LECTURES ON CHIRAL SYMMETRIES AND SOFT PION PROCESSES*

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(Lectures given at the Istanbul Summer School,

August 1966)

EFINS 66-107

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I. INTRODUCTORY REVIEW

At the Istanbul Summer School in 1962 I gave lectures on "Chiral Symmetries in Weak and Strong Interactions."[1] It is only recently, however, that the basic ideas that were started several years ago have begun to bear fruit. We will cover in the present lectures more or less the same general field, but certainly there will be a lot more results to be discussed now than four years ago.

First, let us start with a brief historical review. The concept of chiral symmetry (or γ_5 invariance) originated in the universal V-A theory of weak interactions by Feynmann and Gell-Mann,[2] and others.[3] This theory assumes that the basic Hamiltonian density for weak interactions is of the current-current type

$$H = \frac{G_0}{\sqrt{2}} \sum_{n,m} (j_\mu^{(n)} j_\mu^{+(m)} + h.c.) \tag{1}$$

where n, m run over basic lepton and hadron fields. This is analogous to the electromagnetic interaction between charged particles after the electromagnetic field is eliminated.

The current j involves only the left-handed components of basic fields. By basic fields we mean (e, ν) and (μ, ν') pairs for the leptons, and some fundamental hadronic field for baryons and mesons. A great progress has been made recently with regard to the nature of the last one. We can assume, as

the simplest model, that this fundamental field is the quark triplet[5] $q = (q_1, q_2, q_3)$ which replaces the old Sakata triplet (p, n, Λ).

The strength of the quark model is that it leads to a number of correct predictions in strong, electromagnetic, and weak interactions in comparison with some other models. One way to characterize different models is, as is now well known, through the commutation relations and other algebraic properties of currents. An interesting investigation of different models along this line was done recently by Okubo. There are of course some difficulties with the quark model if we regard quarks as real objects. There are also some models which avoid these difficulties but are otherwise similar in many predictions. It is certainly a very interesting and important problem to test various models in their predictions. We will, however, not attempt it here, although there will be some discussion of models which use phenomenological fields rather than fundamental fields. This is only for the sake of conveniently representing the symmetry properties of real hadrons, and should not be confused with the question of what the fundamental fields are, which in my opinion belongs to a different level of physics.

The fundamental fields we utilize are all Dirac spin 1/2 fields obeying Fermi statistics. (Here again there are other possibilities which we will ignore.) The left- and right-handed components of a Dirac spinor is defined by

$$\psi_L = \frac{1}{2} (1+\gamma_5)\psi$$

$$\psi_R = \frac{1}{2} (1-\gamma_5)\psi$$

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = -\rho_1 \tag{2}$$

A current j_μ is then of the form

$$j_\mu = i\bar{\psi}' \gamma_\mu (1+\gamma_5)\psi = i\bar{\psi}'(1-\gamma_5)\gamma_\mu \psi$$

$$= i\overline{[(1+\gamma_5)\psi']}\gamma_\mu \psi_2, \quad \bar{\psi} = \psi^+ \gamma_4 \tag{3}$$

Since this is not a lecture on weak interactions, we simply state that the currents j_μ are all charged type: $(\bar{\nu}e)$, $(\bar{\nu}'\mu)$ for the leptons, and the Cabibbo mixture $\bar{q}_1 q_2 \cos\theta + q_1 q_2 \sin\theta$ for the hadrons $(\sin\theta = 0.21)$.[5] We are ignoring the problem of well known CP violation as well as the possible existence of neutral $(\Delta Q = 0)$ and $\Delta Q = -\Delta S$ currents.

One of the consequences of the universal current-current theory is the non-renormalization (equality) of all allowed $\Delta S = 0$ Fermi (vector) transition amplitudes, which should be exact in the limit of zero momentum transfer and strict isospin conservation.[2] This is well established between the $\mu \rightarrow e$ decay and $n \rightarrow p$ decay amplitudes to within the somewhat uncertain electromagnetic correction and the Cabibbo factor $\cos\theta$ This non-renormalization is, as is well known, due to the current conservation:

$$\frac{\partial}{\partial x_\mu} j_\mu^{V; \Delta S = 0} = 0 \tag{4}$$

of the vector $\Delta S = 0$ part of the hadronic current since its

fourth component is the isotopic current density. Note that
Eq. (5) is a dynamical statement meaning that the strong
interaction Hamiltonian commutes with isospin. Technically
we can use the Ward identity, like in electrodynamics, to
prove that the renormalization factor $G_V/G_0 \cos\theta = Z_2/Z_1 = 1$.

