Numerical issues for coupling biological models with isopycnal mixing schemes

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Abstract

In regions of sloping isopycnals, isopycnal mixing acting in conjunction with biological cycling can produce patterns in the nutrient field which have negative values of tracer in light water and unrealistically large values of tracer in dense water. Under certain circumstances, these patterns can start to grow unstably. This paper discusses why such behavior occurs. Using a simple four-box model, it demonstrates that the instability appears when the isopycnal slopes exceed the grid aspect ratio ($\Delta z/\Delta x$). In contrast to other well known instabilities of the CFL type, this instability does not depend on the time step or time-stepping scheme. Instead it arises from a fundamental incompatibility between two requirements for isopycnal mixing schemes, namely that they should produce no net flux of passive tracer across an isopycnal and everywhere reduce tracer extrema. In order to guarantee no net flux of tracer across an isopycnal, some upgradient fluxes across certain parts of an isopycnal are required to balance downgradient fluxes across other parts of the isopycnal. However, these upgradient fluxes can cause local maxima in the nutrient field to become self-reinforcing. Although this is less of a problem in larger domains, there is still a strong tendency for isopycnal mixing to overconcentrate tracer in the dense water. The introduction of eddy-induced advection is shown to be capable of counteracting the upgradient fluxes of nutrient which cause problems, stabilizing the solution. The issue is not simply a numerical curiosity. When used in a GCM, different parameterizations of eddy mixing result in noticeably different distributions of nutrient and large differences in biological production. While much of this is attributable to differences in convection and circulation, the numerical errors described here may also play an important role in runs with isopycnal mixing alone. © 1999 Elsevier Science Ltd. All rights reserved.

1. Introduction

In recent years it has become clear that most mixing in the ocean interior occurs along isopycnal surfaces (Jenkins, 1980; McDougall, 1987; Ledwell et al., 1993). The implications of this fact for biogeochemical cycling are just becoming clear, but there is preliminary evidence (Clark,

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1997) that transport of nutrient along isopycnals may be as important as the vertical diapycnal flux of nutrient as a source to the nutrient-poor mixed layer. Since the nutrient flux to the mixed layer is vitally important for a wide range of biogeochemical processes (ecosystem dynamics, carbon uptake, and trace metal cycling to name a few) it is important to include such fluxes in large-scale numerical models. In most published numerical model studies which of global bio-geochemical cycles, however, much or all of the eddy mixing occurs in the horizontal direction (Toggweiler et al., 1989; Najjar et al., 1992).

The reason that large horizontal mixing coefficients were applied to these models was to prevent the solutions from becoming excessively noisy. Although Redi (1982) and Cox (1984) proposed schemes whereby mixing predominantly along isopycnals could be included in ocean models, these schemes were found to require significant background horizontal mixing coefficients (10–40% of the isopycnal mixing coefficient) in order to produce solutions without significant negative values in passive tracers, and excessive noise in active tracers like temperature and salinity. Early efforts to include isopycnal mixing in models with biological cycling also ran into problems with the appearance of noise (Murnane, personal communication). The linkage of mixing to the generation of variance is troubling – since one would expect mixing to eliminate small-scale variability – not to increase it. When modelling nutrients, which are only taken up when they exceed certain threshold values, the appearance of noise can have serious consequences, turning off biological activity altogether in one grid box while enhancing it in the next.

The purpose of this paper is to discuss why general circulation models can have trouble sim-ulating biological cycling in the presence of isopycnal mixing. It argues that truncation errors implicit in attempting to mix along isopycnals using coarse-resolution grids can lead to unrealistic nutrient fields, which under some circumstances can grow unstably. This instability should be distinguished from numerical instabilities familiar to numerical oceanographers, which arise when the time step becomes too large, or when an inappropriate time-stepping scheme is used. Instead, it can be shown to be a natural mode of the equations when they are discretized in space, but continuous in time. It can also be distinguished from recent work which shows how inaccurately coding the process of isopycnal mixing can lead to unstable solutions. The paper proceeds as follows. Section 2 reviews recent work on isopycnal mixing and demonstrates the instability of the nutrient field for a simple case. Section 3 presents a four-box model with isopycnal mixing and nutrient cycling, showing that the nutrients can develop unstably growing patterns. Section 4 shows that the instability can be eliminated by using eddy-induced advection of the form pro-posed by Gent et al. (1995). Section 5 shows that these results are not just important for idealized cases but that eddy mixing parameterizations can have important effects in general circulation models as well. Section 6 concludes this paper.

2. Problems which can arise when running models with isopycnal mixing

2.1. Previous work

In general one can write the evolution of a tracer in the following terms

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x_m} A^{mn} \frac{\partial C}{\partial x_n} + \text{sources} - \text{sinks}, \tag{1}$$

where A^{mn} is a tensor which defines the effect of the eddies on the mean field. In general A^{mn} can have components corresponding to advection or diffusion. The tensor for isopycnal mixing was defined by Gent and McWilliams (1990) as

$$\mathbf{A}_{\text{diff}} = [A^{mn}]_{\text{diff}} = A_I \begin{pmatrix} 1 & 0 & S_x \\ 0 & 1 & S_y \\ S_x & S_y & S_x^2 + S_y^2 \end{pmatrix}, \tag{2}$$

where $S_{x,y}$ are the isopycnal slopes in the x and y directions, respectively and A_I is the isopycnal diffusion coefficient.

