NOTES AND CORRESPONDENCE

Momentum Transport by Quasi-Geostrophic Eddies

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ABSTRACT

Some results due to Kuo concerning momentum fluxes in barotropic flows are generalized so as to apply to quasi-geostrophic flows on a beta-plane. It is shown that linear, amplifying waves on an arbitrary zonal flow cause a net transport of westerly momentum out of that part of the fluid in which Raleigh's stability criterion (as generalized by Charney and Stern, and by Pedlosky) is satisfied locally. Also, it is shown that if quasi-geostrophic eddies are introduced by some "external" agent into a region in which the zonal flow satisfies the stability criterion, then westerly momentum will flow into this region.

1. Introduction

The momentum fluxes in an inviscid barotropic fluid on a beta-plane have been analyzed by Kuo (1951). One basic result is that linear amplifying waves on a zonal flow, $\bar{u}(y)$, transport westerly momentum out of the region where $\beta-\partial^2\bar{u}/\partial y^2$ is positive and into the region where $\beta-\partial^2\bar{u}/\partial y^2$ is negative. There is a simple extension of this result to baroclinic flows. This extension is discussed in the context of Phillips' 2-level model of a baroclinic fluid in Section 2, and with the full equations for quasi-geostrophic flow on a beta-plane in Section 3. Although the analysis is essentially identical to that used by Charney and Stern (1962) and Pedlosky (1964), the information which such an analysis yields about momentum transports has not, apparently, been emphasized in the literature.

In an analogous fashion (that is, by inspection of the mean square eddy potential vorticity balance) the momentum transports due to quasi-geostrophic eddies introduced into a stable zonal flow by a localized stirring mechanism are analyzed in Section 4. The stirring may be thought of as due to either ageostrophic motions (for example, convective activity) or to a truly external mechanical agent. Momentum fluxes generated in this way may be of importance in the maintenance of the solar and Jovian equatorial jets.

2. Momentum fluxes in a 2-level model

The equations of motion for an inviscid, adiabatic, quasi-geostrophic, 2-level model (Phillips, 1951) can be written in the form

$$\left.\begin{aligned} \frac{\partial q_i}{\partial t} &= -J(\psi_i,q_i), \quad i=1,2 \\ q_1 &= \nabla^2\psi_1+\beta y-\frac{1}{2\lambda^2}(\psi_1-\psi_2) \\ q_2 &= \nabla^2\psi_2+\beta y+\frac{1}{2\lambda^2}(\psi_1-\psi_2) \end{aligned}\right\}. \quad (1)$$

The subscripts 1 and 2 refer to the upper and lower layers respectively; q_i is the potential vorticity and ψ_i the streamfunction in the ith layer. J is the horizontal Jacobian, ∇^2 the horizontal Laplacian, and λ the Rossby radius of deformation. The basic state about which the equations are linearized is taken to be a time-independent zonal flow. Zonal means are denoted by an overbar and deviations from zonal means by a prime.

Let u and v be the eastward and northward velocities respectively, i.e.,

$$u_i=-\partial\psi_i/\partial y, \quad v_i=\partial\psi_i/\partial x.$$

The linearized equations become

$$\partial q_i'/\partial t=-\bar{u}_i\partial q_i'/\partial x-v_i'\partial\bar{q}_i/\partial y, \quad (2)$$

where

$$\partial\bar{q}_1/\partial y=\beta-\partial^2\bar{u}_1/\partial y^2+(\bar{u}_1-\bar{u}_2)/2\lambda^2,$$
$$\partial\bar{q}_2/\partial y=\beta-\partial^2\bar{u}_2/\partial y^2-(\bar{u}_1-\bar{u}_2)/2\lambda^2.$$

From (2) we have

$$\frac{1}{2}\frac{\partial}{\partial t}\overline{(q_i')^2}=-\overline{q_i'v_i'}\frac{\partial\bar{q}_i}{\partial y}. \tag{3}$$

Assuming a solution to (2) of the form

$$q_i'=\mathrm{Re}\tilde{q}_i e^{ik(x-ct)},$$

Eq. (3) yields

$$\overline{q_i'v_i'}=-\frac{kc_I}{2}\frac{|\tilde{q}_i|^2e^{2kc_It}}{\partial\bar{q}_i/\partial y},$$

where c_I is the imaginary part of the complex phase speed c. For an amplifying wave, therefore, the eddy potential vorticity flux must be directed down the mean potential vorticity gradient.