Now turning to the axial vector (Gamow-Teller) part of
the current, j^A, we cannot easily make a similar argument since
it is not obvious that the axial vector charge (chirality)
$\bar{\psi}\,\gamma_4\gamma_5\psi = -\psi^+\rho_1\psi$ is a conserved quantity. Nevertheless, the
near-equality of Gamow-Teller and Fermi constants in nucleon
β decay: $-G_A/G_V = 1.18$[6], stimulated this conjecture of chirality
conservation. It appears, however, that there is no necessary
connection between non-renormalization and conservation in the
case of axial vector current. On the other hand, it turned
out that the so-called Goldberger-Treiman relation could be
"derived" from such an assumption independently of detailed
dynamics. Let us briefly sketch the argument.

Take the matrix element of the axial vector current
$j_\mu{}^A \equiv A_\mu = i\bar{\psi}\,\gamma_\mu\gamma_5\psi$ of a spin 1/2 particle. Its general form
is

$$\langle p' | A_\mu | p \rangle = \bar{u}_{p'} G_\mu u_p,$$

$$G_\mu = i\gamma_\mu\gamma_5 G_1 + \gamma_5 q_\mu G_2 + \gamma_5 \sigma_{\mu\nu} q_\nu G_3,$$

$$q = p'-p \tag{5}$$

where u_p, $u_{p'}$ are the initial and final spinors; G_1, G_2, G_3
are real form factors which depend on q^2. This result follows

from Lorentz invariance including parity and time reversal.
Under charge conjugation C, however, the first two terms are
even, and the third odd. If C is a good symmetry, then the
odd term (second class current) must be zero. This is true
of course only if the CP violation is ignored. Now the con-
servation of a_μ means $\partial_\mu a_\mu = 0$ or

$$q_\mu \langle p' | a_\mu | p \rangle = \bar{u}_{p'}(i\gamma \cdot q \gamma_5 G_1 + \gamma_5 q^2 G_2) u_p = 0 \qquad (6)$$

whether $G_3 = 0$ or not. From the Dirac equation

$$(i\gamma \cdot p + m) u_p = \bar{u}_{p'}(i\gamma \cdot p' - m) = 0 \qquad (7)$$

follows that

$$\bar{u}_{p'} i\gamma - q\gamma_5 u_p = -2m \bar{u}_{p'} \gamma_5 u_p.$$

So Eq. (7) will be satisfied if

$$-2m G_1 + q^2 G^2 = 0 \qquad (8)$$

Eq. (6) becomes thus[7]

$$G_\mu = [i\gamma_\mu \gamma_5 + 2m \gamma_5 q_\mu / q^2] G_1 / q^2 \qquad (9)$$

We have essentially only one form factor $G_1(q^2)$. The second
term of Eq. (9) is called the induced pseudoscalar term, which
is proportional to the gradient of a pseudoscalar density.

In the above argument we ignored isotopic spin, but it is
clear that Eq. (9), will hold between any two states p and p'
provided their masses are equal. (Then the G-parity will be
used instead of C regarding the question of G_3.)

Let us pursue the consequences of Eq. (9). For β decay
(proton $\rightarrow$ neutron) q^2 is small, and $G_1(g^2) \approx G_2(0)$ is equal to

1.18. In this static limit we have

$$i\gamma_\mu\gamma_5 + 2m_N\gamma_5 q_\mu/q^2$$

$$-\sigma_{\mu\nu}(\delta_{\mu\nu}-q_\mu q_\nu/q^2), \qquad \mu,\nu = 1,2,3$$

$$-i\sigma\cdot(p+p')/2m_N \qquad \mu = 4.$$

The spatial part is the main term, but it contains a large contribution from the induced pseudoscalar term, cancelling completely the longitudinal part of the Gamow-Teller operator σ. Such an effect is not observed in β decay.

We run into a more drastic trouble if we apply the same argument to $\pi\mu\nu$ decay. The relevant matrix element $\langle 0|a_\mu|\pi(q)\rangle$ $\propto q_\mu$ must vanish if $q_\mu a_\mu = 0$. Therefore, strict axial vectro conservation must be ruled out.

The reason for the above large correction for β decay is that the induced term has a long range from factor $\sim 1/q^2$, which can be produced only by an exchange of massless pseudoscalar field between nucleon and lepton. No such particles are known to exist, but we know that there exist pseudoscalar mesons (pions) which couple to the nucleon source. These mesons will contribute to an induced term with a characteristic factor $1/(q^2+m_\pi^2)$ instead of $1/q^2$. More precisely, the contribution from the pion pole to the weak axial current $(G_0 \cos\theta/\sqrt{2}) \langle n|a_\mu|p\rangle$, between proton and neutron is

$$g_{\pi NN}\ \gamma_5$$

where $\sqrt{2}\, g_{\pi NN}\, \gamma_5$ and $-g_\pi q_\mu$ represent the coupling of the pion to nucleon and the external field (lepton pair) respectively. Thus if we make the assumption that the original $1/q^2$ term is a hypothetical limit $m_\pi \to 0$ of this pion contribution, we obtain by comparing Eqs.(9) and (11)

$$g_{\pi NN}\, g_\pi = 2m_N\, G_1(0)\, G_0\, \cos\theta/\sqrt{2} = 2m_N\, G_A/\sqrt{2} \qquad (12)$$

This is the Goldberger-Trieman[8] relation which relates $g_{\pi NN}(g_\pi{}^2{}_{NN}/4\pi = 14.6)$, G_A $(= 1.18\times10^{-5}/m_p{}^2)$ and g_π, the $\pi - \mu\nu$ decay constant. Originally it was obtained using simple dynamical assumptions and dispersion relations.

