NOTES AND CORRESPONDENCE

New Conservation Laws for Linear Quasi-Geostrophic Waves in Shear

ISAAC M. HELD

Geophysical Fluid Dynamics Laboratory/NOAA, Princeton University, Princeton, NJ 08542

ABSTRACT

There exists an infinite set of quadratic conserved quantities for linear quasi-geostrophic waves in horizontal and vertical shear, the first two members of the set corresponding to the pseudomomentum and pseudo-energy conservation laws that lead to the Rayleigh–Kuo (or Charney–Stern) and the Fjortoft stability criteria. This infinite hierarchy of conservation laws follows from the conservation of the pseudomomentum in each eigenmode of the shear flow.

1. Introduction

Consider the barotropic vorticity equation linearized about a zonal flow $\bar{u}$ with absolute vorticity gradient $\gamma \equiv \beta - \bar{u}_{yy}$:

$$\zeta_t' = -\bar{u}\zeta_x' - \gamma\psi_x'. \tag{1}$$

Assume that the flow is confined to a reentrant channel with $\psi' = 0$ on the northern and southern walls. The well-known conservation laws that lead to the Rayleigh–Kuo and Fjortoft stability criteria take the form

$$\partial_t \int [\gamma^{-1}\overline{\zeta'^2}/2]dy = 0; \tag{2}$$

$$\partial_t \int [\bar{u}\gamma^{-1}\overline{\zeta'^2}/2 + (\overline{\psi'\zeta'})/2]dy$$
$$= \partial_t \int [\bar{u}\gamma^{-1}\overline{\zeta'^2}/2 - e]dy = 0, \tag{3}$$

where $e = (\overline{u'^2 + v'^2})/2$, an overbar refers to a zonal average and the integral is over the channel. In writing (2) and (3) we are assuming that the disturbance vorticity vanishes wherever $\gamma = 0$, so that the disturbance can be thought of as being created by the meridional displacement $\eta' = -\zeta'/\gamma$. Since the first of these conservation laws (pseudomomentum conservation) is known to be related to the translational invariance of the basic state, and the second (pseudo-energy conservation) to its time invariance, and since there are no other obvious symmetries for arbitrary $\bar{u}$, one is tempted to conclude that there are no other similar conservation laws to be found. (See McIntyre and Shepherd, 1987, for a recent discussion of the connection between these symmetries and conservation laws, and the fact that the symmetries insure the existence of nonlinear extensions of the linear conservation laws.)

In fact, (2) and (3) are the first two members of an infinite hierarchy of conservation laws, the next member being

$$\partial_t \int [\bar{u}^2\gamma^{-1}\overline{\zeta'^2}/2 + \bar{u}\overline{\psi'\zeta'} + \gamma\overline{\psi'^2}/2]dy = 0. \tag{4}$$

Equation (4) can be confirmed directly by first noting that the left-hand side equals

$$\int [\bar{u}\gamma^{-1}\overline{\zeta_t'(\bar{u}\zeta' + \gamma\psi')} + \overline{\psi_t'(\bar{u}\zeta' + \gamma\psi')}]dy. \tag{5}$$

Defining ξ' and χ' so that $\xi_x' = \zeta'$ and $\chi_x' = \psi'$, we have

$$\int [\bar{u}\gamma^{-1}\overline{\zeta_t'(\bar{u}\zeta' + \gamma\psi')}] = -\int \bar{u}\gamma^{-1}\overline{\xi_{xt}'\xi_t'} = 0, \tag{6}$$

$$\int [\overline{\psi_t'(\bar{u}\zeta' + \gamma\psi')}] = \int \overline{\chi_{xt}'\xi_t'}$$
$$= -\int \overline{\chi_{xt}'(\chi_{xxt}' + \chi_{yyt}')} = 0, \tag{7}$$

using the fact that $\chi_{xt}' = 0$ at the northern and southern walls of the channel. The following more indirect proof makes clear how (2), (3) and (4) are part of an infinite hierarchy of conservation laws and how this hierarchy is related to the fact that the pseudomomentum in each eigenmode of the shear flow is conserved. The results are easily extended to the quasi-geostrophic case.