From the definitions (1) it also follows that

$$\overline{q_1'v_1'}=-\partial M_1/\partial y-H,$$

$$\overline{q_2'v_2'}=-\partial M_2/\partial y+H,$$

where $M_i\equiv\overline{u_i'v_i'}$ is the northward momentum flux per unit mass in the ith layer, and

$$H\equiv\overline{v_1'(\psi_1'-\psi_2')}/2\lambda^2=\overline{v_2'(\psi_1'-\psi_2')}/2\lambda^2$$

is proportional to the northward eddy flux of heat. Therefore,

$$\left.\begin{aligned}\frac{\partial M_1}{\partial y}+H&=\frac{kc_I}{2}\frac{|\tilde{q}_1|^2e^{2kc_It}}{\partial\bar{q}_1/\partial y}\\ \frac{\partial M_2}{\partial y}-H&=\frac{kc_I}{2}\frac{|\tilde{q}_2|^2e^{2kc_It}}{\partial\bar{q}_2/\partial y}\end{aligned}\right\}, \tag{4}$$

or

$$\frac{\partial M_{\mathrm{TOT}}}{\partial y}=\frac{kc_I}{2}\left(\frac{|\tilde{q}_1|^2}{\partial\bar{q}_1/\partial y}+\frac{|\tilde{q}_2|^2}{\partial\bar{q}_2/\partial y}\right)e^{2kc_It}, \tag{5a}$$

where $M_{\mathrm{TOT}}\equiv M_1+M_2$. Since

$$\tilde{q}_i=\frac{-\tilde{\psi}_i}{\bar{u}_i-c}\partial\bar{q}_i/\partial y,$$

one can also write

$$\frac{\partial M_{\mathrm{TOT}}}{\partial y}=\frac{kc_I}{2}\left(\frac{|\tilde{\psi}_1|^2}{|\bar{u}_1-c|^2}\partial\bar{q}_1/\partial y+\frac{|\tilde{\psi}_2|^2}{|\bar{u}_2-c|^2}\partial\bar{q}_2/\partial y\right)e^{2kc_It}. \tag{5b}$$

It is assumed that v' vanishes at the northern and southern boundaries, so that the eddy flux of momentum into or out of the region is zero. Pedlosky's necessary condition for instability, that the mean flow potential vorticity gradient take both positive and negative values somewhere in the fluid, then follows from (5), since $\partial M_{\mathrm{TOT}}/\partial y$ cannot be of the same sign everywhere.

A basic result easily obtained from (5) is that $\partial M_{\mathrm{TOT}}/\partial y$ for an amplifying wave will be positive (negative) wherever both $\partial\bar{q}_1/\partial y$ and $\partial\bar{q}_2/\partial y$ are positive (negative).

To discuss the implications of this result, consider first a basic state in which $\partial\bar{q}_1/\partial y>0$ everywhere in the fluid. Let $\mathcal{R}$ denote the region in which $\partial\bar{q}_2/\partial y<0$. (There must be such a region for an unstable wave to exist.) We also assume that there are regions in which $\partial\bar{q}_2/\partial y>0$ since otherwise no simple conclusions can be drawn. Then, for an amplifying wave, $\partial M_{\mathrm{TOT}}/\partial y$ is positive outside of $\mathcal{R}$ and, by conservation of momentum, there is a net flux of westerly momentum into $\mathcal{R}$. Further, if $\mathcal{R}$ is contained within the interval $y_1<y<y_2$, then $M_{\mathrm{TOT}}<0$ at $y=y_2$ and $M_{\mathrm{TOT}}>0$ at $y=y_1$. There is, in other words, a net flux of westerly momentum into $\mathcal{R}$ at both its northern and southern limits.

If $\beta\gg\partial^2\bar{u}_2/\partial y^2$ then $\mathcal{R}$, in the case above, is a region of large positive vertical shear. To consider instead a flow which is unstable due to large negative vertical shear, let $\partial\bar{q}_2/\partial y$ be positive everywhere in the fluid. An unstable wave in such a zonal flow will transport westerly momentum out of regions in which $\partial\bar{q}_1/\partial y>0$ (if such regions exist) and, therefore, into that part of the fluid in which $\partial\bar{q}_1/\partial y<0$. If $\beta\gg\partial^2\bar{u}_1/\partial y^2$ this implies that westerly momentum will be transported into regions of large negative vertical shear.

A flow which is unstable due to large, localized, negative vertical shear occurs, for example, in the numerical integrations by Lin (1974) of a 2-level (primitive equation) model of a rectangular ocean. In response to an applied wind stress, a northern boundary current forms with a westward counter-current on its southern flank. This latter current becomes baroclinically unstable, and, although the picture is somewhat obscured by the lateral boundaries, the net effect of the baroclinic waves is to decrease the vertically-integrated transport by this current, as one would expect from the above argument.