Because in actuality $m_\pi \neq 0$, the real axial vector current cannot satisfy $\langle n|\partial_\mu a_\mu|p\rangle = 0$ (unless $G_1(q^2)$ has the form $[q^2/(q^2+m_\pi{}^2)]\, G_1'(q^2)$, $G_1'(0) \neq 0$, in which case, however, $G_1(0) = 0$ or $G_A = 0$, in contradiction with β decay. Thus $\partial_\mu a_\mu \simeq 0$ only to the extent that $m_\pi{}^2$ can be ignored compared to other parameters such as q^2 or $m_p{}^2$. In this sense it is known as the partially conserved axial vector current (PCAC) hypothesis. To show this situation more clearly, we use here a formulation due to Gell-Mann and Lévy.[9] Take the isotopic axial vector current $a_\mu{}^i = i\bar{q}\gamma_\mu\gamma_5\tau^i q$. Its divergence $\partial_\mu a_\mu{}^i$ has the same quantum numbers as the pion field ϕ^i. This is true at least if there are no hidden quantum numbers that can distinguish them. Then we can use $\partial_\mu a_\mu{}^i$ as the _definition_ of pion field after proper normalization

$$\partial_\mu a_\mu{}^i(x) = C\phi^i(x) \qquad (13)$$

where C is a constant. It is determined by the condition that the asymptotic part, or the Fourier component at $q^2 = -m_\pi^2$, of ϕ^i has the properly quantized value for a free meson field. Now the pion-nucleon coupling constant $g_{\pi NN}$ is defined by

$g_{\pi NN} i \bar{u}_2 \gamma_5 \tau^i u_1 = (q^2 + m_\pi^2)\langle 2|\phi^i|1\rangle$ (taken at the extrapolated value $q^2 = -m_\pi^2$), but from Eq. (13) this is also $= -(1/C)iq_\mu \langle 2|a_\mu|1\rangle = (1/C) i\bar{u}_2\gamma_5\tau^i u_1 (q^2+m_\pi^2)[G_1 - q^2 G_2]$ by using Eq. (6). Taking the limit $q^2 \to 0$ (instead of $q^2 = -m_\pi^2$), we get

$$g_{\pi NN}(0) = 2m_N m_\pi^2 G_1(0)/C$$

or

$$C = 1.18\, m_\pi^2/f_{\pi NN}(0),$$

where

$$f_{\pi NN}(0) \equiv g_{\pi NN}(0)/2m_N \tag{14}$$

In a similar fashion, the $\pi - \mu\nu$ decay constant is defined by $-g_\pi q_\mu = G_0(\cos\theta/\sqrt{2})\langle 0|a_\mu^i|\pi_q^i\rangle \sqrt{2E_\pi}$ so that using (13), $-g_\pi q^2 = -m_\pi^2 g_\pi = G_0 C(\cos\theta/\sqrt{2})\langle 0|\phi^i|\pi^i\rangle \sqrt{2E_\pi} = C G_0\cos\theta/\sqrt{2}$. This time g_π is defined at $q^2 = -m_\pi^2$.

Thus

$$C = \sqrt{2}\, m_\pi^2\, g_\pi/G_0\cos\theta \tag{15}$$

From (14) and (15) we obtain the Goldberger-Tremain relation (12), provided that $g_{\pi NN}(0)$ is used there. Note that $C \propto m_\pi^2$, meaning that $\partial_\mu a_\mu = 0$ in the limit $m_\pi \to 0$. In this formulation, therefore, one would not have obtained any useful information had we put $m_\pi = 0$ at the beginning. The definition (13) fails to define the pion field. Here it is important to distinguish between

external pion mass q^2 and internal pion mass m_π^2.

