# A Scaling Theory for Horizontally Homogeneous, Baroclinically Unstable Flow on a Beta Plane

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## ABSTRACT

The scaling argument developed by the authors in a previous work for eddy amplitudes and fluxes in a horizontally homogeneous, two-layer model on an $f$ plane is extended to a $\beta$ plane. In terms of the nondimensional number $\xi = U/(\beta\lambda^2)$, where $\lambda$ is the deformation radius and $U$ is the mean thermal wind, the result for the rms eddy velocity $V$, the characteristic wavenumber of the energy-containing eddies and of the eddy-driven jets $k_j$, and the magnitude of the eddy diffusivity for potential vorticity $D$, in the limit $\xi \gg 1$, are as follows:

$$V/U \approx \xi, \quad k_j\lambda \approx \xi^{-1}, \quad D/(U\lambda) \approx \xi^2.$$

Numerical simulations provide qualitative support for this scaling but suggest that it underestimates the sensitivity of these eddy statistics to the value of $\xi$. A generalization that is applicable to continuous stratification is suggested that leads to the estimates

$$V \approx (\beta T^2)^{-1}, \quad k_j \approx \beta T, \quad D \approx (\beta^2 T^3)^{-1},$$

where $T$ is a timescale determined by the environment; in particular, it equals $\lambda U^{-1}$ in the two-layer model and $N(f\partial_z U)^{-1}$ in a continuous flow with uniform shear and stratification. This same scaling has also been suggested as relevant to a continuously stratified fluid in the opposite limit, $\xi \ll 1$. Therefore, the authors suggest that it may be of general relevance in planetary atmospheres and in the oceans.

## 1. Introduction

An understanding of the mechanisms that control the amplitudes of baroclinic eddies and their transport properties is basic to developing theories for the general circulation of planetary atmospheres and oceans and for incorporating the effects of mesoscale eddies in large-scale ocean models. Different classes of idealized models are needed to improve our understanding of various aspects of this problem. We focus on one such class—models of horizontally homogeneous quasigeostrophic (QG) turbulence, with imposed large-scale potential vorticity gradients. These models are useful in bridging the gap between studies of homogeneous two-dimensional turbulence and studies of baroclinic eddy fluxes in inhomogeneous flows of physical interest. Analyses of these baroclinic homogeneous models, and work on closely related problems,

can be found in Rhines (1977), Salmon (1978, 1980), Haidvogel and Held (1980), Vallis (1983), Hoyer and Sadourny (1982), Panetta (1993), and Held and O'Brien (1992).

Recently, Larichev and Held (1995, LH hereafter) have considered the statistically steady state of a two-layer model on an $f$ plane, with an imposed environmental vertical shear, or interface slope, that results in equal and opposite potential vorticity gradients in the two layers. Following the lead of Rhines, Salmon, and Hoyer and Sadourny, they describe a picture of the barotropic and baroclinic energy cascades that leads to simple scaling arguments for the energy level and potential vorticity fluxes in this baroclinically unstable system, given the scale to which the inverse energy cascade extends.

On a $\beta$ plane, the barotropic inverse energy cascade is halted when the characteristic overturning time for the energy-containing eddies becomes comparable to the inverse of the Rossby wave frequency (Rhines 1975), at which scale the energy is channeled into zonal jets. In a baroclinically unstable flow, these jets also organize the instability, so that the eddies form a "storm track" on each jet, as described by Williams

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(1979) and Panetta (1993). By combining the scaling argument of LH with the Rhines scale at which the inverse cascade halts, we obtain a qualitative theory for the eddy amplitudes and fluxes on a $\beta$ plane. We discuss the results for the two-layer model, describe some numerical solutions that provide partial support for the theory but also point to some deficiencies, and then indicate how the argument generalizes to continuous stratification and arbitrary vertical structure in the mean flow.

