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SOFT PION PROCESSES*

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(Invited talk at the American Physical Society Meeting, Chicago, January 1968)

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I. My talk is concerned with a review, not necessarily of the latest theoretical developments, but rather of an old idea which has contributed to recent theoretical activities. By soft pion processes I mean processes in which low energy pions are emitted or absorbed or scattered, just as we use the word soft photon in a similar context. Speaking more quantitatively, we may call a pion soft if its energy is small compared to a natural scale in the reaction. This scale is determined by the particular dynamics of pion interaction, and one may roughly say that a pion is soft if its energy is small compared to the energies of the other individual particles that participate in the reaction. It is important to note at this point that pion is by far the lightest member of all the hadrons, and much of the success of the soft pion formulas depends on this fact. This also means that the same procedure cannot be expected to work as well for the K meson. Whether this is an accident of nature, or it has a more deep-seated significance, is where people may have different opinions.

The basic problem is to derive a kind of low energy theorem which would become exact if the pion had zero energy. This is similar to the well-known low energy theorems for soft photon problems, which state that the electromagnetic properties of a system may be characterized by a few parameters such as the charge and the magnetic moment, in the soft photon limit.

The low energy theorems for photon follow from the peculiar nature of electromagnetic interaction; namely, the photon is massless, and its source, the electromagnetic current, is conserved. One characterizes this by saying that the electromagnetic Lagrangian satisfies gauge invariance.

In an analogous way, one can derive low energy theorems for pion by postulating a certain invariance, or a symmetry, of the dynamical laws that govern strong interactions. This symmetry is called chiral symmetry. It is not, however, an exact symmetry as was the case with the electromagnetic interaction, primarily because the actual pion is not massless. One can apply the theorem rigorously only after the unphysical extrapolation to zero energy and zero momentum.

II. Historically, what amounts to the chiral symmetry involving the pions appears in Schwinger's work (in 1957). But the V-A theory of weak interactions actually gave a more direct stimulation for a search of the new symmetry, or if not a symmetry, at least a search for a new formulation of pion interaction.

One recalls the argument of Feynman and Gell-Mann for the non-renormalization of the weak vector current, based on the assumption that it is proportional to the isotopic spin current, which is conserved in strong interactions. In contrast, the weak axial vector current for the nucleon, for example, is not divergenceless because of the nucleon mass, and therefore, there is no reason to expect non-renormalization. In fact we know $-G_A/G_V = 1.18$.

Nevertheless, the simple structure of the leptonic current suggests that for the hadrons too, the basic weak current must be simple. This idea later found its explicit formulation in Gell-Mann's quark currents and their commutator algebra.

The real clue, in my opinion, to the symmetry involving the axial vector currents came from the so-called Goldberger-Treiman relation

$$m_N G_A \approx g_{\pi N} \; g_\pi \tag{1}$$

where	m_N	nucleon mass
	G_A	weak axial const in β decay
	$g_{\pi N}$	$\pi - N$ coupling (strong)
	g_π	$\pi - \mu\nu$ decay coupling

Since both G_A and g_π are supposed to be proportional to the basic weak interaction constant, it is a statement about the nature of strong interactions. It was soon realized that the relation would follow if there was a partial conservation of the axial current, broken only by the non-zero pion mass. This is the PCAC hypothesis.

It is usually formulated as

$$\partial_\mu a_\mu^\pm = c\phi^\pm \equiv \frac{m_\pi^2}{f}\phi^\pm \qquad (2)$$

$$f = G_V/2g_\pi \approx f_{\pi N}/1.18 \approx 0.8/m_\pi$$

where $\phi^\pm$ charged pion field

$a_\mu^\pm$ axial current that appears in the weak interaction, expressed for example in terms of the quark fields.

One can obviously include the neutral counterpart as well, and treat all the pions on an equal footing. The quantum numbers for the pion are defined by (2), which certainly seems consistent with its known properties.