Problems can arise with isopycnal mixing schemes in a number of ways:

1. In convective regions $S_{x,y}$ are infinite. This was handled in the original implementation of the MOM model by setting the slope to some maximum value, following Cox (1984). Unfortunately, such a solution has the effect of producing negative diffusion in the vertical (Gerdes et al., 1991; Gough, 1997; Griffies et al., 1998)! This is completely unphysical and may have significant consequences (Gough, 1997). Gerdes et al. (1991) suggested that this might be removed by scaling the diffusion coefficient so that the product $A_I^{\text{Scaled}} * (S_x^2 + S_y^2)$ goes to A_I as the slopes become large.
2. The time step may be so large that the diffusion causes a CFL violation. This can be eliminated, however, by treating the diffusion implicitly in the vertical.
3. The scheme may not reduce the variance of tracer fields which are not aligned with the isopycnals. This can happen because of inaccuracies relating to how the slopes $S_{x,y}$ are estimated along a coarse-resolution grid (Griffies et al., 1998).
4. The scheme may not reduce all maxima and minima. The reasons for this and effects of such a failure are considered in more detail later in this paper.

Although the first three problems can be solved by more accurately coding the diffusion equation, as discussed in Griffies et al. (1998), solving the final problem is much harder. In fact, Beckers et al. (1998) argue that there is *no* solution of the problem which reduces variance, does not flux tracers across isopycnals, and eliminates spurious minima and maxima. The purpose of this paper is to alert numerical modellers to these problems, which are unfortunately linked to the incompatibility between mixing along isopycnals and casting the resulting fluxes on a Cartesian grid.

2.2. An example of unstably growing nutrient fields

Consider the evolution of a nutrient field subject *only* to isopycnal mixing and nutrient cycling. Nutrient is incorporated into biota which are either eaten and incorporated into fecal pellets which sink into the deep ocean, or die and sink into the abyss. As they sink, bacterial and physical processes 'remineralize' the nutrient, returning it to the water. If one does a thought experiment about how such a system should evolve, the conclusion would be that isopycnal mixing would bring nutrient to the surface along one isopycnal, then biological production and remineralization would transport the nutrient to a deeper isopycnal. Over time, the nutrient would tend to become concentrated in the densest water, and the production would drop to low values except where the densest water outcropped at the surface.

Discrete models of this process, however, do not necessarily give a final state where all the nutrient is concentrated in the densest water and the production is low. This can be illustrated using a simple idealized model. The grid used has 300 km horizontal resolution and 100 m vertical resolution. The initial nutrient field (shown in Fig. 1(A)) is linearly correlated with the temperature field, so that isolines of temperature and nutrient coincide. The surface nutrient is then restored to 0, producing a flux of nutrient out of the mixed layer F_{bio}. If C is the nutrient concentration, F_{bio} is then

$$F_{bio} = -C/T_{rest} \ * \ dz = -\lambda \ * \ C \ * \ dz, \quad C > 0, \tag{3a}$$

$$F_{bio} = 0, \quad C \leqslant 0, \tag{3b}$$

where T_{rest} is the restoring time and $\lambda = 1/T_{rest}$ is the restoring rate. For the simulation in Fig. 1, time constant T_{rest} of 20 days was used. Eqs. (3a) and (3b) are commonly used representations of the effects of biology, which will take up nutrient when it exceeds a certain threshold, but will not take it up when it drops below this threshold (in this case, zero nutrient). The resulting flux of nutrient is remineralized over the water column.

$$Q_{remin} = \frac{dF_{bio}}{dz} = F_{bio}/L_{remin}, \tag{4}$$

where Q_{remin} is the source of nutrient due to remineralization and L_{remin} is the length scale over which remineralization occurs. For all the runs in this paper $L_{remin} = 2$ km. Any nutrient reaching the bottom is remineralized in the bottom box.
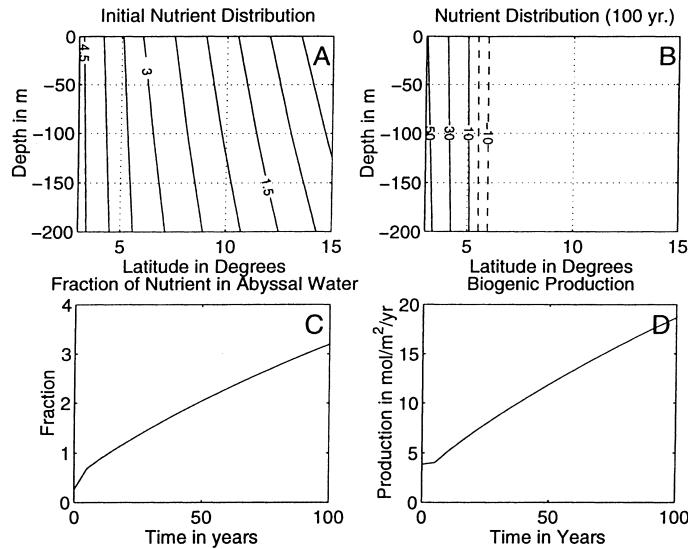

Fig. 1. Unstable evolution of the nutrient field in the presenceof isopycnal mixing. (A) Initial nutrient field. Isolines areidentical to the isotherms which are held fixed throughout therun. (B) Nutrient field after 100 years. Note that the concentrationin most of the domain is now negative. (C) Fraction of the nutrient inthe leftmost column. Note the severe overconcentration of nutrient.(D) Biogenic production.