## 2. The two-layer model

We consider a two-layer QG model with equal-depth layers when at rest. The barotropic and baroclinic streamfunctions are denoted by $\psi \equiv (\psi_1 + \psi_2)/2$ and $\tau \equiv (\psi_1 - \psi_2)/2$, where the subscripts 1 and 2 refer to the upper and lower layers, respectively. The corresponding barotropic and baroclinic velocities are referred to as $(u_\psi, v_\psi)$ and $(u_\tau, v_\tau)$. The mean flows in the two layers $U_i = $ const are specified to provide a vertical shear of

$$U \equiv (U_1 - U_2)/2 > 0, \tag{1}$$

which results in the mean potential vorticity gradients $\beta \pm U\lambda^{-2} = U\lambda^{-2}(\xi^{-1} \pm 1)$, where $\lambda$ is the internal radius of deformation and where

$$\xi \equiv U/(\beta\lambda^2). \tag{2}$$

The plus sign refers to the upper layer, and the minus sign to the lower layer. The lower-layer potential vorticity (PV) gradient is reversed if the shear is large enough that $\xi > 1$, which is the criterion for instability in an inviscid flow.

The argument in LH for the $f$ plane can be reformulated as follows. Start by assuming that the kinetic energy of the flow is predominately barotropic on scales much larger than the internal radius of deformation $\lambda$. On these scales the barotropic mode evolves as in a two-dimensional flow, cascading energy to larger scales. Assume that this inverse energy cascade halts on average at the wavenumber $k_0$. Given the Kolmogorov energy spectrum of $k^{-5/3}$, or any spectrum steeper than $k^{-1}$, the bulk of the energy will be contained on the scale $k_0$. The *baroclinic* potential vorticity is advected by this nearly barotropic flow on large scales. Since it does not induce a significant part of the flow by which it is advected, the baroclinic potential vorticity will behave as a passive tracer and will be mixed downgradient by the "turbulent diffusion" engendered by the barotropic flow. This mixing will be dominated by the largest scales in the flow. Since the magnitude of the ambient potential vorticity gradient is $U\lambda^{-2}$, the typical size of the eddy baroclinic potential vorticity will be

$$q' \approx k_0^{-1}(U\lambda^{-2}) \tag{3}$$

so that

$$\overline{v'q'} \approx -D(U\lambda^{-2}) \tag{4}$$

with

$$D \approx Vk_0^{-1}, \tag{5}$$

where $V$ is the rms *barotropic* velocity and $D$ is the diffusivity. Thus, (3) is equivalent to the statement that the baroclinic velocities on the energy-containing scale $(v_\tau' \approx k_0\lambda^2 q')$ are of the order of $U$.

The baroclinic potential vorticity, acting as a passive scalar, will cascade to smaller scales. On scales larger than $\lambda$, the variance of this potential vorticity is dominated by thickness fluctuations and is therefore proportional to the potential energy in the baroclinic mode. The rate of eddy energy production per unit mass in a horizontally homogeneous two-layer model is

$$\begin{aligned}\epsilon &= -(U_1\overline{v_1'q_1'} + U_2\overline{v_2'q_2'})/2 \\ &= U\overline{v_\psi'\tau'}/\lambda^2 \approx Vk_0^{-1}(U/\lambda)^2.\end{aligned} \tag{6}$$

In this homogeneous model, the potential vorticity fluxes in the two layers are equal and opposite and proportional to the thickness flux.

The baroclinic eddy energy is produced on the largest scale $k_0^{-1}$, since it is proportional to the potential vorticity flux. It then cascades down to the deformation radius. Energy cannot cascade farther since the layers decouple at smaller scales, leaving 2D flow in which energy cannot proceed downscale; instead, it is converted to barotropic energy, in which form it cascades back upscale, eventually to be dissipated.

The barotropic energy level is determined by the requirement that at equilibrium the rate at which energy flow to larger scales in the barotropic mode is equal to the baroclinic energy production. If the energy cascade extends to the scale $k_0$ and the rms barotropic velocity is $V$, then by dimensional analysis

$$\epsilon \approx V^3 k_0, \tag{7}$$

an expression familiar from 3D turbulence (e.g., Tennekes and Lumley 1972). It is equivalent to the assumption utilized by LH that an inertial range exists. From (6) and (7), one finds

$$V \approx (k_0\lambda)^{-1}U. \tag{8}$$

The barotropic velocities are larger than the mean shear by the factor $(k_0\lambda)^{-1}$. We have assumed that this factor is much greater than unity so as to have a significant inverse energy cascade. Barotropic velocities are also larger than baroclinic velocities on the scale $k_0$ by the same factor, so the dominance of advection by the barotropic flow on large scales is assured. The corresponding diffusivity is of the magnitude

$$D \approx (k_0\lambda)^{-1}Uk_0^{-1}. \tag{9}$$

We also have the result that the eddy available potential energy is proportional to the barotropic kinetic energy:

$$\frac{\tau^2}{\lambda^2} \approx \frac{U^2}{(k_0\lambda)^2} \approx v_\psi^2 + u_\psi^2. \qquad (10)$$

The argument above is equivalent to that provided by LH.