An easier way, however, to see the role of PCAC in the Goldberger-Treiman relation is to write down explicitly the matrix element for a_μ in the β decay:

$$\frac{Go}{\sqrt{2}} \langle p|a_\mu|n\rangle = u_p^+ [\frac{G_A}{\sqrt{2}} i\gamma_\mu\gamma_5 + \sqrt{2}\, G_{\pi N} g_\pi \gamma_5 \frac{q_\mu}{q^2 + m_\pi^2}] u_n,$$

$$q = p_p - p_n \qquad (3)$$

where the second term represents the virtual process

$$n \rightarrow p + \pi, \quad \pi \rightarrow \text{leptons}.$$

By taking the divergence $q_\mu a_\mu$, and making the two terms to cancel each other for $m_\pi = o$, we get (1).

So one might say that the nature tries to make the axial current as well conserved as possible by invoking the pion. In the limit $m_\pi \rightarrow o$ we would have a new conservation law and a quantum number associated with it. This quantum number should be defined by

$$\chi^i = \int a_o^i \, d^3x, \qquad i = 1, 2, 3 \tag{4}$$

which we call isotopic chirality. We realize, however, that it is a rather strange quantum number because it has odd parity. Thus a particle at rest with a definite parity is not an eigenstate of χ unless the eigenvalue is zero. Otherwise it implies there must be a parity degeneracy of states. Actually, (3) shows that $<\chi^i>_N$ depends on the velocity of the nucleon, given by

$$<\chi^i>_N = -<\tau^i h v> \tag{5}$$

where h is the helicity. Only in the limit $v \to c$ can a nucleon be an eigenstate of χ since then the nucleon mass may be ignored and the velocity operator becomes diagonal.

What then, is the meaning of chirality conservation? One may interpret it by saying that although the nucleon is not an eigenstate of χ, the nucleon plus the surrounding medium conserves χ in such a way that when the nucleon is accelerated, the change in $\langle\chi\rangle_N$ is compensated by the emission of massless pion excitation.

Since chirality is thus such an elusive observable, it has to be more precisely defined. This, however, is not unique, as might be expected. A typical way to characterize a symmetry is to characterize it in terms of its group structure. There exist two different possibilities, which can be illustrated with particular models.

1) Massless pion with pseudovector coupling with the nucleon: (PS-PV theory).

$$a_\mu^{\ i} = i\bar{\psi}\,\gamma_\mu\gamma_5\tau^i\psi + \frac{1}{f}\,\partial_\mu\phi^i$$
$$\equiv \chi^i_N + \chi^i_\pi \qquad (6)$$

The invariance is with respect to a displacement $\phi^i \rightarrow \phi^i + c^i$, which, combined with isospin transformation, gives the inhomogeneous SU(2) group. The divergence $\partial_\mu a_\mu^{\ i}$ amounts to a wave equation for ϕ^i, and leads to the PCAC relation (2) if a mass term is added.

2) Quark model. We define the isospin transformation for the left-handed and right-handed quarks separately, with the currents $v_{\mu\ L}^{\ i}$, $v_{\mu\ R}^{\ i}$ respectively.

$$= i\bar{q}_2\ \gamma_\mu\ \tau^i\ q_L \qquad\qquad = i\bar{q}_R\ \gamma_\mu\ \tau^i\ q_R$$

We have thus a group $SU(2)_L \otimes SU(2)_R$ (which of course can readily be extended to $SU(3)_L \otimes SU(3)_R$). The vector and axial vector currents are then

$$v_\mu^{\ i} = v_{\mu\ L}^{\ i} + v_{\mu\ R}^{\ i}, \quad a_\mu^{\ i} = v_{\mu\ L}^{\ i} - v_{\mu\ R}^{\ i} \qquad (7)$$

These currents can also be defined in terms of phenomenological fields.