In the case shown in Fig. 1, there are some isopycnals which do not reach from the surface to the bottom. Therefore one would expect biology and mixing acting in tandem to export all the nutrient from those isopycnals which outcrop at the surface to those isopycnals which do not outcrop. Over time, then, the deep concentrations of nutrient would increase and the biological production would drop to near zero. This does not happen. Instead, nutrient is drained out of the light water, producing negative concentrations over most of the domain (Fig. 1(B)). The fraction of nutrient in boxes which do not outcrop is shown in Fig. 1(C). As time progresses more nutrient ends up in these boxes than in the system as a whole – so that the negative concentrations in some parts of the domain are balanced by impossibly high concentrations in other parts of the domain. As a result the biological production actually rises over the course of the simulation.

In realistic GCMs the instability shown in Fig. 1 may be counteracted to some degree by explicit horizontal diffusion as well as numerical diffusion caused by the advection scheme. In runs with the Modular Ocean Model, the author found that that unstably growing patterns similar to that seen in Fig. 1 were largely limited to shallow, marginal seas. Increasing the strength of the diffusive coefficient was found to greatly increase the size of negative concentrations and to increase the biological production in these marginal seas. In other parts of the ocean negative concentrations were found at the surface, dense waters were overly enriched with nutrient, and biogenic production was extremely large. Although advection and vertical diffusion did seem to reduce the strength of the instability, it was by no means clear that this would always be the case.

It is disturbing (and somewhat counterintuitive) that a diffusive process can lead to unstably growing patterns. It should be emphasized that the pattern found in Fig. 1 is not the classic CFL-type of instability, which arises when the time step becomes too large. In fact the unstable pattern in Fig. 1 is found regardless of the time step. How can diffusion, which reduces variance, be linked to such large increases in tracer variance? The simple answer is that forcing isopycnal mixing to occur on a discrete grid introduces truncation errors. Because isopycnal mixing does not flux tracer across isopycnal surfaces in the mean, some of these truncation errors cause downgradient diapycnal fluxes, while others cause upgradient diapycnal fluxes. The latter type of fluxes can cause maxima to increase and minima to decrease. In some cases, which are detailed in the following section, this can lead to unstably growing patterns.

3. Description of the instability

3.1. A simple analytical problem

In order to analyze what causes the unstably growing patterns seen in Fig. 1, it is helpful to simplify the system to minimal complexity. This can be done by reducing the problem to a simple four box model such as that shown in Fig. 2(A), in which the isopycnals slope up towards the left with slope S_ρ. The initial condition is such that there is no nutrient in the surface box in the right-hand column (box 2) and that the nutrient field is such that are no fluxes of nutrient due to isopycnal mixing. In order for this to be true on a discrete grid, $S_\rho = S_C$ so that tracer and density have the same slopes. This can only be true if there is a locally linear relationship between nutrient and density.
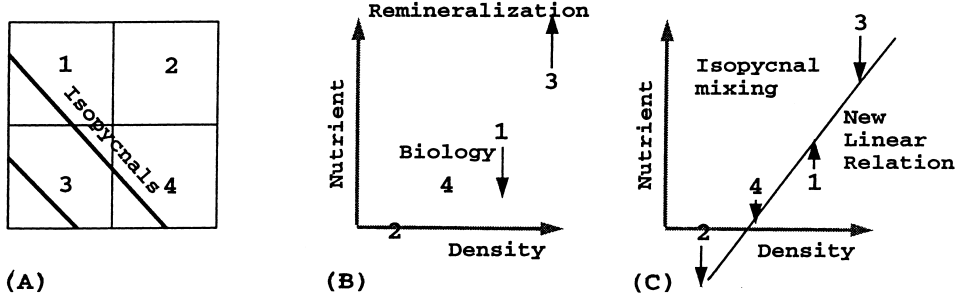
Fig. 2. Schematic of how isopycnal instability can react unstably with nutrient cycling. (A) Schematic of a simple four-box model. (B) If the isopycnals slope more steeply than the grid and the nutrient field is initially in equilibrium with isopycnal mixing, then there will be a simple linear relationship between the nutrient field and the isopycnals as shown. Biology will act to drive down the concentration in box 1, while remineralization will act to increase it in box 2 as shown. (C) Isopycnal mixing then tries to restore the nutrient field to a new linear relationship with density. This new relationship has a steeper slope than the old one, and thus there is more nutrient in the denser water column for the biology to work on.