In the basic $f$-plane numerical experiment analyzed in LH, $k_0^{-1}$ is determined by the size of the domain. On a $\beta$ plane, we expect the inverse cascade to stop at the Rhines scale, at which rms barotropic velocities $V$ are comparable to Rossby wave phase speeds:

$$k_0^{-2} = V/\beta. \qquad (11)$$

We ignore the anisotropy of the dispersion relation in this scaling. By combining (7) and (11), one obtains the expression for $k_0$ in terms of energy cascade rate discussed by Pelinovskiy (1978) or Vallis and Maltrud (1991): $k_0 \approx (\beta^3/\epsilon)^{1/5}$. (If the lower boundary is not flat, one should replace $\beta$ by the effective $\beta$ felt by the barotropic mode in the presence of large-scale bottom relief, $H|\nabla(f/H)|$). As the cascade is halted, the barotropic energy is organized into jets of this meridional scale. Panetta (1993) has shown that this relation accounts quite well for the number of jets produced in a model of homogeneous geostrophic turbulence in the presence of a mean shear, if $V$ is set equal to the square root of the total kinetic energy (see his Fig. 4).

Our assumption that $(k_0\lambda)^{-1} \gg 1$ translates into the statement that $\xi \gg 1$. We assume that the presence of $\beta$ does not modify the previous scalings when $\xi$ is large. Combining (8) and (11), we have

$$V/U \approx \xi \qquad (12)$$

and

$$k_0\lambda \approx \xi^{-1} \qquad (13)$$

with the consequence that

$$D \approx U\lambda\xi^2. \qquad (14)$$

Since $D$ is proportional to $U^3$, the potential vorticity flux is proportional to $U^4$, while the energy generation is proportional to $U^5$.

## 3. Some numerical results

These scaling relations can be tested against numerical experiments such as those described by Haidvogel and Held (1980) and Panetta (1992), for two-layer, QG homogeneous turbulence on a $\beta$ plane with imposed mean vertical shear. Figure 18 in Panetta (1993) shows that the scale with the maximum energy and the scale of maximum energy generation are both approximately linear in $\xi$, consistent with (13). [In the notation of Panetta, $\xi = (2\beta)^{-1}$.] The predicted eddy amplitudes and fluxes are not that well verified, however. The eddy energy should be proportional to $\xi^2$. The results tabulated by Panetta suggest that the energies and

fluxes are even more sensitive to $\xi$. To confirm this result, we have obtained numerical solutions to the same model as considered by Haidvogel and Held (1980) and Panetta (1993) but with 256 × 256 resolution that allows more scale separation between the radius of deformation and the energy-containing eddy scale.

The numerical model is identical to that used in the $f$-plane simulations of LH. We choose a value of the lower-layer Ekman damping, $\kappa = 0.16(U/\lambda)$, that is identical to that in LH and set the deformation radius so that $2\pi\lambda = L/50$, where $L$ is the size of the square doubly periodic domain. Small-scale damping is modeled with a $\nu\nabla^8$ diffusion operator, with $\nu = 0.08(U\lambda^7)$. Statistically steady states have been obtained for the values of $\xi$ shown in Table 1. Instability actually persists for values of $\xi$ slightly less than 1, due to dissipative destabilization, but these are not displayed in the table.

Figure 1 shows the horizontal spectra of the barotropic component of the meridional velocity and of the baroclinic eddy energy generation (or equivalently, of the potential vorticity flux in either layer, or of the buoyancy flux) for each of the statistically steady states obtained. The spectra are plotted as a function of total horizontal wavenumber, after averaging over angle in $k$ space. *The energy generation clearly moves to larger scales when the energy is allowed to cascade to larger scales.* Also shown by arrows is the Rhines scale (11), computed using the rms barotropic meridional velocity for $V$, for each value of $\xi$. This prediction of the scale is seen to be in reasonable agreement with the scale of the maxima in energy and energy generation, for large values of $\xi$ down to $\xi \approx 2$.