For example, the pion part of χ may be defined as

$$\chi_\pi^{\ i} = \int [\frac{1}{f}\ \pi^i + 2f\ \phi^i\ \underset{\sim}{\pi} \cdot \underset{\sim}{\phi} - f\pi^i\ \underset{\sim}{\phi} \cdot \underset{\sim}{\phi}]\ d^3x \qquad (8)$$

where $\pi^i(x)$ is the canonical conjugate to $\phi^i(x)$. Another way is to use a neutral scalar field σ in conjunction with π^i to form a 4-dimensional representation of $SU(2)_L \times SU(2)_R$. The difference between the above two groups shows up in the commutation relations between χ^i's:

$$
\begin{aligned}
[\chi^i, \chi^j] &= 0 && \text{Case (1)} \\
&= 2\epsilon^{ijk} I^k, && \text{Case (2)},
\end{aligned}
\tag{9}
$$

where I^k is the isospin operator. In other words, one can tell the difference in processes involving two soft pions. The Adler-Weissberger relation for $\pi - N$ scattering was indeed such a test, and confirmed the group $SU(2)_L \otimes SU(2)_R$. This relation sum rule involving $\pi - N$ scattering total section at all energies, but it is also equivalent to a statement about zero energy scattering amplitude. It is essentially a low energy theorem. Thus, the quark model seems to be the correct way to define chirality.

III. THE SOFT-PION FORMULA

The standard procedure for deriving the specific formula for soft-pion emission is the Lehmann-Symanzik-Zimmerman technique and the PCAC relation (2). A simple and more physical way to see its meaning is to write out the conservation law

$$\chi^i_{out} = \chi^i_{in}$$

or

$$\chi^i_{in} S - S \chi^i_{in} = o \qquad (10)$$

and use, for example, (6) for χ^i. If χ^i is not strictly conserved, but violated by a perturbing term H', such as the electromagnetic and weak interactions, a modified form is,

$$\chi^i_{in} S - S\chi^i_{in} = -i S \int_{-\infty}^{\infty} [\chi^i(t), H'(t)] dt \qquad (11)$$

Taking (11) between two states a and b, and inserting intermediate states, we realize that $\chi^i_\pi \sim \frac{1}{f} \partial_\mu \phi^i$ causes transitions to states with an extra zero-energy pion emitted. In this way, we get a relation between the amplitude $M_{A \to B}$ and the emission amplitude $M_{A \to B + \pi}$, which is given by

$$\frac{1}{f} M_{A \to B + \pi} = \chi^{i'} P M_{A \to B} - M_{A \to B} P \chi^{i'}$$

$$+ \; S [\chi^i, H'] \qquad (12)$$

I inserted the energy-momentum conservation projection P. $\chi^{i'}$ is $\chi^i - \chi^i_\pi$. The last term represents direct pion emission due to the disturbance H'. Since χ^i is the generator of the chiral group, this commutator can be specified if we know the transformation property of H' under the chiral group. This is the reason why the formula is so useful in weak and electromagnetic processes where the quark model tells you how H' transforms. And it is also the reason why the formula was not so useful before the quarks.

For processes involving more than one soft pion, the formula is more complicated. Not only do the commutators of two axial currents (which is equal to a vector current) come in, but also other quantities which are not specified by the group alone. This reflects the fact that a soft pion is not really soft compared to other soft pions in the process.

The following is a representative list of processes to which the formula has been applied. Brackets mean there are still problems which have not been resolved yet.