$$C = D\rho + E, \tag{5}$$

where D and E are undetermined constants. Since the isopycnals slope upwards to the left, box 1 will be denser than box 2 and will have some nutrient in it. Biology acts on the nutrient field as in Eqs. (3a) and (3b), reducing it to zero in the surface layer when it is greater than zero, and the resulting flux is then exported to the lower box. Supposing that the isopycnal slopes on all faces are the same, then one may define for notational convenience

$$\delta_x = \frac{A_I}{\Delta x^2}, \quad \delta_z = \frac{A_I S_\rho^2}{\Delta z^2}, \quad \delta_{xz} = \frac{A_I S_\rho}{\Delta x \Delta z}. \tag{6}$$

Assuming that biological fluxes are parametrized as in Eq. (6), the system of linear equations is then

$$\frac{\partial C_1}{\partial t} = -(\lambda + \delta_x + \delta_z - \delta_{xz})C_1 + \delta_x C_2 + \delta_z C_3 - \delta_{xz} C_4, \tag{7a}$$

$$\frac{\partial C_2}{\partial t} = \delta_x C_1 - (\delta_x + \delta_z + \delta_{xz})C_2 + \delta_{xz} C_3 + \delta_z C_4, \tag{7b}$$

$$\frac{\partial C_3}{\partial t} = (\lambda + \delta_z)C_1 + \delta_{xz} C_2 - (\delta_x + \delta_z + \delta_{xz})C_3 + \delta_x C_4, \tag{7c}$$

$$\frac{\partial C_4}{\partial t} = -\delta_{xz} C_1 + \delta_z C_2 + \delta_x C_3 - (\delta_x + \delta_z - \delta_{xz})C_4, \tag{7d}$$

when the diffusion tensor is coded following Griffies et al. (1998). The biology does not act in the right-hand column (boxes 2 and 4), because the surface nutrient is already zero there.

Because the system of Eqs. (7a)–(7d) is linear, it can be decomposed into a system of eigenvalues and eigenvectors. Defining **C** as the vector of concentrations

$$\frac{\partial \mathbf{C}}{\partial t} = \sum_{n=1}^{4} a_n \frac{\partial \mathbf{V}_n}{\partial t} = \sum_{n=1}^{4} a_n g_n \mathbf{V}_n, \tag{8}$$

where $\mathbf{V}_n$ is the nth eigenvector associated with eigenvalue g_n. Analysis of the eigenvalues shows that there are positive eigenvalues when the isopycnal slope S_ρ exceeds the grid aspect ratio $\Delta z / \Delta x$. This means that for such cases, some pattern or normal mode will grow unstably with an exponential growth rate given by g_n. Over time the pattern associated with the largest positive eigenvalue will come to dominate the spatial distribution. The largest positive eigenvalue defines the growth rate of the most unstably growing mode.

The growth rate of the fastest-growing mode $(\max(g_n))$ is shown in Fig. 3 for two grids, one with aspect ratio of 1:1000, the other with aspect ratio 1:2000. The growth rate is a strong function of slope and of the restoring time. Maximum growth rates are found when S_ρ, the isopycnal, slope is twice the aspect ratio. Decreasing the damping time (thus increasing the damping rate) results in a faster growth of instability. The unstably growing modes have the structure shown in Fig. 4, with high concentrations in the dense water and low concentrations in the light water. Over time, the unstably growing patterns shown in Fig. 4 will come to dominate the tracer field, with large positive concentrations in the dense water and large negative concentrations in the light water.

3.2. Conceptual picture

A conceptual understanding of the instability may be gained by returning to Fig. 2(B). If the isopycnals slope more steeply than the grid aspect ratio, then both boxes on the right-hand side will be lighter than both boxes on the left-hand side. Biology will then reduce the concentration in box 1 and remineralization will increase the concentration in box 3. This in turn increases the mean least-squares slope between nutrient and density as shown in Fig. 2(B) and (C). Isopycnal mixing will then act on the field to restore it towards the new linear relationship as shown in
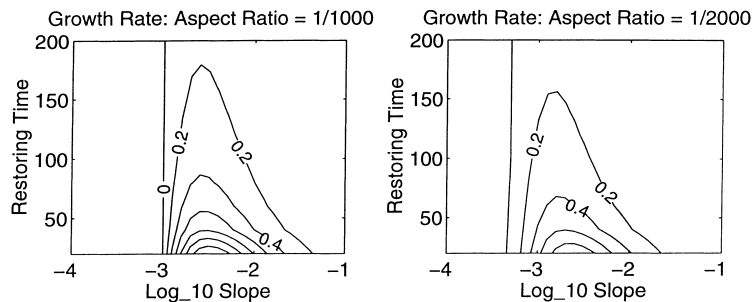

Fig. 3. Instability of the four box model for two different grid aspect ratios as a function of restoring time T and isopycnal slope S_ρ for the basic scenario illustrated in Fig. 2. Growth rates for the most unstably growing normal mode are in year^{-1}. Horizontal axis is $\log_{10}$ isopycnal slope and vertical axis is restoring time. (A) Growth rate for a grid aspect ratio of 1:1000 (boxes 100 m thick in the vertical and 100 km wide in the horizontal) given $A_I = 1000$ m^2/s. (B) Growth rate for a grid aspect ratio of 1:2000 (boxes 100 m thick in the vertical and 200 km wide in the horizontal) given $A_I = 1000$ m^2/s. Note that growth rates are smaller, but the range of slopes for which instability occurs is larger.