As Fig. 2 shows, the energies and the energy generation increase more rapidly with increasing $\xi$ than predicted by our scaling arguments. Instead of the expected $\xi^2$ power law, we obtain a dependence that is between $\xi^3$ and $\xi^4$. This result confirms the impression obtained from Panetta's results. We cannot rule out the possibility that even the largest values of $\xi$ are not yet large enough to attain the asymptotic regime predicted, but this appears to be unlikely since there is no trend

TABLE 1. Statistics from the numerical model as a function of $\xi$: $\epsilon_{diss}$ is the dissipation of energy in the barotropic mode by Ekman friction; $k_0$ is computed from (11) using the rms meridional velocity of the barotropic mode for $V$.

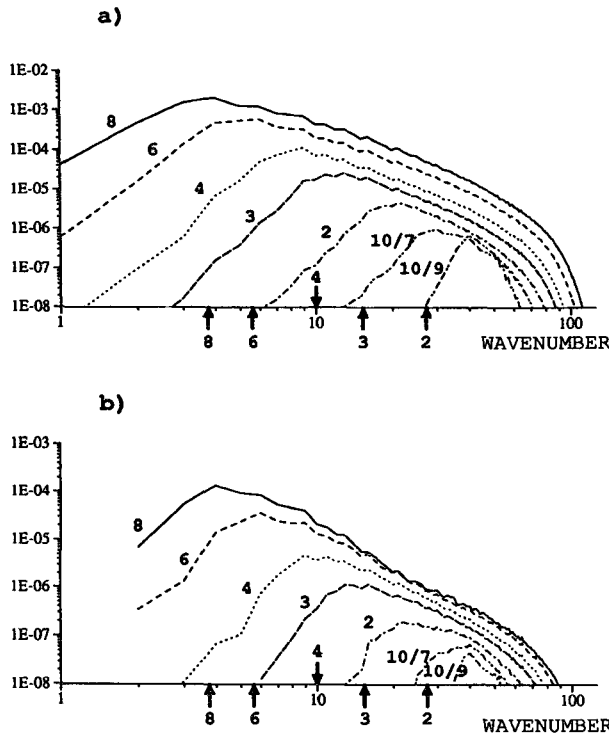| $\xi$ | $\overline{v_\psi'^2}/U^2$ | $\overline{v_\psi'^2}/\overline{u_\psi'^2}$ | $\epsilon/(U^3\lambda^{-1})$ | $\epsilon_{diss}/\epsilon$ | $(k_0\lambda)^{-1}$ |
|---|---|---|---|---|---|
| 8 | 488 | 0.92 | 104 | 0.74 | 13.3 |
| 6 | 175 | 0.88 | 38.4 | 0.70 | 8.91 |
| 4 | 38.6 | 0.76 | 8.70 | 0.69 | 5.00 |
| 3 | 12.7 | 0.62 | 3.01 | 0.69 | 3.27 |
| 2 | 2.85 | 0.48 | 0.717 | 0.69 | 1.84 |
| 10/7 | 0.77 | 0.40 | 0.211 | 0.64 | 1.12 |
| 10/9 | 0.28 | 0.33 | 0.085 | 0.58 | 0.77 |

FIG. 1. The horizontal spectra of (a) the barotropic meridional velocity and (b) the baroclinic eddy energy production rate, for different $\xi$ at equilibrium. The arrows indicate the Rhines scale computed from (11) using the rms barotropic meridional velocity for $V$.

in the results that suggests that the $\xi^2$ behavior is being approached. (The comparison with theory is also made more complicated by the fact that the size of the domain is beginning to make itself felt at the largest value of $\xi$ used.) This discrepancy is also seen in the $f$-plane results in LH, in which the scale of the energy generation was arbitrarily constrained.

To examine the sources of this discrepancy, it is convenient to define

$$k_R \equiv (\beta/V)^{1/2}, \tag{15}$$

$$k_M \equiv V\epsilon^{-1}(U/\lambda)^2, \tag{16}$$

and

$$k_B \equiv \epsilon/V^3. \tag{17}$$

We set $V$ equal to the rms meridional velocity in the barotropic mode. The preceeding scaling argument then follows from the assumption that $k_R \approx k_M \approx k_B$. The nondimensional length scales $(k\lambda)^{-1}$ corresponding to these three characteristic wavenumbers are plotted in Fig. 3 as a function of $\xi$. For clarity, the effective mixing scale $(k_M^{-1})$ and the scale related to the inverse cascade $(k_B^{-1})$ have been multiplied by constants so that they equal the Rhines scale $(k_R^{-1})$ at $\xi = 4$.