	E.M.	W.	S + MS
1π	$\gamma N \to N\pi$	$\nu N \to N e \pi$	
	$eN \to eN\pi$	$(B \to N\pi)$	
		$K \to \pi e \nu$	
2π		$K \to 2\pi$	$\pi N \to N\pi$ (A-W)
		$K \to 2\pi e \nu$	$(\rho \to 2\pi)$
3π	$(\eta \to 3\pi)$	$K \to 3\pi$	
4π			$(\pi\pi \to \pi\pi)$

IV. NON-LINEAR CHIRAL LAGRANGIAN MODELS AND GAUGE FIELDS

In order to understand the peculiar role of the chiral group in pion dynamics, it is instructive to make use of a phenomenological Lagrangian. There are various models of chiral symmetry, and an important question here is how the pion field is to transform under the chiral group. The most appropriate one seems to be to assign a non-linear transformation property to the pion field. This is due to the fact that:

1) there does not seem to exist a well-defined neutral scalar meson σ which would go with the pions to form a 4-dimensional representation of the chiral group; and

2) the nucleon mass in any case precludes the conventional correspondence between a multiplet of single particle states and a representation of a group.

A convenient way for this purpose is to follow a procedure due to Gürsey. Consider the nucleon (proton and neutron) field ψ_α, $\alpha = 1,2$, and associate the chiral group $SU(2)_L \otimes SU(2)_R$ with the isospin transformations of its left-handed and right-handed components. (L and R distinguish the two different irreducible representations of the Lorentz group.) In the spirit of the quark model, one should actually consider massive quark fields, but the basic argument is the same. The mass term in the Lagrangian,

$$-m\,\bar{\psi}\,\psi \;=\; -m(\psi_R^{+}\,\psi_L + \psi_L^{+}\,\psi_R) \qquad (13)$$

is not invariant, but behaves like

$$D_L(\tfrac{1}{2}) \otimes D_R^{*}(\tfrac{1}{2}) \oplus D_R(\tfrac{1}{2}) \otimes D_L^{*}(\tfrac{1}{2})$$

of the chiral group. We introduce a 2×2 matrix field $\mathcal{M}(\phi_i(x))_{\alpha\beta}$ which is a function of the pion field $\phi_i(x)$, and postulate that the indices α and β transform like $D_L(\frac{1}{2})$ and $D_R^{*}(\frac{1}{2})$ respectively. We can then form an invariant

$$-m(\psi_R^{+}\,\mathcal{M}^{+}\,\psi_L + \psi_L^{+}\,\mathcal{M}\,\psi_R) \qquad (14)$$

Under $SU(2)_L \otimes SU(2)_R$, the transformation will be

$$\psi_L \rightarrow U_L\,\psi_L, \quad \psi_R \rightarrow U_R\,\psi_R,$$
$$\mathcal{M} \rightarrow U_L\,\mathcal{M}\,U_R^{+}, \quad \mathcal{M}^{+} \rightarrow U_R\,\mathcal{M}^{+}\,U_L^{+} \qquad (15)$$

Since a product of 2×2 matrices is another 2×2 matrix, (15) defines a new $\mathcal{M}' = \mathcal{M}(\phi')$, and besides, since the U's are unitary, we may impose the same unitary condition on $\mathcal{M}$:

$$\mathbb{M}\,\mathbb{M}^+ = \mathbb{M}^+\,\mathbb{M} = \text{const} = \mathbb{1} \tag{16}$$

Under parity, $L \leftrightarrow R$, $\phi_i \to -\phi_i$, $\mathbb{M} \leftrightarrow \mathbb{M}^+$, so that $\mathbb{M}(\phi)^+ = \mathbb{M}(-\phi)$. A simple example of such $\mathbb{M}$ is

$$\mathbb{M}(\phi) = \exp\left[\, 2if\underset{\sim}{\phi} \cdot \underset{\sim}{\tau} \,\right] = \exp\left[\, 2if\not{\phi} \,\right] \tag{17}$$

By expanding $\mathbb{M}$ in ϕ in (14), we see that the first two terms give the nucleon mass and $\pi - N$ interaction terms with the conventional coupling constant $G_{\pi N} = 2fm_N$.

According to (17), ϕ's will behave like the parameters of finite rotation (the Eulerian angles) in the isotopic space, so their transformation is a quite complicated non-linear one. Only under the ordinary isotopic transformation $U_L = U_R$, does it undergo the familiar linear transformation.