0.16	-0.65
0.71	-0.21

Slope=-0.002, g=0.86 /yr
Restoring time=30 days

(A)

0.21	−0.66
0.67	−0.22

Slope =-0.002, g=0.17 /yr
Restoring time = 200 days

(B)

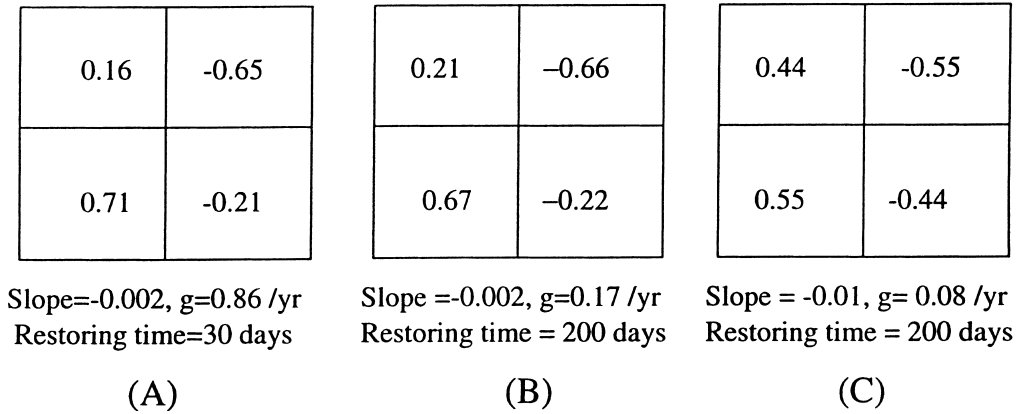| 0.44	-0.55
0.55	-0.44

Slope = -0.01, g= 0.08 /yr
Restoring time = 200 days

(C)

Fig. 4. Structure of the fastest-growing unstable mode for various combinations of restoring time and isopycnal slope. A_i is 1000 m^2/s, Δx is 100 km and Δz is 100 m. The modes shown have a normalized amplitude, actual fields would be some multiple of that shown. Growth rate of each mode g is listed underneath each table. (A) Slope = −0.002, restoring time = 30 days. (B) Slope = −0.002, restoring time = 200 days. Note that while the growth rate of the most unstable mode has changed a lot (from 0.86^{-1} to 0.17^{-1}, a factor of 5), the structure of the unstable mode has not changed significantly relative to that in A. (C) Slope = −0.01, Restoring time = 200 days. Now the structure of the fastest-growing unstable mode is what changes, with a greater contrast between the boxes in the horizontal and a smaller contrast in the vertical.

Fig. 2(C) (see Appendix A for a proof that isopycnal mixing restores the nutrient in the four-box model towards the least-squares slope and that this slope is in fact increased by the production–remineralization cycle). Note that the arrows for boxes 2 and 4 both point downwards in Fig. 2(C), so that the effect of isopycnal mixing is to decrease the concentration in both boxes in the lighter column. This means that the horizontal isopycnal flux is upgradient. The result will be a larger contrast in nutrient between the left-hand and right-hand columns, further increasing the nutrient available for biological production. Additionally, since the new linear relationship implies a higher concentration for box 1 than initially, the flux in the dense water will increase, resulting in a positive feedback.

In the event that the isopycnal slope is less steep than the grid aspect ratio, the positions of boxes 1 and 4 in Fig. 2(A) will be reversed. This means that after the production and remineralization cycle, the effect of the isopyncal mixing will be to restore the tracer in box 1 to a *lower* value than the initial. This situation is governed by a negative feedback loop and the final state is one with high tracer concentrations in both of the deep boxes and low biological production. This in turn explains why the maximum growth rate is zero for this range of slopes in Fig. 3.

The problem appears to be fundamental to discretizing the equations on a Cartesian grid. Beckers et al. (1998) recently pointed out that isopycnal diffusion should obey a number of constraints:
1. It should not flux tracer when tracer isolines and isopycnals are identical. Over the four-box domain in Fig. 3 this amounts to requiring a linear relationship between tracer and density at diffusive equilibrium.
2. It should reduce variance over the domain.
3. It should not create minima and maxima.

The Griffies et al. (1998) scheme obeys the first and second constraints, but not the third, unless the isopycnals line up with the grid. The essential reason is that the diffusion equation corresponds to the following matrix equation

$$\frac{\partial C_i}{\partial t} = B^{ik} C_k, \tag{9}$$

where B^{ik} represents an exchange coefficient between two boxes i and k. Beckers et al. (1998) demonstrate that in order for condition 1 to be satisfied, some off-diagonal B^{ik} have to be negative when the grid does not line up with the isopycnals. However in order for condition 3 to be satisfied no off-diagonal element can be less than zero. Looking at Eq. (7a), it can easily be seen that either B^{12} or B^{14} must be negative, unless the slope is zero. This means that any discretization which has no diapycnal fluxes has the potential to develop the basic problem in Fig. 2.