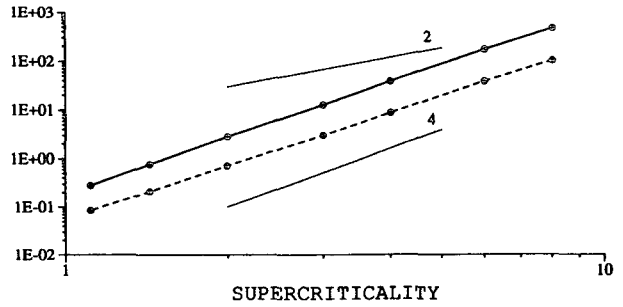At first glance, the assumption that these scales are proportional appears to be a useful first approximation,



FIG. 2. A log–log plot of $\overline{v_\psi'^2}$, twice the meridional barotropic energy (solid line), and $\epsilon$, the baroclinic eddy energy production rate (dashed line), as functions of the supercriticality $\xi$. Also shown are slopes consistent with $\xi^2$ and $\xi^4$ dependencies.

but it is these differences that account for the inaccuracy of our scaling argument. The Rhines scale is found to increase less rapidly with supercriticality than the other two scales. A log–log plot of the ratios suggests the crude scalings

$$k_R/k_B \approx \xi^{1/2} \tag{18}$$

$$k_R/k_M \approx \xi^{1/3}. \tag{19}$$

From the definitions (15)–(17), one has the exact relations

$$(V/U)^2 = \xi^2(k_R/k_M)^2(k_R/k_B)^2, \tag{20}$$

$$(k_M\lambda)^{-1} = \xi(k_R/k_M)^{3/2}(k_R/k_B)^{1/2}, \tag{21}$$

$$\frac{\epsilon}{(U^3/\lambda)} = \xi^2(k_R/k_M)^{5/2}(k_R/k_B)^{3/2}. \tag{22}$$

The estimates (18)–(19) then yield $(V/U)^2 \approx \xi^{3.7}$ and $\epsilon/(U^3\lambda^{-1}) \approx \xi^{3.6}$, as well as $(k_M\lambda)^{-1} \approx \xi^{1.75}$, $(k_B\lambda)^{-1} \approx \xi^{1.9}$, and $(k_R\lambda)^{-1} \approx \xi^{1.4}$. These are all roughly consistent with the results in Figs. 2 and 3.

Errors that appear to be small in the individual approximations combine to create a large discrepancy in the final dependence of energy on $\xi$. The underlying reason for this sensitivity is the strong positive feed-
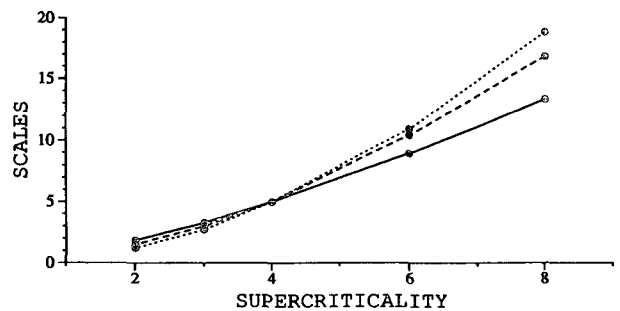


FIG. 3. The length scales $(k_R\lambda)^{-1}$ (solid line), $(k_M\lambda)^{-1}$ (dashed line), and $(k_B\lambda)^{-1}$ (dotted line), as defined in (15)–(17), plotted as a function of supercriticality $\xi$.

back inherent in the system: as the energy level increases, the length scale of the energy-containing eddies increases, which increases the mixing length and $V$, increasing the energy generation.

The biggest discrepancy in scales is between $k_B$ and $k_R$. One can reduce this difference if one uses the total (zonal plus meridional) rms eddy velocity for $V$ in (17) while retaining the use of the meridional velocity variance in (15). This choice is admittedly somewhat arbitrary but it does indicate that changes in the degree of anisotropy (see the ratio of rms eddy $u_\psi$ and $v_\psi$ in the table), which we have ignored in our simple scaling, could play a role in the discrepancy in Fig. 3.

