquad (67)$$

Once again it is possible to estimate the order of magnitude of these coefficients. For a bottom stress of a dyne per centimeter squared, equation (22) can be used to show

$$\gamma = \frac{\tau}{Uf} = 5 \times 10^{-2} \; g/cm^2 .$$

And this then suggests that

$$N_2 = \frac{\gamma}{\rho o H} \approx 2 \times 10^{-2} . \qquad (68)$$

To evaluate N_3, we note that

$$\beta = \frac{\partial f}{\partial y} = \frac{f}{R \tan \phi} \quad ,$$

where ϕ is the latitude and R is the radius of the Earth. Substituting this into the expression for N_3 and evaluating it for a latitude of 60° gives

$$N_3 = \frac{L}{R \tan \phi} \approx 2 \times 10^{-2} \quad . \tag{69}$$

It is now necessary to non-dimensionalize (34) and substitute the results into (66). This can be carried out by the straightforward substitution of the defined non-dimensional variables and gives

$$Ty' = - \tau x' - d'\frac{\partial \zeta'}{\partial x} + N_1 \left(\frac{\partial \Delta'}{\partial x'} + \alpha'(d')\frac{\partial d'}{\partial x'}\right) - N_2\left(\frac{\partial \zeta'}{\partial x'} + \frac{\partial \zeta'}{\partial y'}\right)$$
$$- N_1 N_2 \left(\frac{\partial \alpha'(d')}{\partial x'} + \frac{\partial \alpha'(d')}{\partial y'}\right) \quad . \tag{70}$$

Substituting this into (66) gives the final non-dimensional form of the model equation

$$N_2\nabla^2\zeta' + N_1N_2 \nabla^2\alpha'(d') - J(\zeta',d') - N_1J(\alpha'(d'), d')$$
$$- \vec{k}'\cdot\nabla x\vec{\tau}'(\zeta') - N_3\tau x' - N_3d'\frac{\partial \zeta'}{\partial x'} + N_1N_3\left(\frac{\partial \Delta'}{\partial x'} + \alpha'(d')\frac{\partial d'}{\partial x'}\right)$$
$$- N_2N_3\left(\frac{\partial \zeta'}{\partial x'} + \frac{\partial \zeta'}{\partial y'}\right) - N_1N_2N_3\left(\frac{\partial \alpha'(d')}{\partial x'} + \frac{\partial \alpha'(d')}{\partial y'}\right) = 0 \quad . \tag{71}$$

In this form the equation can be easily attacked using numerical techniques, and the consequences of the geophysical scaling are clearly evident.

5. APPLICATION OF MODEL EQUATIONS

Thus far, the model has been derived and its demands for boundary conditions and input data have been considered. In this section the proposed application of the model to a real geophysical situation will be outlined. The actual detailed results of that application will be the subject of the next technical report in this series.

The region first to be studied with this model is a portion of the Continental Shelf in the Gulf of Alaska. Within this region, roughly 80 by 150 miles, a series of oceanographic stations have been laid out in an inner and outer array. The inner array extends well into deep water (fig. 6). Obviously the inner array was designed to attempt some minimal resolution of the complex bathymetry associated with the shelf break and the outer array is anticipated to supply offshore boundary conditions. STD casts will be made at each of these stations approximately six times within a year. For each of these, α and Δ will be calculated and used as input into the model. Wind data will be compiled from daily weather maps prepared by the National Weather Service regional office in Anchorage. Monthly values will be summarized and a representative stress field calculated. All input data will be reduced to values at station locations. Each of the variable fields will be interpolated assuming continuous linear variations over the triangular elements whose vertices are the station location (fig. 7).

The first solution will use the formulation given in (71), and for the first model tests N_3 will be assumed to be zero. The necessary boundary conditions will initially be estimated from assumed transports along a line roughly perpendicular to the coastline. Later phases of the study will use data from moored current meter arrays and pressure gauge records within the study area (fig. 8) for boundary conditions. The effects of various boundary formulations will be carefully investigated using some range of values for the parameter N_2.