This trouble may be avoided if it is possible to write down a_μ^i as

$$a_\mu^i(x) = a_\mu'^i(x) + \frac{1}{f} \partial_\mu \phi^i(x) \tag{16}$$

where the right-hand side is expressed in terms of phenomeno-logical fields, $a_\mu'^i$ being appropriately defined currents for baryons and other sources. The condition

$$\partial_\mu a_\mu^i = \partial_\mu a_\mu'^i + \frac{1}{f} \Box \phi^i = 0 \tag{17}$$

serves both as a conservation law and as a wave equation for massless pion with coupling strength f. If $m_\pi \neq 0$, the wave equation becomes

$$\partial_\mu a_\mu^i = \frac{m_\pi^2}{f} \phi^i = C \phi^i \tag{18}$$

which is equivalent to Eq. (13), with

$$f = f_{\pi NN}(0)/G_1(0) = G_0 \cos\theta/\sqrt{2} \ g_\pi \tag{19}$$

In the following, we shall often use the notations of Eqs. (18) and (19). Experimentally

$$f = G_0 \cos\theta/\sqrt{2} \ g_\pi = G_V/\sqrt{2} \ g_\pi = 1.05/m_\pi$$

and

$$f_{\pi NN}(m_\pi)/G_1(0) = 0.8^4/m_\pi.$$

II. INTERPRETATION AND CRITIQUE OF PCAC RELATION

There remains the question of interpreting and justifying the PCAC hypothesis on a theoretical basis. We will discuss it from various angles.

1). First look at the Gell-Mann-Lévy ansatz $\partial_\mu a_\mu{}^i = C\phi^i$. As we have remarked already, this in itself should be considered a definition rather than an assumption. It is known that there is no unique way of defining a phenomenological field for a particle. An appropriate local operator like $\partial_\mu a_\mu{}^i$ will do if it has the right quantum numbers and is properly normalized.[10] We could, for example, also use $i\bar{q}\gamma_5\tau^i q$ with equal justification. Different definitions of a field agree by necessity on the mass shell of the particle, and may differ only as we go off the mass shell. Unless we know precisely what a pion field is (e.g., we know what a bare pion field is in the fundamental Lagrangian), there is no unique way of defining ϕ^i. Now it so happens that the pion is the lightest member of all hadrons, and especially the next states having the same quantum numbers are 3π configurations with mass $\geq 3m_\pi$. These belong to the off-mass-shell contributions. But since the mass ratio $(m_\pi/3m_\pi)^2 = 1/9$ is small. it may be reasonable to expect that near the pion mass shell $0 \lesssim q^2 \lesssim m_\pi{}^2$, the ambiguity, if there is any, of q^2 dependence will not be great. This ambiguity would show up in what we mean by $g_{\pi NN}(0)$ in Eq. (14) since it is an extrapolation from

the unambiguous mass shell value $g_{\pi NN}(q^2 = -m_\pi^2)$. Their difference involves contributions from higher states if we write down the dispersion relation for $g_{\pi NN}(q^2)$:

$$(1/C)(q^2+m_\pi^2)[F_1(q^2) - q^2 F_2(q^2)]$$

$$= g_{\pi NN}(-m_\pi^2) + (q^2+m_\pi^2) \int_{9m_\pi^2}^{\infty} \frac{\rho(\kappa^2)}{q^2+\kappa^2} d\kappa^2 \qquad (20)$$

Thus the PCAC assumption actually means the assumption about the smallness of off-mass-shell deviations.[13]

2) According to the first interpretation, PCAC was an accident in the sense that m_π happened to be small compared to all other hadron masses. Thus the same idea would not work nearly as well for K mesons, where the q^2 extrapolation ranges $0 \leq q^2 \leq m_K^2$, $m_K = 500$ Mev, and the next state $K\pi\pi$ has $m \geq 780$ Mev. A more radical view is that m_π is small not by accident, but for good reason. We have seen that if we let $m_\pi \to 0$ (not $q \to 0$), then $\partial_\mu a_\mu = 0$ according to (14) or (15); or we could have postulated it as in (17). So two things must go together: a_μ is conserved, and there exist massless "pions".

$\partial_\mu a_\mu^i = 0$ means that the Hamiltonian (at least the strong interaction part) commutes with isotopic chirality

$$\chi^i = \int a_0^i(x) \, d^3x \qquad (21)$$

and this in turn suggests that χ^i is a meaningful operator of

certain symmetry under which the Hamiltonian is invariant. If we use the quark representation, this operation

$$\chi^i = -\int q^+ \rho_1 \tau^i q \; d^3x \tag{22}$$

means that the left-handed and right-handed quarks are given isotopic rotations with opposite phases. Together with the ordinary isotopic rotation

$$I^i = \int q^+ \tau^i q \; d^3x \tag{23}$$

the combinations

$$\chi_{\pm}^i = (I^i \pm \chi^i)/2 \tag{24}$$

correspond respectively to isospin rotation on left-handed and right-handed components. The corresponding group is $SU(2)_L$ x $SU(2)_R$. Since all hadrons are made up of quarks according to our fundamental assumption, Eq. (24) defines chirality transformation for all hadrons. The fundamental Hamiltonian, involving only quark fields, must somehow have the property that if we ignore certain small terms, it becomes invariant under the operations (24), and at the same time the pion mass will come down to zero, but no other dramatic change will take place.