The kinetic energy of the pion in the Lagrangian can also be made chiral invariant by taking

$$-\frac{1}{16f^2}\, T_r\, \partial_\mu \mathbb{M}^+\, \partial_\mu\, \mathbb{M} \tag{18}$$

which again contain non-linear terms in ϕ representing pion-pion interaction.

The conserved currents that follow from (18) are, in a matrix notation

$$v_\mu = \frac{i}{4f^2} (\partial_\mu \mathbb{M}^+ \mathbb{M} - \mathbb{M} \partial_\mu \mathbb{M}^+)$$
$$a_\mu = \frac{i}{4f^2} (\partial_\mu \mathbb{M}^+ \mathbb{M} + \mathbb{M}^+ \partial_\mu \mathbb{M}) \tag{19}$$

Again, by expansion, we see that

$$v_\mu = i (\partial_\mu \not\phi \not\phi - \not\phi \partial_\mu \not\phi) + \cdots$$
$$a_\mu = \frac{1}{f} \partial_\mu \not\phi + \cdots \tag{20}$$

Finally, we can add a symmetry breaking pion mass term. But the choice is not unique; we have to specify how it transforms. An example is $\frac{\mu^2}{8f^2} T_r (M + M^+)$, which also contain non-linear interaction terms.

The same procedure may be extended to $SU(3)_L \otimes SU(3)_R$ of three quarks by considering a 3×3 matrix $\mathbb{M}(\phi_n)$ of nine $(8+1)$ pseudoscalar fields ϕ_n.

The basic prescription is to use the above Lagrangian and the currents, making a perturbation expansion in $f\phi$, and collecting consistently terms of the same and lowest order contributing to a given multipion process, but ignoring all higher order effects. Because of the non-linear relations between v, a and ϕ, for example, we get automatically many-pion emission amplitudes when these currents are substituted in the basic weak interaction. In this way all of the results obtained from the PCAC and current algebra may be reproduced.

Now what is the meaning of all this? We will make a few remarks in this regard.

1) The physical pion state and the pion field $\phi^i(x)$ must be sharply distinguished. The latter is an auxiliary operator which, for each fixed x, undergoes a non-linear transformation under $SU(2)_L \otimes SU(2)_R$. In the language of group theory, ϕ_i is a 3-dimensional representation of the subgroup SU(2), which is used as a carrier space to induce a linear representation $M(\phi_i)$ of the entire group. The precise functional form of $M(\phi_i)$ does not matter, as long as it has an expression $1 + 2if\phi + \cdots$. f is the basic dynamical parameter, and ϕ defines the physical pion state. Higher order processes are uniquely determined from this. But the perturbation expansion is a non-invariant procedure since the "free" and "interaction" Lagrangian are not separately chiral invariant.

2) As the non-linear transformation of ϕ_i is a local one, one actually has a group of local chiral transformation or chiral gauge transformation, although the kinetic energy terms in the Lagrangian are not invariant

under such gauge transformation. It would seem natural, then, to introduce gauge fields of the Yang-Mills type to make the whole thing gauge invariant. This extended principle of chiral invariance, initiated by Weinberg, and carried out by many people, has led to some more interesting results. The basic idea is to introduce two fields, $V_{\mu L}$ and $V_{\mu R}$, again in the 2 x 2 matrix notation, related to the vector and axial vector fields V_μ and A_μ by

$$\begin{aligned} V_{\mu L} + V_{\mu R} &= V_\mu \\ V_{\mu L} - V_{\mu R} &= A_\mu \end{aligned} \tag{21}$$

They transform under local $SU(2)_L \otimes SU(2)_R$ as

$$(V_\mu)_{\substack{L\\R}} \to (U\, V_\mu\, U^+ + (i/g)\partial_\mu U U^+)_{\substack{L\\R}} \tag{22}$$