In our scaling argument, we have assumed that the energy dissipation due to Ekman damping at large scales in the barotropic mode is equal, or at least proportional, to the eddy energy production. But as indicated in the table, this is not strictly true, since the fractional energy lost due to subgrid diffusivity and the Ekman damping of baroclinic energy is a function of supercriticality. The dissipation in the barotropic mode is a better estimate of the rate at which energy is cascading to larger scales. If one substitutes this rate for $\epsilon$ in (17), the discrepancy between $k_B$ and $k_R$ is reduced somewhat but only at small supercriticality. Although we have compromised our resolution at small scales in order to provide room for a substantial inverse cascade, we feel that it is unlikely that a model with better resolution of the direct enstrophy cascade and weaker diffusion would produce qualitatively different results.

The relevance of the classic picture of a self-similar inverse energy cascade when $\beta = 0$ has been questioned, owing to the formation of coherent vortices, and it has been found that the inverse energy cascade can produce spectra much steeper than $-5/3$ in a very large domain (Borue 1994 and references therein). Our guess is that an important part of the discrepancy between our scaling and the numerical model is a result of a naive treatment of this barotropic dynamics.

Although our qualitative arguments clearly require some modification in order to fit numerical results, we feel that they are a useful starting point and therefore have examined how they might be generalized to horizontally homogeneous flows with arbitrary stratification and mean vertical shears.

## 4. Continuous stratification

Consider a Boussinesq QG flow with stratification $N^2(z)$ and mean flow $U(z)$, confined between flat horizontal boundaries at $z = 0$, $H$. The energy production per unit mass $\epsilon$ can be written in the form

$$\epsilon = H^{-1} \int_0^H \frac{f\overline{v'b'}}{N^2} \frac{\partial U}{\partial z} \, dz. \qquad (23)$$

Here, $\overline{v'b'}$ is the meridional buoyancy flux, related to the potential vorticity flux by

$$\overline{v'q'} = f \partial_z (N^{-2} \overline{v'b'}). \qquad (24)$$

Consistent with our assumption of horizontal homogeneity, the relative vorticity flux, which equals the convergence of the eddy momentum fluxes, is ignored. The mean wind shear is related to the mean buoyancy gradient by

$$f \partial_z U = -\partial_y B. \qquad (25)$$

We assume again that the inverse cascade is substantial enough that the flow advecting the potential vorticity can be taken as barotropic. The downgradient fluxes can then be set equal to a diffusivity $D$ times the negative of the mean gradient, with the diffusivity independent of $z$. In this case of a vertically uniform diffusivity, the same diffusivity can be applied to buoyancy and to potential vorticity. The result is

$$\epsilon = DT^{-2}, \qquad (26)$$

where

$$T^{-2} \equiv H^{-1} \int_0^H \frac{f^2}{N^2} \left( \frac{\partial U}{\partial z} \right)^2 dz$$

$$= H^{-1} f^2 \int_0^H Ri^{-1} dz. \qquad (27)$$

For the two-layer model of section 2, $T = \lambda/U$, where $U = (U_1 - U_2)/2$, from which we can rederive the two-layer scaling. For a continuous flow with $N$ and $\partial_z U$ constants, we have $T = N(f \partial_z U)^{-1} = f^{-1} Ri^{1/2}$. In both cases, $T$ is proportional to the $e$-folding time for baroclinic waves in the limit that the effect of $\beta$ is small. We again assume that the diffusivity $D$ is related to the energy-containing scale $k_0$ and the barotropic velocity scale $V$, as in (5), and that the energy production must balance the upscale transport of energy by the barotropic flow (7), so that

$$\epsilon = Vk_0^{-1}T^{-2} = V^3 k_0, \qquad (28)$$

or simply,

$$T = 1/(k_0 V). \qquad (29)$$

Thus, in addition to being the timescale associated with the linear dynamics of deformation scale eddies, $T$ is also the timescale of the energy-containing eddies on much larger scales. Combined with the estimate of $k_0$ as the Rhines scale (11), we have

$$k_0 = \beta T, \qquad (30)$$

$$V = 1/(\beta T^2), \qquad (31)$$

$$D = 1/(\beta^2 T^3). \qquad (32)$$

## 5. Relationship to more weakly unstable flows

The preceding analysis is intended for the asymptotic limit of small $\beta$, in which there is an inverse energy cascade over a substantial range of scales. It is in this