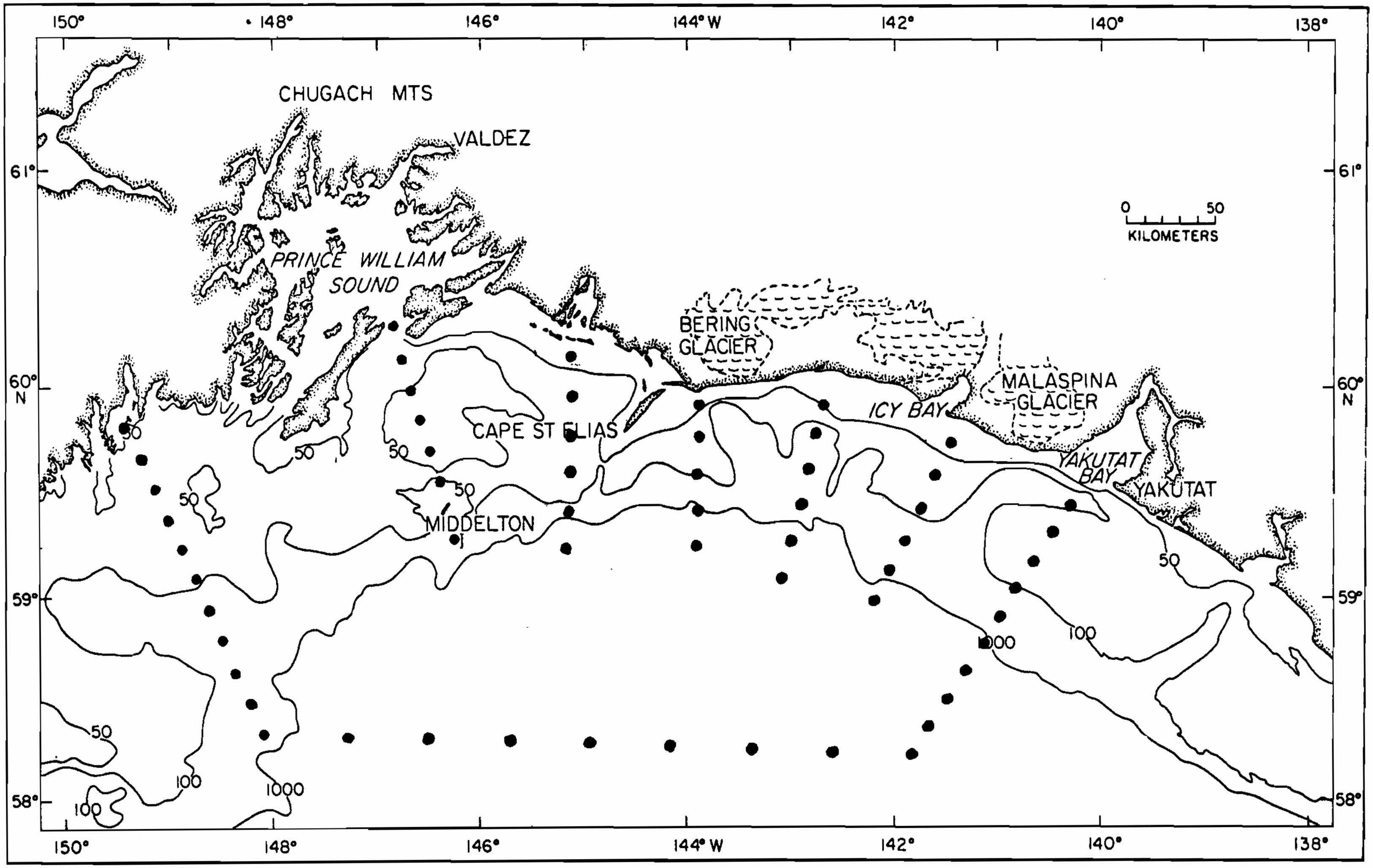

Figure 6. Test study area in the Gulf of Alaska.