Then the combinations

$$(D_\mu)_{\substack{L\\R}} = (\partial_\mu - ig\, V_\mu)_{\substack{L\\R}}$$

$$\begin{aligned} (F_{\mu\nu})_{\substack{L\\R}} &= (D_\mu V_\nu - D_\nu\, V_\mu)_{\substack{L\\R}} \\ &= (\partial_\mu\, V_{\mu\nu} - \partial_\nu V_\mu - ig\,[\,V_\mu, V_\nu\,])_{\substack{L\\R}} \end{aligned}$$

$$\mathbb{D}_\mu \mathbb{M} = \partial_\mu \mathbb{M} - ig\, \mathbb{V}_{\mu L}\, \mathbb{M} + ig\, \mathbb{M}\, \mathbb{V}_{\mu R}$$

$$\mathbb{D}_\mu \mathbb{M}^+ = \partial_\mu \mathbb{M}^+ - ig\, \mathbb{V}_{\mu R}\, \mathbb{M}^+ + ig\, \mathbb{M}\, \mathbb{V}_{\mu L}$$

(23)

$$(\mathbb{v}_\mu)_{L} = \frac{-i}{4f_2}\, \mathbb{D}_\mu \mathbb{M}\cdot\mathbb{M}^+ , \quad (\mathbb{v}_\mu)_R = \frac{-i}{4f_2}\, \mathbb{D}_\mu \mathbb{M}^+\cdot\mathbb{M}$$

R

$$\text{or} \quad \mathbb{v}_\mu = \frac{-i}{4f_2}\left(\mathbb{D}_\mu \mathbb{M}\,\mathbb{M}^+ + \mathbb{D}_\mu \mathbb{M}^+\,\mathbb{M}\right),$$

$$\mathbb{a}_\mu = \frac{-i}{4f_2}\left(\mathbb{D}_\mu \mathbb{M}\,\mathbb{M}^+ - \mathbb{D}_\mu \mathbb{M}^+\,\mathbb{M}\right),$$

transform under local chiral transformation in the same way as under constant chiral transformation. So a typical Lagrangian is of the form

$$\begin{aligned}\mathbb{L} = &-\frac{1}{16f^2}\,\mathrm{Tr}\,\mathbb{D}_\mu \mathbb{M}^+\,\mathbb{D}_\mu \mathbb{M} \\ &-\frac{1}{2}\,\mathrm{Tr}\,(\mathbb{F}_{\mu\nu}\mathbb{F}_{\mu\nu})_L - \frac{1}{2}\,\mathrm{Tr}\,(\mathbb{F}_{\mu\nu}\mathbb{F}_{\mu\nu})_R \\ &-\frac{m_o^2}{2}\,\mathrm{Tr}\,(\mathbb{V}_\mu\mathbb{V}_\mu)_L - \frac{m_o^2}{2}\,\mathrm{Tr}\,(\mathbb{V}_\mu\mathbb{V}_\mu)_R \\ &-(\bar{\psi}\gamma_\mu\mathbb{D}_\mu\psi)_L - (\bar{\psi}\gamma_\mu\mathbb{D}_\mu\psi)_R - m(\psi_R^+\mathbb{M}^+\psi_L + \psi_L^+\mathbb{M}\psi_R) \\ &-\epsilon(i\bar{\psi}\gamma_\mu\mathbb{v}_\mu\psi + i\psi\gamma_\mu\gamma_5\mathbb{a}_\mu\psi)\end{aligned} \qquad (24)$$

Only the vector meson mass terms $\sim m_o^2$ break gauge invariance. Without them, actually $\mathbb{M}$, $\mathbb{M}^+$ can be completely transformed away by a (parity non-conserving) gauge transformation.