The problem could be solved by demanding that the horizontal grid be fine enough so that the isopycnal slope never exceeds the grid aspect ratio. However, this is not feasible in the general case. Isopycnals often slope quite steeply, especially in boundary currents and in the Southern Ocean, where slopes of 1:500 are not uncommon. However, in order to resolve the thermocline, grid boxes near the surface have small Δz, as small as 50 m. This would then require 25 km resolution in the horizontal. Given that models run at 200 km resolution require over 1000 h of supercomputer time to come to equilibrium such high resolution in the horizontal is simply impractical for long-term climate runs.

Given this fact, it is clear that some additional physics must be added. Given that the main problem is the upgradient horizontal flux which leads to the nutrient in the dense column increasing and that in the light column decreasing, the additional physics would need to counteract this flux. A solution which has been used in the past has been to impose background horizontal mixing. This has the basic problem that it leads to large diapycnal fluxes in regions where the isopycnals slope steeply. The effective diapycnal diffusivity goes as

$$K_{\text{diapyc}} = A_{\text{horiz}} * S_\rho^2. \tag{10}$$

Given slopes of order 0.002, and $A_{\text{horiz}} \sim 200$ m²/s, the effective diapycnal diffusivities which result are of order 10^{-3} m²/s, two orders of magnitude larger than those associated with observed diapycnal diffusion (Ledwell et al., 1993). Alternatively, one can limit the total fluxes out of the mixed layer to some value. Again this is not completely physical and does not address the underlying problem of upgradient fluxes. In the following section it is argued that there are more physically justifiable alternatives for addressing this problem.

4. Solutions

4.1. Eddy thickness advection

Eddies not only act as diffusive agents, but as advective agents as well (Plumb and Mahlman, 1987), producing residual velocities or Stokes drifts. One recent parameterization postulates that these advective velocities act to homogenize the thickness of isopycnal layers (Gent and

McWilliams, 1990; Gent et al., 1995). Griffies et al. (1998) noted that the advective effect of eddies on the tracer field could be expressed by defining the antisymmetric tensor

$$\mathbf{A}_{adv} = [A^{mn}]_{adv} = A_{thick} \begin{pmatrix} 0 & 0 & -S_x \\ 0 & 0 & -S_y \\ S_x & S_y & 0 \end{pmatrix}. \tag{11}$$

If $A_{thick} = A_I$ the total tensor becomes the sum of the tensors in Eqs. (12) and (11).

$$\mathbf{A}_{tot} = A^{mn}_{adv} + A^{mn}_{diff} = A_I \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2S_x & 2S_y & S_x^2 + S_y^2 \end{pmatrix} \tag{12}$$

so that the horizontal fluxes

$$F_C^x = -A_I \frac{\partial C}{\partial x}, \tag{13a}$$

$$F_C^y = -A_I \frac{\partial C}{\partial y}, \tag{13b}$$

are equivalent to those given by downgradient horizontal diffusion. The Gent–McWilliams flux thus acts to cancel the portion of the isopycnal flux which causes the instability, namely the horizontal upgradient fluxes.

Fig. 5 shows the effect of including the eddy-induced advective flux on the instability in the four-box model. The growth rate when the isopycnal slope is twice the grid aspect ratio is shown as A_{thick} is varied. For $A_{thick} > 0.5A_I$ the instability is essentially eliminated for the four-box model.

It should be pointed out, however, that although the GM parameterization may produce good results in the present case, the eddies which it parametrizes may actually behave quite differently. In particular near boundaries and boundary currents eddy advection may behave quite differently than in the idealized case discussed here (Starr, 1953). In some cases it may even be characterized
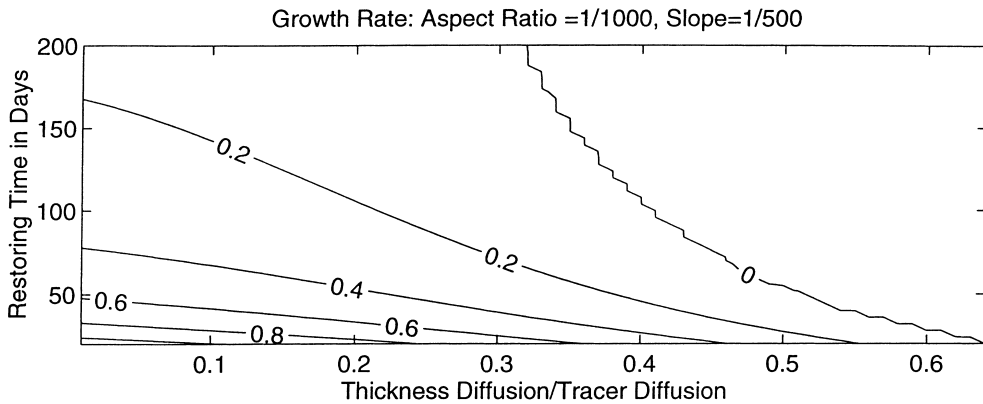

Fig. 5. Growth rate in the four-box model as a function of the ratio of the thickness diffusion coefficient A_{th} to the isopycnal diffusion coefficient A_I and the restoring time in days.

by a negative diffusion coefficient (whereas by definition the stirring terms will always have a positive diffusion coefficient).