This is a very strange requirement, and does not look at all easy to realize. Perhaps it is unnatural, and the last interpretation above is the more reasonable one. But the fact is that it is possible to set up model systems which do satisfy the required conditions. We will postpone the details of such

examples to a later section.   Here we point out some features of axial vector conservation.

a).   Even if χ^i commutes with the Hamiltonian, physical eigenstates of H cannot in general be eigenstates of χ^i.   This is because, first of all, χ^i is a pseudoscalar.   So for example, a nucleon at rest, or a pion at rest cannot be an eigenstate. (Of course in our limit the pion will be massless so one should take an S wave packet.)   At any rate, χ^i can still commute with H if, for example, χ^i has eigenvalue 0, or there are degenerate states of opposite parities.   In fact there might exist a scalar σ meson into which π^i will be transformed by χ^i.   The nucleon might be coupled to the S_{11} resonance by χ^i.   For quarks themselves, also a similar situation should exist.   (If quarks are massless, they can be eigenstates of χ^i.)   Another possibility makes use of the concept of degenerate vacuum, as will be treated later.

b).   The amount of mixing of other states under the operation χ^i depends on the state of motion.   For example, take a single bare massive quark obeying Dirac equation.   We have

$$\langle \chi^i \rangle = - \langle \rho_1 \tau^i \rangle = - \langle \sigma \cdot v \tau^i \rangle = - h v \tau^i \tag{25}$$

where h is the helicity $\langle \sigma \cdot p \rangle / p$.   The Lorentz transformation enhances positive helicity component over the other, and as $v \to 1$ only the positive one survives.   Thus χ^i becomes diagonal. For real quarks and baryons there will be a cloud of (bare) quarks and antiquarks moving inside, relative to each other, so

the situation will be more complicated. But it shows the advantage of formally dealing with particles with infinite momenta in considering chiral symmetry.

c). The extension of chiral symmetry to SU(3) is of course a natural step. We will then switch off the K and η masses as well, which is a more drastic approximation than switching off the pion mass. The relevant $SU(3)_L \times SU(3)_R$ group was considered, among others, by Gell-Mann[11] and by Marshak et al.[12] There is, in addition, one more chiral transformation which is possible on the quark field. That is the simple γ_5 transformation corresponding to Eq. (1), and goes with the baryon number group to make $U(1)_L \times U(1)_R$. The SU(3) singlet pseudoscalar meson associated with this axial current may be the $\eta'(960)$. If this is the case, however, it must be a very bad symmetry since the η' mass is so high. It is a curious fact that the simpler symmetry should be the more approximate one, although it is again possible to cook up a model for this. (At any rate the large splitting between pseudoscalar octet and singlet, which does not seem to be the case with vector and higher mesons, cannot easily be understood in the quark model.)

3) We have mentioned two possible views of the PCAC. Can we distinguish between them? What other predictions will they lead to? We do not know a satisfactory answer to these, but is rather clear that the two are more or less equivalent as long as we consider only a single current operator, namely a single vertex function which involves only one pion. There may be differences if we

consider correlations between more than one current. As we
switch off the pion mass in each current one by one, we run
into the problem of non-commutative infrared pions. Namely
the result of switching of of pion masses involved depends on
the order in which the limiting processes are carried out, and
whether we take $q^2 \to 0$ or $m_\pi \to 0$, etc. This is understandable
since we cannot apply the earlier argument for the insensitivity
of matrix elements on q^2 or m_π^2 when there are other soft pions
in the process. The detailed behavior of these amplitudes will
depend on more than just the ordinary PCAC assumptions. The
current commutation relations will of course play an important
role in this respect, but more dynamics will have to be invoked,
especially for cases with three or more pions.

III. MODELS OF CHIRAL SYMMETRY

Before we embark on practical applications of PCAC relations,
we would like to look into some model Hamiltonians having chiral
symmetry. This may not be necessary if we take the standpoint 1)
of Section II. But we get more insight doing this, especially in
dealing with many-meson problems. Also it provides us with a gene-
ral way of finding currents satisfying the chiral commutator alge-
bra which we can retain even if the currents are not conserved in
reality.

The first problem we have to face in such an attempt is
that the mass term $\sim \overline{\psi}\psi$ of a Dirac field violates γ_5 symmetry

since it mixes left- and right-handed components:

$$\overline{\psi}\psi = \psi^+ \rho_3 \psi = \psi^+_L \psi_R + \psi^+_R \psi_L.$$

Different models solve it in different ways.