Useful information about eddy heat transports can also be obtained from this analysis if $\partial M_2/\partial_y\ll H$. In this case, from (4), heat transport will be equatorward (poleward) in regions where $\partial\bar{q}_2/\partial y$ is positive (negative). Stone (1974) has observed this property in particular solutions and has used it in developing a parameterization of eddy heat transports. To obtain his solutions, Stone assumes that the radius of deformation is much smaller than the meridional scale of the zonal flow. The aforementioned property of H does not, however, depend on this assumption; it is sufficient that momentum fluxes in the lower layer be small.

3. Momentum fluxes in a continuous baroclinic fluid

For simplicity, consider a Boussinesq fluid confined between two rigid horizontal surfaces. Analogous results can be obtained for an unbounded, compressible atmosphere if suitable conditions are imposed as $z \to \infty$.

The equation of motion for inviscid, adiabatic, quasi-geostrophic motion on a beta-plane is

$$\partial q/\partial t = -J(\psi, q),$$

where

$$q \equiv \nabla^2\psi + \beta y + f_0^2 \frac{\partial}{\partial z}\left(\frac{1}{N^2}\frac{\partial \psi}{\partial z}\right), \tag{6}$$

f_0 is the Coriolis parameter at a standard latitude, and N the Brunt-Väisälä frequency. At the rigid horizontal surfaces we require that

$$\frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial z}\right) = -J\left(\psi, \frac{\partial \psi}{\partial z}\right), \quad z = 0, D. \tag{7}$$

Following the procedure in Section 2, we linearize, form the mean square eddy potential vorticity equation, and assume a solution of the form

$$\left.\begin{aligned} q' &= \mathrm{Re}\,\tilde{q}e^{ik(x-ct)} \\ \psi' &= \mathrm{Re}\,\tilde{\psi}e^{ik(x-ct)} \end{aligned}\right\},$$

with the result that

$$\overline{v'q'} = -\frac{kc_I}{2}\frac{|\tilde{q}|^2}{\partial\bar{q}/\partial y}e^{2kc_It},$$

where

$$\frac{\partial \bar{q}}{\partial y} = \beta - \partial^2\bar{u}/\partial y^2 - f_0^2\frac{\partial}{\partial z}\left(\frac{1}{\bar{N}^2}\frac{\partial \bar{u}}{\partial z}\right).$$

But from (6),

$$\overline{v'q'} = -\partial M/\partial y + \partial H/\partial z,$$

where

$$M \equiv \overline{u'v'} \quad \text{and} \quad H \equiv \frac{f_0^2}{\bar{N}^2}\overline{\frac{\partial \psi'}{\partial z}v'}.$$

This formula has been discussed by Green (1970), among others.

Therefore,

$$\frac{\partial M_{\mathrm{TOT}}}{\partial y} \equiv \int_0^D \frac{\partial M}{\partial y}dz = \frac{kc_I}{2}e^{2kc_It}\int_0^D \frac{|\tilde{q}|^2}{\partial \bar{q}/\partial y}dz + H\Big|_0^D.$$

From the boundary conditions (7),

$$\frac{1}{2}\frac{\partial}{\partial t}\overline{\left(\frac{\partial \psi'}{\partial z}\right)^2} = -\overline{v'\frac{\partial \psi'}{\partial z}}\frac{\partial}{\partial y}\frac{\partial \bar{\psi}}{\partial z} = \overline{v'\frac{\partial \psi'}{\partial z}}\frac{\partial \bar{u}}{\partial z}, \quad z = 0, D,$$

or

$$\overline{v'\frac{\partial \psi'}{\partial z}} = \frac{kc_I}{2}\frac{|\partial\tilde{\psi}/\partial z|^2}{\partial \bar{u}/\partial z}e^{2kc_It}, \quad z = 0, D.$$

Therefore,

$$\frac{\partial M_{\mathrm{TOT}}}{\partial y} = \frac{kc_I}{2}\left[\int_0^D \frac{|\tilde{q}|^2}{\partial\bar{q}/\partial y}dz + \frac{f_0^2}{\bar{N}^2}\frac{|\partial\tilde{\psi}/\partial z|^2}{\partial\bar{u}/\partial z}\Big|_0^D\right]e^{2kc_It}. \tag{8a}$$