Additionally, the Gent–McWilliams scheme will not be sufficient to fix problems arising from inaccuracies in the advective scheme. Gough (1997) and Shchepetkin et al. (1998) provide a discussion of such errors, which occur when the advection is discretized in a form which allows for the creation of maxima and minima in the tracer field. Such overshoots frequently occur in coarse-resolution models near topography due to the poor representation of the transition from cross-shore geostrophic flow in the model interior to along-shore frictional flow in the boundary current.

4.2. Other possible solutions and their problems

A number of other solutions to the instability were tried, and found to be relatively unsuccessful. These included the following:

1. Increasing the vertical diffusion. For the four-box model, it was found that very large values of vertical diffusion were needed to suppress the instability (of order 10 cm^2/s). Such large values of diffusion would have major effects on the thermohaline circulation and are not acceptable.
2. Horizontal mixing within the mixed layer alone. Relatively modest values of horizontal mixing within the mixed layer were found to stabilize the four-box model. The diffusion coefficients required were approximately 50% of the isopycnal mixing coefficient. Such mixing is physically justifiable on the grounds that particles do not follow density surfaces within the mixed layer but interact strongly with the atmosphere as they are stirred from place to place. However, when horizontal mixing was added within the ocean general circulation model, it was found to have a relatively minor effect on final solution The model did reach a final state somewhat sooner than with isopycnal mixing alone but otherwise the differences were small. In a few areas, adding horizontal mixing actually made the situation worse by causing convection which increased the local isopycnal slope.
3. Smoothing the remineralization horizontally so that particles produced at one point are remineralized at surrounding points. Siegel et al. (1990) noted that turbulent motions cause a spreading of particles produced by individual blooms. In regions where the isopycnals slope steeply this has the effect of counteracting some of the upgradient horizontal transport. It has been found, however, that this effect is not strong enough to eliminate the instability (even within the four-box model), although it does reduce the growth rates.

5. Effect of the parameterizations in a full OGCM

One reason that these effects are not purely numerical curiosities is that eddy parameterization can have important effects on the biological productivity predicted by general circulation models. Gnanadesikan (1999) presents results from one such set of simulations, examining the oceanic cycle of silicon. The following cases were run:

1. A case with horizontal mixing alone, following Toggweiler et al. (1989) referred to as HOR.
2. A case with isopycnal mixing using the parameterization of Griffies et al. (1998) with a coefficient of 1000 m^2/s with a background horizontal mixing coefficient of 10 m^2/s, referred to as ISO.
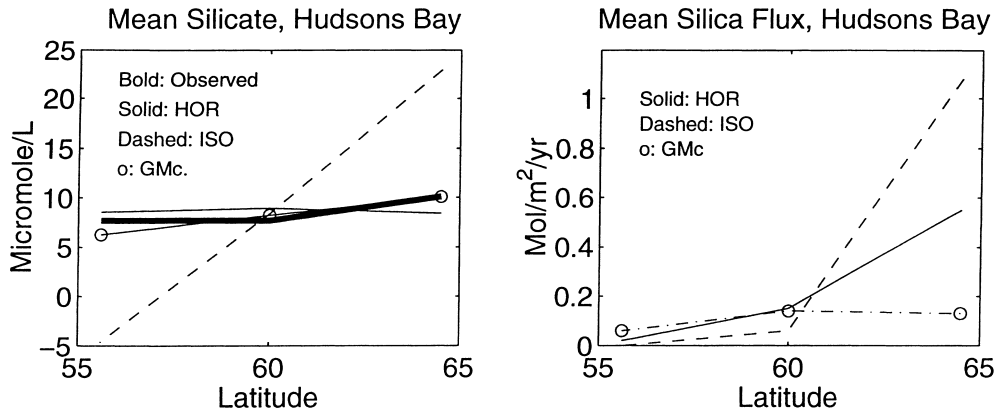
Fig. 6. Illustration of the effect of mixing parameterization in a shallow closed basin. Left: Averaged silica down to 150 m. Bold lines are observed, solid are for the HOR model, dashed for the ISO model mixing (note negative concentration in south), chain-dot with o is for the GMc model. Right: Silica fluxes in mol/m^2/yr predicted for the three model cases. Lines as at left.

3. A case with Gent–McWilliams eddy-induced advection, referred to as GMc.

This work showed that the GMc model produced the most realistic export flux and nutrient distribution. There were clear physical reasons for some of the improvement. In particular, a better simulation of Antarctic surface water resulted in reducing convection and vertical exchange in the Southern Ocean. Similar results were reported by Robitaille and Weaver (1995). As a result the fluxes in this region were far smaller in the GMc model than in the HOR or ISO models, producing predictions far closer to observations.