1) Gell-Mann-Lévy type model.[9,13] Consider quark field q, and a 3 x 3 complex matrix meson field $\mathcal{M}$ of mixed parity which operates on q. (The meson-baryon system is more complicated, but it does not change the essentials.) The Lagrangian is given by

$$- \mathcal{L} = q^+_L \gamma_\mu \partial_\mu q_L + q^+_R \gamma_\mu \partial_\mu q_R + g(q^+_R \mathcal{M} q_L + q^+_L \mathcal{M}^+ q_R)$$

$$+ \frac{1}{2} \mathrm{Tr} \partial_\mu \mathcal{M}^+ \partial_\mu \mathcal{M} + \frac{\mu_0{}^2}{2} \mathrm{Tr} \mathcal{M}^+ \mathcal{M} \tag{26}$$

This will be invariant under $U(3)_L$ x $U(3)_R$ if we define the transformation property by

$$q_L \to U_L q_L, \quad q_R \to \quad U_R q_R$$

$$\mathcal{M} \to U_R \mathcal{M} U^+_L \tag{27}$$

where U is a constant U(3) transformation matrix

$$U_L = \exp[i \sum_{i=0}^{8} \alpha_I{}^i \lambda^i],$$

$$U_R = \exp[i \sum_{i=0}^{8} \alpha_R{}^i \lambda^i] \tag{28}$$

Eq. (27) is always possible since no condition is imposed on $\mathcal{M}$. It indicates that $\mathcal{M} = (\mathcal{M}_{ik})$ must behave like $\mathcal{M}_k{}^i \sim (3_R, 3_L^*)$ with respect to the indices i, k. Under ordinary U(3), we have $U_L = U_R$, so $\mathcal{M} \to U \mathcal{M} U^+$. Under γ_5 U(3), we have $U_R = U_L^+$, so we

have $\mathcal{m} \to U^+ \mathcal{m} U^+$. In infinitesimal form, this means

$$\delta\mathcal{m} = i[\alpha \cdot \lambda, \mathcal{m}] \qquad (U(3))$$

$$\delta\mathcal{m} = -i\{\alpha \cdot \lambda, \mathcal{m}\} \qquad (\gamma_5 U(3)) \qquad (29)$$

Writing $\mathcal{m}$ as a sum of Hermitian and anti-Hermitian parts:

$$\mathcal{m} = S + iP = \sum_{i=0}^{8} \lambda^i S_i + i\lambda^i P_i,$$

we find that S and P behave as scalar and pseudoscalar respectively, with the transformation rule

$$\delta S = i[\alpha \cdot \lambda, S], \quad \delta P = i[\alpha \cdot \lambda, P] \qquad (U(3))$$

$$\delta S = \{\alpha \cdot \lambda, P\}, \quad \delta P = -\{\alpha \cdot \lambda, S\} \qquad (\gamma_5 U(3)) \qquad (30)$$

The Lagrangian (26) becomes, in terms of S and P,

$$-\mathcal{L} = \bar{q}\gamma_\mu \partial_\mu q + g(\bar{q}\lambda^i q S_i + i\bar{q}\gamma_5 \lambda^i q P_i)$$

$$+ \frac{1}{2}(\partial_\mu S_i \partial_\mu S_i + \partial_\mu P_i \partial_\mu P_i)$$

$$+ \frac{\mu_0^2}{2}(S_i S_i + P_i P_i) \qquad (31)$$

The conserved currents are:

$$v_\mu^i = i\bar{q}\gamma_\mu \lambda^i q + \frac{1}{2} T_r \partial_\mu \mathcal{m}^+ [\lambda^i, \mathcal{m}] - \frac{1}{2} T_r [\lambda^i, \mathcal{m}^+] \partial_\mu \mathcal{m}$$

$$= i\bar{q}\gamma_\mu \lambda^i q + i T_r \partial_\mu S[\lambda^i, S] + i T_r \partial_\mu P [\lambda^i, P]$$

$$a_\mu^i = i\bar{q}\gamma_\mu \gamma_5 \lambda^i q - \frac{1}{2} T_r \partial_\mu \mathcal{m}^+ \{\lambda^i, \mathcal{m}\} - \frac{1}{2} T_r \{\lambda^i, \mathcal{m}^+\} \partial_\mu \mathcal{m}$$

$$= i\bar{q}\gamma_\mu \gamma_5 \lambda^i q + T_r \partial_\mu S\{\lambda^i, P\} - T_r \partial_\mu P\{\lambda^i, S\} \qquad (32)$$

The bare quark mass must be zero, but the bare meson mass need not be zero. We can create quark mass $m \neq 0$ by putting in a bare

mass term $m_0\bar{q}q$ so that

$$\partial_\mu a_\mu{}^i = 2m_0\bar{q}\gamma_5\lambda^i q = -\frac{2m_0}{g}(\Box - \mu_0{}^2)P^i;$$

or we can assume the real vacuum to be of such a nature that
$\langle S^o(x)\rangle = \text{const} \neq 0$, so $m = g\langle S^o\rangle$, and $a_\mu{}^i$ splits into two
parts:

$$a_\mu{}^i = a_\mu{}^{i\prime} - 2\langle S^o\rangle \partial_\mu P^i = a_\mu{}^{i\prime} - \frac{2m}{g}\partial_\mu P^i.$$