2. The hierarchy of conservation laws

Let $\psi' = \tilde{\psi}e^{ik(x-ct)} + \text{cc}$ and consider the eigenvalue problem

$$\bar{u}\tilde{\zeta} + \gamma\tilde{\psi} = c\tilde{\zeta}. \tag{8}$$

In terms of the meridional particle displacements η, (8) can be written in the form

$$R\gamma\tilde{\eta} = c\gamma\tilde{\eta}; \quad R \equiv \bar{u} + \gamma\nabla^{-2}, \tag{9}$$

where ∇^{-2} is the inverse of the operator $\partial_{yy} - k^2$. Let the modal decomposition of the disturbance $\eta' = \eta e^{ikx} + \text{cc}$ be

$$\eta = \sum_i a_i \eta_i(y) e^{-ikc_i t} + \sum_j [b_j \eta_j(y) e^{-ikc_j t} + d_j \eta_j^*(y) e^{-ikc_j^* t}],$$

$$(10)$$

where the first sum is over the neutral modes and the second over the amplifying and decaying modes. One should consider the first sum as including an integral over the continuous spectrum. (For the following discussion, one can think of the continuous spectrum as discretized by some finite-differencing of the original equation.) Using the notation

$$\{A, B\} = \int A^* B \, dy \qquad (11)$$

the total pseudomomentum in the wave field, which is conserved according to (2), is proportional to $M = \{\eta, \gamma\eta\}$. As described in Held (1985), $\{\eta_i, \gamma\eta_j\} = 0$ unless $c_i = c_j^*$.

Consider first the case in which there are no unstable modes, so that all eigenvalues c are real. In this case, $M = \sum m_i$, where

$$m_i = |a_i|^2 \{\eta_i, \gamma\eta_i\}. \qquad (12)$$

Each of the m_i is conserved by the flow since the modal amplitudes, a_i, are constants. But from (9), for each positive integer n,

$$\{\eta, R^n \gamma\eta\} = \sum_{ij} a_i^* a_j \{\eta_i, R^n \gamma\eta_j\}$$

$$= \sum_{ij} a_i^* a_j c_j^n \{\eta_i, \gamma\eta_j\} = \sum_i c_i^n m_i. \qquad (13)$$

Since the m_i are conserved, the combination $\sum_i c_i^n m_i$ must also be conserved. Therefore, $\{\eta, R^n \gamma\eta\}$ is conserved for each n.

In the case when unstable modes are present, there is also a contribution to the total pseudomomentum from the product of each amplifying mode with its decaying partner:

$$M = \sum_i m_i + 2 \operatorname{Re} \sum_j m_j; \quad m_j = b_j d_j^* \{\eta_j^*, \gamma\eta_j\}. \qquad (14)$$

One can show, however, that

$$\{\eta, R^n \gamma\eta\} = \sum_i c_i^n m_i + 2\operatorname{Re} \sum_j c_j^n m_j,$$

which is again conserved. For $n = 1$,

$$\{\eta, R\gamma\eta\} = \{\eta, (\bar{u}\gamma + \gamma\nabla^{-2}\gamma)\eta\} = \int (\bar{u}\gamma\overline{\eta'^2} + \overline{\psi'\zeta'})/2,$$

which reduces to the pseudo-energy conservation law (3). For $n = 2$,

$$\{\eta, R^2\gamma\eta\} = \{\eta, (\bar{u}^2\gamma + \gamma\nabla^{-2}\bar{u}\gamma + \bar{u}\gamma\nabla^{-2}\gamma$$

$$+ \gamma\nabla^{-2}\gamma\nabla^{-2}\gamma)\eta\} = \int (\bar{u}^2\gamma\overline{\eta'^2} + 2\bar{u}\,\overline{\psi'\zeta'} + \gamma\overline{\psi'^2})/2,$$

which is the conserved quantity in (4). The $n = 3$ conservation law involves such quantities as $\psi'\nabla^{-2}\psi'$.

One can obtain (2) from (3) by using Galilean invariance. Substituting $\bar{u} + \Delta$ for $\bar{u}$ in (3) and requiring that the same conservation law hold in the new reference frame, the term multiplying Δ must vanish, yielding (2). In a similar way, one can obtain (2) *and* (3) from the $n = 2$ conservation law (4). The substitution of $\bar{u} + \Delta$ for $\bar{u}$ now produces terms linear and quadratic in Δ. The vanishing of the quadratic term yields (2) and the vanishing of the linear term yields (3). In this way, each member of this hierarchy of conservation laws can be said to contain within itself all of the preceeding members of the hierarchy.