Using the linearized equations of motion and boundary conditions, this formula can be rewritten

$$\frac{\partial M_{\mathrm{TOT}}}{\partial y} = \frac{kc_I}{2}\left[\int_0^D \frac{|\tilde{\psi}|^2}{|\bar{u}-c|^2}\frac{\partial\bar{q}}{\partial y}dz + \frac{f_0^2}{\bar{N}^2}\frac{|\tilde{\psi}|^2}{|\bar{u}-c|^2}\frac{\partial\bar{u}}{\partial z}\Big|_0^D\right]e^{2kc_It}. \tag{8b}$$

The special case of an internal jet, in which the boundary term in (8) is zero (or negligible), is of particular interest. The necessary condition for instability in this case follows from (8)—$\partial\bar{q}/\partial y$ must change sign somewhere in the fluid, as discussed by Charney and Stern (1962). From (8) it is also evident, for an amplifying wave, that $\partial M_{\mathrm{TOT}}/\partial y$ will have the same sign as a positively weighted vertical average of $\partial\bar{q}/\partial y$. In the simplest case, $\partial\bar{q}/\partial y$ is positive everywhere except in the vicinity of an unstable jet. There must, in this case, be a net flux of westerly momentum into the latitudinal zone containing that part of the fluid in which $\partial\bar{q}/\partial y$ is negative. $\partial\bar{q}/\partial y$ can be either positive or negative at the center of an unstable westerly jet, depending on the relative importance of the horizontal and vertical shears. However, $\partial\bar{q}/\partial y$ will generally be negative at the center of an unstable easterly jet since the horizontal and vertical shear contributions are both negative at such a point.

Burpee (1972) has analyzed the momentum fluxes due to "African waves" which apparently form as instabilities on an easterly jet in the lower troposphere south of the Sahara in Northern Hemisphere summer. The data, though sparse, suggests that these waves transport westerly momentum into the jet, as expected from the discussion above.

The usefulness of (8) is considerably reduced when the heat flux at the lower (or upper) boundary is not negligible. Many unstable flows of interest have $\partial\bar{q}/\partial y > 0$ in the interior and $\partial\bar{u}/\partial z > 0$ at the lower boundary, so that the integral and the lower boundary term in (8) will be of opposite sign. If there do exist regions of the fluid in which 1) $\partial\bar{q}/\partial y > 0$ throughout a vertical column, and in which 2) $\partial\bar{u}/\partial z$ is non-positive at the lower boundary and non-negative at the upper boundary, then westerly momentum flows out from these regions and into the rest of the fluid.

4. Momentum transport by forced quasi-geostrophic eddies in a stable zonal flow

Consider a zonal flow, with $\partial\bar{q}/\partial y > 0$ everywhere, in an infinite Boussinesq fluid on a beta-plane. We introduce a quasi-geostrophic disturbance localized in a latitudinal span, $y_1<y<y_2$, and then allow the flow to evolve. If the disturbance is sufficiently small, the linearized equations are valid, so that

$$\frac{\partial}{\partial t}[\tfrac{1}{2}\overline{(q')^2}]=-\overline{v'q'}\,\partial\bar{q}/\partial y. \tag{9}$$

Outside of the interval $y_1<y<y_2$, there will, in general, be some response to the disturbance introduced within the interval: $\overline{(q')^2}$, initially zero, will be positive at any later time t. Integrating (9) over time from 0 to t, one finds that the time-averaged value of $\overline{v'q'}$ outside of the interval $y_1<y<y_2$ will be negative, and, therefore, $\partial M_{\mathrm{TOT}}/\partial y$ will be positive. Thus, there is a net flux of westerly momentum into the region in which the disturbance is introduced.

The barotropic analogue of this result has been discussed by Kuo (1951) and is mentioned by Green (1970), who, in turn, attributes it to Eady.

Alternatively, one can consider a localized, statistically steady source in a fluid with some unspecified form of dissipation. If one again assumes that the linear equations are sufficiently accurate, then

$$0=-\{\overline{v'q'}\}\,\partial\bar{q}/\partial y+\{Q\}-\{D\},$$

where Q is the source and D the sink of mean square eddy potential vorticity (and braces denote a time average). In a region in which $\{Q\}\approx 0$, one has the balance

$$\{\overline{v'q'}\}\approx-\frac{\{D\}}{\partial\bar{q}/\partial y}.$$

Presumably, $\{D\}>0$, and the conclusion again is that momentum flows into the region of forcing.

These arguments suggest the following explanation for the equatorial jets in the atmospheres of Jupiter and the Sun: small-scale convective motions, most intense near the equator, force geostrophic eddies which transport the momentum necessary to maintain the jet. A less speculative explanation would require some understanding of how large-scale geostrophic eddies are forced in these atmospheres.

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