In the case of $U(2)_L \times U(2)_R$, we can restrict ourselves to
a 2 x 2 matrix $\mathcal{M}$ operating on an isotopic doublet q or ψ_N.
$\mathcal{M}$ and $\mathcal{M}^+$ will behave as (2,2*) and (2*,2). Because $2 \approx 2^*$,
we obtain two separate real quartets, one consisting of a pseudo-
scalar isotriplet and a scalar isosinglet (π^i, σ), and the other
consisting of a scalar isotriplet and a pseudoscalar isosinglet
$(\pi^{i\prime}, \sigma')$. The former is the simplest assignment for the pion in
the context of $SU(\mathbf{2})_L \times SU(\mathbf{2})_R$, and was considered by Gell-Mann
and Lévy.[9] This group is equivalent to SO(4).

There is a different possibility in a model of this type.
Instead of $\mathcal{M}$ and $\mathcal{M}^+$, we may define Hermitian matrices $\mathcal{M}_L$
and $\mathcal{M}_R$ which transform as

$$\mathcal{M}_L \to U_L \mathcal{M}_L U_L{}^+, \quad \mathcal{M}_R \to U_R \mathcal{M}_R U_R{}^+ \tag{33}$$

and couple to quarks like

$$if(q_L{}^+\gamma_\mu \partial_\mu \mathcal{M}_L q_L + q_R{}^+\gamma_\mu \partial_\mu \mathcal{M}_R q_R).$$ The $\mathcal{M}$'s then
belong to (8,1) + (1,8) rather than (3,3*) + (3*,3). At the
SU(2) x SU(2) level, the pions will belong to (3,1) + (1,3).
This model, however, does not provide a natural relation between

quark mass and the mesons. As soon as the quark mass is created, mesons of the $(3,3^*)$ type will be induced.

2) Nishijima-Gürsey type model.[14] We can subject the meson matrix $\mathcal{M}(x)$ to the unitarity condition

$$\mathcal{M}^+\mathcal{M} = \text{const} = 1 \tag{34}$$

without spoiling the transformation law (27). Eq. (34) may be satisfied, for example, by setting

$$\mathcal{M} = \exp[2if \sum_{i=0}^{8} \lambda^i \phi_i] \tag{35}$$

or

$$\mathcal{M} = (1 + i f \sum_{i=0}^{8} \lambda^i \phi_i)/(1 - i f \sum_{i=0}^{8} \lambda^i \phi_i) \tag{35a}$$

where the ϕ's are 9 pseudoscalar fields. (If we want only $SU(3)_L \times SU(3)_R$, 8 fields are sufficient with the form (35). This does not work with (35a), however.) Then

$$- \mathcal{L} = \bar{q}\gamma_\mu \partial_\mu q + m_0(q_R^+ \mathcal{M} q_L + q_L^+ \mathcal{M}^+ q_R)$$

$$+ \frac{1}{2} \text{Tr} \partial_\mu \mathcal{M}^+ \partial_\mu \mathcal{M} \tag{36}$$

By expanding $\mathcal{M}$ in powers of ϕ_i, we find that it is a highly non-linear system, of which the lowest order terms are

$$- \mathcal{L} = \bar{q}\gamma_\mu \partial_\mu q + m_0\bar{q}q + 2m_0 f_i \bar{q}\gamma_5 \lambda_n q \phi_n$$

$$+ \frac{1}{2} \partial_\mu \phi_n \partial_\mu \phi_n + \cdots \tag{37}$$

In this model, we do not need independent scalar fields, but the ϕ's must undergo a complicated non-linear transformation in such

a way as to satisfy (27). The conserved currents are also non-linear.

Recently M. Sugawara[15] posed the question of constructing vector and axial vector currents which satisfy the Gell-Mann algebra but are expressed in terms of pseudoscalar meson (and baryon) fields only. The Gürsey-Nishijima-type model mentioned above of course possesses currents of this nature, but he constructs, working from the commutation relations, the following form of axial charge for pseudoscalar mesons:

$$(a_0)_a{}^b = \frac{1}{f} (\Pi)_a{}^b + f(\phi\Pi\phi)_a{}^b \tag{38}$$

Here ϕ and Π are the meson field and its canonical conjugate, each being regarded as a 3 x 3 matrix (nonet of mesons).

The vector currents are of course given by the usual expression

$$(\upsilon_0)_a{}^b = i(\phi\Pi - \Pi\phi)_a{}^b \tag{39}$$

It is easy to verify the commutation relations among υ's and a's. However, singularities due to the non-linearity of a_0 makes these relations somewhat superficial. Turning Sugarawa's question around, we may ask: Is there a Lagrangian that will lead to these conserved currents? J. Cronin[14] has studied this problem, and has found that the Lagrangian (36) with the choice (35a) of $\mathcal{m}$ leads exactly to the Sugarawa form of meson currents. Whether this Cayley form is physically more meaningful than the exponential form (35) is not clear.