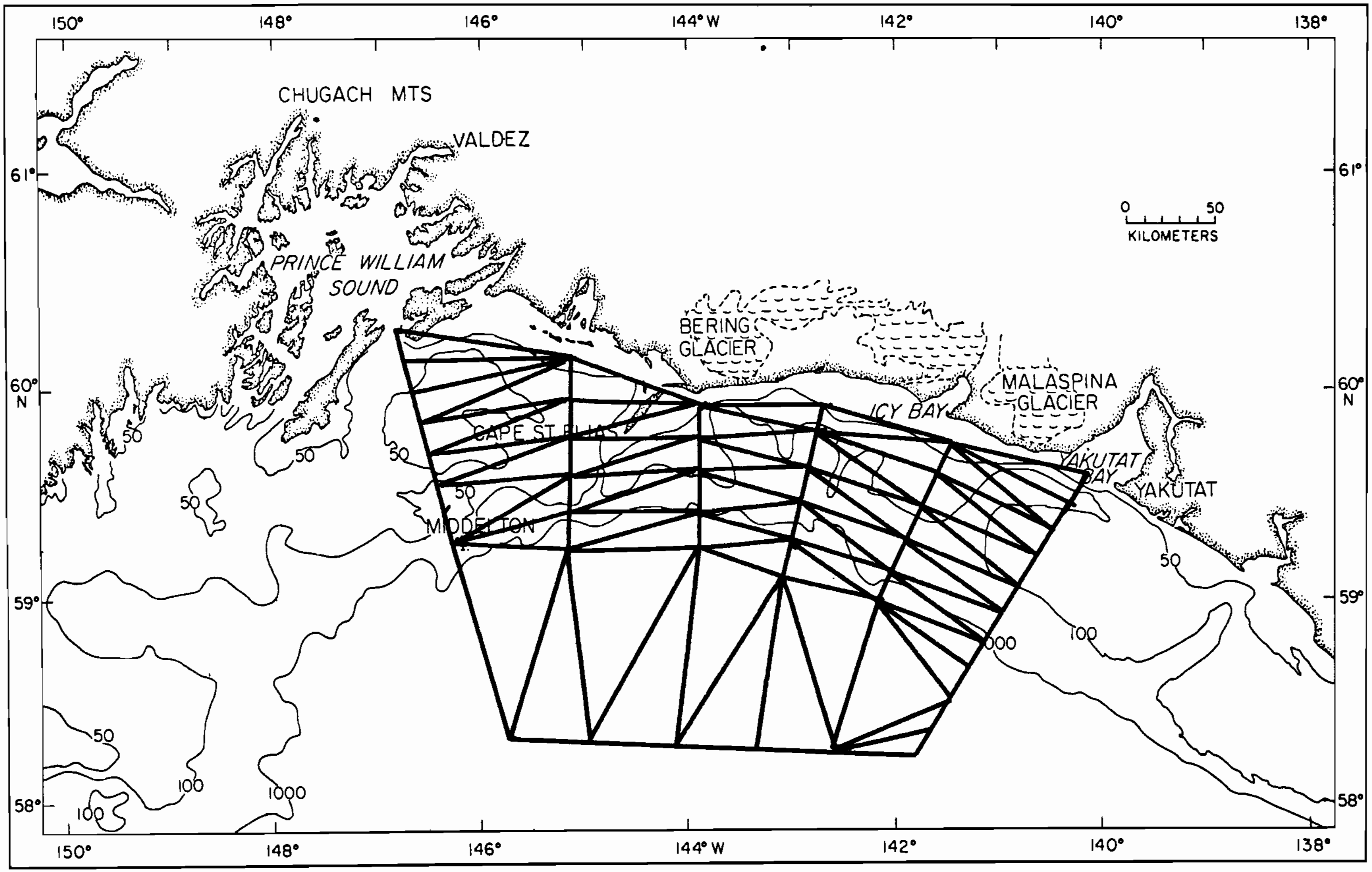

Figure 7. Triangular elements used for interpolation of field data.