The pion kinetic energy term contains a combination $\partial_\mu \phi - \frac{g}{2f} A_\mu$, which results in a mixing of $\partial_\mu \phi$ and A_μ. By separating out the pion normal mode, we obtain renormalization of m_A and f, given by

$$f_r = f/z ,$$

$$m_A = m_v/z = m_o/z \qquad (25)$$

$$z^2 = 1 - g^2/4m_o^2 f_r^2 = {}^1/(1+g^2/4m_o^2 f^2)$$

The celebrated Weinberg relation $m[A_1] = \sqrt{2}\, m[\rho]$ and the Kawarabayash-Suzuki-Fayyazuddin-Riazuddin relation $g^2/2m_v^2 f_r^2 = 1$ are tied together, and correspond to $z = {}^1/\sqrt{2}$. This value, however, does not seem to follow from the chiral group alone.

3) The prescription in the chiral Lagrangian method that we expand in the pion field and keep only the lowest terms contributing to a given process suggests the phenomenological character of the method. In terms of Feynman diagrams this means that we pick up only "tree diagrams" which do not contain any internal loops. Thus rescattering, or unitarity correction, is ignored, among other things, as may be appropriate for low energy phenomena. Actually, we can characterize the procedure as a classical, or WKB, approximation to the S-matrix. To be more explicit, we write the S-matrix in the interaction representation in the form

$$S = T \exp [\frac{i}{\hbar c} \int g L_{int}\, d^4 x] = : \exp [- \frac{i}{\hbar c} Ⓗ]: \qquad (26)$$

where Ⓗ is a functional of free field operators representing all connected diagrams. As is well known, a T-product can be converted into a normal product by repeated contractions using Feynman propagators, expressed in the concise symbolical form

$$T \exp [\frac{i}{\hbar c} \int g L_{int}\, d^4 x]$$

$$= : \exp [\hbar c \iint \frac{\delta}{\delta \phi_{(x)}} \Delta \ (x - x') \frac{\delta}{\delta \phi_{(x')}} d^4 x\, d^4 x']$$

$$\exp [\frac{i}{\hbar c} \int g L_{int} (\phi)\, d^4 x'']:$$

$$\equiv : U \exp [\frac{i}{\hbar c} \int g L_{int}\, d^4 x] : \qquad (27)$$

(with obvious generalizations when L contains more than one field, obeying either statistics.) From this follows, by differentiating S with respect to g,

$$- i \hbar c \partial S/_{\partial g} = : \int L_{int} (\tilde{\phi})\, d^4 x\, S : = - : \frac{\partial Ⓗ}{\partial g} S : ,$$

$$\tilde{\phi} (x) = U \phi(x)\, U^{-1} = \phi(x) + \hbar c \int \Delta (x - x') \frac{\delta}{\delta \phi (x')} d^4 x'$$

or $$\frac{\partial Ⓗ}{\partial g} + \int L_{int} [\phi (x) - i \int \Delta (x-x')(\frac{\delta Ⓗ}{\delta \phi_{(x')}} + i\hbar c \frac{\delta}{\delta \phi_{(x')}})\, d^4 x']\, d^4 x = 0,$$

(28)

where $\delta/\delta\phi$ applies to fields standing to its right in the functional L.

To drop this term, which is proportional to $\hbar$, amounts to making the WKB approximation in a Schrödinger equation, and we arrive at a "classical" Hamilton-Jacobi equation

$$\frac{\partial \textcircled{H}_{tot}}{\partial g} + \int L_{int}\left[\frac{\delta \textcircled{H}_{tot}}{\delta Q_{(x)}}\right] d^4x = o,$$

$$\textcircled{H}_{tot} = \textcircled{H} - \frac{1}{2}\int \phi\, L_o\, \phi\, d^4x, \quad (L_o = i\,\Delta^{-1})$$

$$Q_{(x)} = L_o\,\phi_{(x)} \tag{29}$$

This equation may be formally integrated in a straight-forward manner, and yields a classical solution to the field equation derived from L_{tot}. If the solution is expressed as a perturbation expansion, it can be shown that only the tree diagrams are generated.

A consequence of this analysis is that a symmetry of a Lagrangian is not lost in the approximation if it is good at the level of c-number fields. The formalism also enables one to handle many-particle processes in a systematic manner.