limit that the flow responsible for diffusing the potential vorticity is primarily barotropic, so that we can remove the diffusivity $D$ from the integral in the expression for the energy generation (21). In the two-layer model, these scaling relations cannot remain qualitatively valid for all $\xi$, since the flow is stable once $\xi$ drops below a critical value (although it is impressive how well the two-layer results in Fig. 2 are approximated by a power law even as $\xi$ approaches unity). However, these expressions may retain their relevance over the full range of $\xi$ in certain continuous flows. Held (1978) suggests scaling arguments for Charney-like continuous models in the limit that $h/H \ll 1$, where $H$ is the depth of the fluid, or the scale height in the non-Boussinesq case, while

$$h \equiv \frac{f^2 \partial_z U}{\beta N^2} . \tag{33}$$

In this limit the vertical scale of the most unstable waves in Charney's model is proportional to $h$ and their horizontal scale to

$$Nh/f = \frac{f \partial_z U}{\beta N} \equiv 1/(\beta T), \tag{34}$$

where we have used the expression (25) for $T$. This is identical to the scaling in (28). If these scales dominate the statistically steady state, surface buoyancy perturbations will be of magnitude $b' \approx (Nh/f)\partial_y B$, or, using equipartition of eddy kinetic and available potential energies,

$$V \approx b'/N \approx h\partial_z U \approx \beta^{-1}(f\partial_z U/N)^2, \tag{35}$$

which is identical to (29). The result is again a diffusivity proportional to the cube of the vertical shear.

In Held (1978), this scaling was assumed to break down as $h/H$ becomes comparable to unity. Following Stone (1972), it was assumed that the dominant horizontal scale relevant for the eddy transports would then be $NH/f = \lambda$ and the characteristic velocity $H\partial_z U$, leading to a diffusivity that is proportional to the first power of the vertical shear. The argument of LH suggests instead that once $h/H$ ($=\xi$) rises above unity the inverse energy cascade sets in and the appropriate eddy scale and mixing length continue to increase in proportion to $\xi$. With this picture in mind, one can imagine that the scalings (28)–(30) are relevant for the full range of $\xi$.

In the weakly unstable case one cannot assume that the diffusivity is independent of height. In fact, if $h/H \ll 1$, the mixing should be confined to a depth proportional to $h$ (Held 1978) in a Charney-like environment. More generally, a theory for the vertical structure of the diffusivity is required as one moves away from the strongly unstable limit.

## 6. Final remarks

The qualitative theory presented for the eddy amplitudes and scales in horizontally homogeneous, baro-

clinically unstable flows can be most simply described by the statement that the environmental shear and stratification determine a timescale $T$, which can be combined with $\beta$ to determine a length $(\beta T)^{-1}$, a velocity $(\beta T^2)^{-1}$, and a diffusivity $(\beta^2 T^3)^{-1}$ in only one way. The remainder of the argument provides the rationale for thinking that the shear and stratification provides a timescale only, in the limit in which the flow is energetic enough that there exists a substantial inverse energy cascade. The result coincides in form to that proposed by Held (1978) for a Charney-like environment, in the opposite limit in which the eddies are very weak. This leads us to suggest that these scalings may in fact be useful for a wide range of energy levels in certain flows, although we do not understand if there are fundamental reasons why this should be so.

In practice, one must keep in mind that there may be insufficient room for $\beta$ to halt the cascade, in which case the eddy scale is determined by the domain size or the size of the baroclinic region. This leads to the diffusivity proposed by Green (1970), as described by LH.

Several limitations to these arguments are evident. Even in the limit of very strong supercriticality in the two layer model, there is only partial agreement with numerical simulations. Our scaling arguments suggest that an accurate theory for eddy amplitudes and fluxes will be difficult to obtain because of the delicate balance resulting from a positive feedback in which potential vorticity fluxes increase as eddy velocity and length scales increase, leading to higher energy levels and a stronger inverse energy cascade, leading, in turn, to an increase in eddy velocity and length scales.

Even if the limitations of the asymptotic theory are ignored, one still needs a theory for the vertical structure of the diffusivity, or of the eddy PV fluxes, when the inverse energy cascade is not so strong as to generate a nearly barotropic eddy velocity field. These arguments also do not tell us how to treat the dependence of eddy statistics on the strength of the surface drag. Also, the bottom topography so far ignored in the analysis may strongly effect the cascades, rendering the flows more baroclinic (Rhines 1977; Cox 1985; Treguier and Hua 1988).