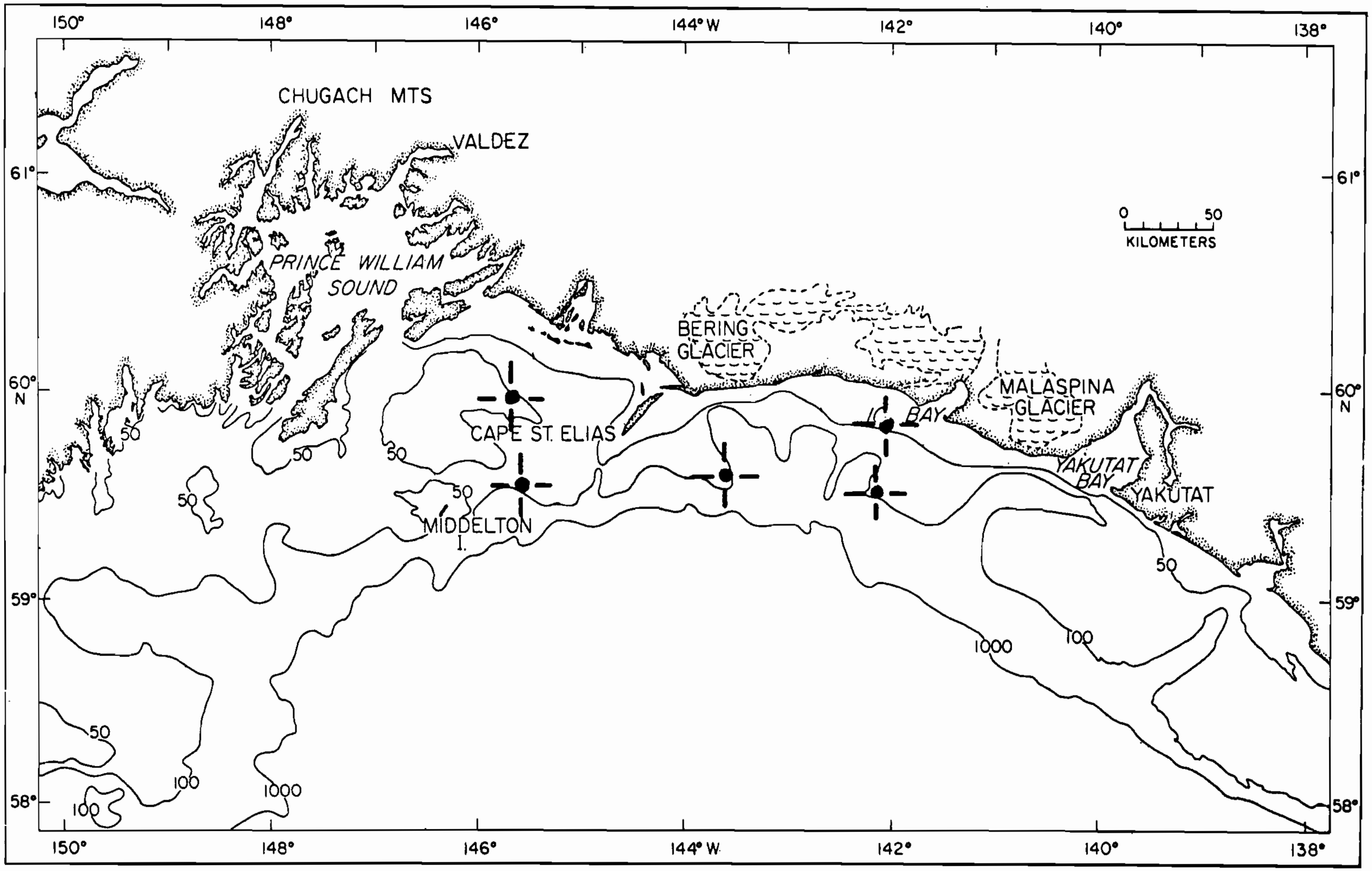

Figure 8. Positions of moored current meter arrays for use in the formulation of boundary conditions.

6. ACKNOWLEDGMENTS

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