However, numerical issues still played a role in some regions. An example is shown in Fig. 6, for Hudson's Bay. In the HOR and GMc models, the concentrations are within reasonable limits, and the total production is small. In the ISO model is used, large positive concentrations develop on the northern side of the basin and the fluxes are quite large. For larger values of mixing coefficient, the nutrient field was found to become numerically unstable when isopycnal mixing alone was used. This demonstrates that even with an isopycnal scheme which is stable for active tracers (temperature and salinity) biological cycling can still generate unrealistic nutrient fields.

It should also be pointed out that in general the ISO models overconcentrated nutrient in the densest waters. Dissolved silicate has a maximum at about 2500 m in the North Pacific. While the HOR and GMc models tended to reproduce this feature, the ISO model tended to produce the largest values of dissolved silicate right on the bottom. At present it is unclear how much of a role numerical truncation errors played in this phenomemon.

6. Conclusions

This paper has demonstrated that isopycnal mixing with no background diffusion can lead to the overconcentration of nutrients in dense water. For some cases this overconcentration can

result in exponential growth of patterns of nutrient. This instability is intrinsic to modelling steeply sloping isopycnals on coarse-resolution Cartesian grids. The instability can be controlled with the Gent and McWilliams eddy-induced advection parameterization which has some physical justification. The results are of considerable importance, since parameterization of subgridscale mixing by eddies can have a major effect on the biological production predicted by an OGCM (Gnanadesikan, 1999). Numerical models which aim to predict biological production on basin or global scales must consider these effects. Areas of particular sensitivity include the deep ocean, the Southern Ocean, and shallow marginal seas.

It is appropriate to inquire at this point whether the solution to these problems is simply to switch to an isopycnal coordinate numerical model. Unfortunately, such models also have problems in regions where the isopyncals slope steeply. In most isopycnal models, if more than one isopycnal layer outcrops between two grid boxes, it is effectively insulated from the surface boundary. Examples of this can be seen in Jia and Richards (1996) who simulate tritium penetration into the Atlantic thermocline. Certain of their isopycnal layers are completely isolated from the atmosphere. For simulations with biology, this would mean that nutrients would tend to build up in some interior layers. Although this problem could be resolved in isopycnal models by reducing the number of layers, such a solution would create the opposite problem of giving too fast a connection between the deep layers where remineralization occurs and the surface mixed layer. In general truncation errors will be a problem in both classes of models, unless some sort of thickness mixing is used.

Acknowledgements

Appendix A. Proof that isopycnal mixing drives the nutrient towards the local least-squares slope

There are two eigenvectors of the four-box system with zero eigenvalues.

$$\mathbf{V}_1 = (1/4, 1/4, 1/4, 1/4), \tag{A.1a}$$

$$\mathbf{V}_2 = \frac{(\rho_1 - \rho_m, \rho_2 - \rho_m, \rho_3 - \rho_m, \rho_4 - \rho_m)}{\sqrt{(\rho_1 - \rho_m)^2 + (\rho_2 - \rho_m)^2 + (\rho_3 - \rho_m)^2 + (\rho_4 - \rho_m)^2}}, \tag{A.1b}$$

where ρ_m is the mean density. The first eigenvector corresponds to a uniform field, the second to a linear slope between density and nutrient. The other eigenvectors have negative eigenvalues. Thus when the system is changed to a state vector $\mathbf{C}$ it goes towards

$$\mathbf{C}_{\text{final}} = (\mathbf{C} \cdot \mathbf{V}_1) * \mathbf{V}_1 + (\mathbf{C} \cdot \mathbf{V}_2) * \mathbf{V}_2. \tag{A.2}$$

But $\mathbf{C} \cdot \mathbf{V}_2$ is just

$$\mathbf{C} \cdot \mathbf{V}_2 = \left(\langle C * \rho \rangle - \langle C \rangle \langle \rho \rangle\right) \Big/ \sqrt{\langle \rho^2 \rangle - \langle \rho \rangle^2}, \tag{A.3}$$

where the brackets note averaging. This is just the least-squares slope. The effect of biology and remineralization is to increase this slope. Suppose that the initial values for

$$\mathbf{C}_{\text{init}} = (C_1, C_2, C_3, C_4) = R_{\text{init}} \ * (\rho_1 - \rho_m, \rho_2 - \rho_m, \rho_3 - \rho_m, \rho_4 - \rho_m) + S \tag{A.4}$$

so that

$$\mathbf{C} \cdot \mathbf{V}_2 = R_{\text{init}}. \tag{A.5}$$

When biology reduces the nutrient in box 1 by δC and remineralization increases that in box 3 by δC.

$$\mathbf{C}_{\text{new}} = (C_1 - \delta C, C_2, C_3 + \delta C, C_4). \tag{A.6}$$

The new slope

$$R_{\text{new}} = \mathbf{C}_{\text{new}} \cdot \mathbf{V}_2 = R_{\text{init}} + \delta C(\rho_3 - \rho_1) > R_{\text{init}} \tag{A.7}$$

since $\rho_3 > \rho_1$. Thus the effect of biology and remineralization in the four-box model is to increase the slope of the linear relationship between nutrient and density and isopycnal mixing will drive the tracer towards the new slope.

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