Equation (4) can be manipulated into the alternative form

$$\partial_t \int [\bar{u}^2\gamma^{-1}\overline{\zeta'^2}/2 - 2\bar{u}e + \beta\overline{\psi'^2}/2] = 0 \qquad (16)$$

by noting that

$$\int \bar{u}\overline{\psi'\zeta'} = -\int [2\bar{u}e + \bar{u}_y(\overline{\psi'^2})_y/2]$$

$$= -\int [2\bar{u}e - \bar{u}_{yy}\overline{\psi'^2}/2]. \qquad (17)$$

In this form, the signs of the different terms in the conserved quantity become evident.

3. The quasi-geostrophic case

If no temperature gradient exists on the lower and upper boundaries of the fluid, the conserved quantities for quasi-geostrophic waves in horizontal and vertical shear take precisely the same form as in the barotropic case, with the mean potential vorticity gradient $\bar{q}_y$ replacing γ and the perturbation potential vorticity q' replacing ζ'. Temperature gradients at the boundaries can be incorporated in the usual way by including δ-function contributions in $\bar{q}_y$ and q'. In the case of a lower boundary at $z = 0$, $q' \to q' + \epsilon\psi_z'\delta(0+)$, and $\bar{q}_y \to \bar{q}_y - \epsilon\bar{u}_z\delta(0+)$, where $\epsilon = (N/f)^2$. For example, the $n = 2$ conserved quantity becomes

$$\iint [\bar{u}^2\bar{q}_y\overline{\eta'^2}/2 + \bar{u}\overline{\psi'q'} + \bar{q}_y\overline{\psi'^2}/2]\rho \, dy \, dz$$

$$- \epsilon \int [\bar{u}^2\bar{u}_z\overline{\eta'^2}/2 - \bar{u}\overline{\psi'\psi_z'} + \bar{u}_z\overline{\psi'^2}/2]_{z=0} dy, \qquad (18)$$

where $\rho = \exp(-z/H)$, in $\log(p)$ coordinates. Equation (18) can also be written in a form analogous to (16):

$$\iint [\bar{u}^2\bar{q}_y\overline{\eta'^2}/2 - 2\bar{u}e + \beta\overline{\psi'^2}/2]\rho \, dy \, dz$$

$$- \epsilon \int \bar{u}^2\bar{u}_z\overline{\eta'^2}/2|_{z=0} dy, \qquad (19)$$

where the eddy energy density e now includes the available potential as well as the kinetic energy.

4. Concluding remarks

As an example of an application of the $n = 2$ conservation law, consider a barotropic flow in the absence of rotation ($\beta = 0$), for which (2), (3) and (17) yield the conserved quantities

$$\int V, \quad \int (\bar{u}V - e), \quad \text{and} \quad \int (\bar{u}^2 V - 2\bar{u}e), \quad (20)$$

where $V = \overline{\zeta'^2}/(2\gamma)$. Suppose the disturbance is initially localized in a narrow region over which $\bar{u}$ and γ have negligible variation. If this disturbance remained localized and propagated into another region with a substantially different value of $\bar{u}$, the quantities in (20) could not be conserved simultaneously. Therefore, if $\beta = 0$ it is impossible for a disturbance in a shear flow to remain localized while moving into a region of different $\bar{u}$. If $\beta \neq 0$, this is certainly possible; Rossby wavepackets have just this property.

Analogous conservation laws can be obtained for gravity waves in shear, but the expressions are sufficiently complicated that they do not appear to be particularly useful. Also, it seems unlikely that any of the new conservation laws have nonlinear extensions analogous to those that exist for pseudomomentum and pseudoenergy.

REFERENCES

Held, I. M., 1985: Pseudomomentum and the orthogonality of modes in shear flows. *J. Atmos. Sci.,* **42,** 2280–2288.
McIntyre, M. E., and T. G. Shepherd, 1987: An exact conservation theorem for finite-amplitude disturbances to nonparallel shear flows, with remarks on Hamiltonian structure and on Arnold's stability theorems. Submitted to *J. Fluid Mech.*