The final and most fundamental limitation relates to the assumption of horizontal homogeneity. For example, if the barotropic decay is due to momentum fluxes in the presence of a mean horizontal shear, and if this horizontal shear is imposed by other constraints rather than being internally generated by the eddies themselves, then the dependence of the length and velocity scales on environmental parameters would clearly be changed. However, the recent study of Pavan and Held (1996) suggests that diffusivities obtained from a homogeneous model can be surprisingly useful for jetlike flows with substantial horizontal shears.

Despite these limitations and misgivings, we believe that these scaling arguments provide a useful background for developing closure schemes for simple dif-

fusive atmospheric models as well as for non-eddy-resolving ocean models.

## REFERENCES

Borue, V., 1994: Inverse energy cascade in stationary two-dimensional homogeneous turbulence. *Phys. Rev. Lett.,* **72,** 1475–1478.

Cox, M. D., 1985: An eddy-resolving numerical model of the ventilated thermocline. *J. Phys. Oceanogr.,* **15,** 1312–1324.

Green, J. S. A., 1970: Transfer properties of the large-scale eddies and the general circulation of the atmosphere. *Quart. J. Roy. Meteor. Soc.,* **96,** 157–185.

Haidvogel, D. B., and I. M. Held, 1980: Homogeneous quasi-geostrophic turbulence driven by a uniform temperature gradient. *J. Atmos. Sci.,* **37,** 2644–2660.

Held, I. M., 1978: The vertical scale of an unstable baroclinic wave and its importance for eddy heat flux parameterizations. *J. Atmos. Sci.,* **35,** 572–576.

——, and E. O'Brien, 1992: Quasigeostrophic turbulence in a three-layer model: Effects of vertical structure in the mean shear. *J. Atmos. Sci.,* **49,** 1861–1870.

Hoyer, J.-M., and R. Sadourny, 1982: Closure modeling of fully developed baroclinic instability. *J. Atmos. Sci.,* **39,** 707–721.

Larichev, V. D., and I. M. Held, 1995: Eddy amplitudes and fluxes in a homogeneous model of fully developed baroclinic instability. *J. Phys. Oceanogr.,* **25,** 2285–2297.

Maltrud, M. E., and G. K. Vallis, 1991: Energy spectra and coherent structures in forced two-dimensional and beta-plane turbulence. *J. Fluid. Mech.,* **228,** 321–342.

Panetta, R. L., 1993: Zonal jets in wide baroclinically unstable regions: Persistence and scale selection. *J. Atmos. Sci.,* **50,** 2073–2106.

Pavan, V., and I. M. Held, 1996: The diffusive approximation for eddy fluxes in baroclinically unstable jets. *J. Atmos. Sci.,* in press.

Pelinovskiy, Ye. N., 1978: Wave turbulence on a beta-plane. *Oceanology,* **18,** 126–128.

Rhines, P. B., 1975: Waves and turbulence on a beta-plane. *J. Fluid. Mech.,* **69,** 417–443.

——, 1977: The dynamics of unsteady currents. *The Sea,* Vol. 6, E. A. Goldberg, I. N. McCane, J. J. O'Brien, and J. H. Steele, Eds., J. Wiley and Sons, 189–318.

Salmon, R., 1978: Two-layer quasi-geostrophic turbulence in a simple special case. *Geophys. Astrophys. Fluid Dyn.,* **10,** 25–52.

——, 1980: Baroclinic instability and geostrophic turbulence. *Geophys. Astrophys. Fluid Dyn.,* **15,** 167–211.

Stone, P. H., 1972: A simplified radiative-dynamical model for the static stability of rotating atmospheres. *J. Atmos. Sci.,* **29,** 405–418.

Tennekes, H., and J. Lumley, 1972: *A First Course in Turbulence.* The MIT Press.

Treguier, A. M., and B. L. Hua, 1988: Influence of bottom topography on stratified quasi-geostrophic turbulence in the ocean. *Geophys. Astrophys. Fluid Dyn.,* **43,** 265–305.

Vallis, G. K., 1983: On the predictability of quasi-geostrophic flow: The effects of beta and baroclinicity. *J. Atmos. Sci.,* **40,** 10–27.

Williams, G. P., 1979: Planetary circulations: 2. The Jovian quasi-geostrophic regime. *J. Atmos. Sci.,* **36,** 932–968.