3) Marshak-Okubo type model.[12] If the quark is the only funda-
mental field, we must write down the fundamental Lagrangian in
terms of q. A typical one is the Heisenberg type non-linear
model

$$- \mathcal{L} = \bar{q}_L \gamma_\mu \partial_\mu q_L + \bar{q}_R \gamma_\mu \partial_\mu q_R + \lambda [(\bar{q}_L \gamma_\mu q_L)(\bar{q}_L \gamma_\mu q_L)$$

$$+ (\bar{q}_R \gamma_\mu q_R)(\bar{q}_R \gamma_\mu q_R)]$$

$$= \bar{q} \gamma_\mu \partial_\mu q + \lambda [(\bar{q} \gamma_\mu q)(\bar{q} \gamma_\mu q) + (\bar{q} \gamma_\mu \gamma_5 q)(\bar{q} \gamma_\mu \gamma_5 q)]$$

$$(40)$$

Since q_L and q_R are completely independently of each other,
there is no communication between them. The communication must
be established, for example, by a "small" bare mass term which
will break $U(3)_L \times U(3)_R$. Actually, $\mathcal{L}$ possesses a larger
symmetry, but its real significance is not clear. As this is
only a model, perhaps we should not attach too much significance
to the particular form of interaction, especially since we do
not know whether the real hadrons actually can dynamically arise
from such a Lagrangian.

4) Spontaneous breakdown of chiral symmetry. Adopting the
quark model as we do, it is clear that no bare mass is allowed
if exact chiral symmetry is to hold. If quarks exist as par-
ticles, however, they are probably very massive; at least they
cannot be massless. On the other hand, we have approximate
axial vector conservation where the only visible violation is

symbolized by the pion (and κ, η) mass. So the question is whether a large real quark mass is compatible with a small violation of γ_5 symmetry.

One answer to that question is the concept of spontaneous breakdown.[16] It means, essentially, that the system is dynamically unstable against a small perturbation , such as a bare mass term, which breaks the γ_5 symmetry. Thus if we introduce a bare mass m_0, it will create a finite quark mass m which, in perturbation theory will be

$$m = m_0 + C_1 m_0 + C_2 m_0^2 + C_3 m_0^3 + \ldots \qquad (41)$$

where C_1, C_2, ..., depend on the interaction constant. If this expansion converges, then $m \to 0$ as $m_0 \to 0$.

However, it may not converge for any finite m_0, and there are examples of this. One is the energy gap (which corresponds to m) in superconductivity. In such a case, the successive terms of expansion become larger and larger, so the system is unstable against forming a finite mass, or tends to create mass spontaneously. We have to treat the problem in a different way, namely first guess the final answer, and then see whether it is is self-consistent. The expansion should be done not in terms of m_0 but in terms of assumed m (like in renormalization theory), and the self-consistency takes the form

$$m = m_0 + C_1^{1} m + C_2^{1} m^2 + C_3^{1} m^3 + \ldots = m_0 + mF(m) \qquad (42)$$

where $F(m)$ will now converge. Then in the limit $m_0 \to 0$ we can

get a trivial solution $m = 0$ as well as a non-trivial one $F(m_1) = 0$, $m_1 \neq 0$. For a small finite m_0, m will deviate only slightly from the equilibrium position m_1. The solution m_1, however, is not unique when $m_0 = 0$. This is because in this limit there is no communication between q_L and q_R; consequently the phase difference between them loses physical meaning. Of course a mass term

$$m\bar{q}q = m(q_R^+ q_L + q_L^+ q_R)$$

of the real particle will change into

$$m(\bar{q}q \cos 2\alpha + i\bar{q}\gamma_5 q \sin 2\alpha) \tag{43}$$

under the γ_5 phase transformation

$$q_L \to \exp(i\alpha)q_L, \quad q_R \to \exp[-i\alpha]q_R,$$

but it does not mean parity violation. The parity operator must be redefined after the γ_5 transformation.

The phase α can be fixed only by the phase of the external perturbation m_0: first give a phase α to m_0, and then let $m_0 \to 0$, we obtain a finite mass term with a definite phase α as in Eq. (43).

This continuous degeneracy of solutions is associated with a corresponding degeneracy of "vacuum". It turns out that once a phase is chosen, we pick up a particular vacuum state, and starting from it we get a complete set of physical states. Solutions with different phases cannot coexist, namely they belong to different physical Hilbert spaces.

The situation is similar to ferromagnetism where there are an infinity of ferromagnetic states with different orientations of magnetization. The orientation cannot be changed from within. Only an external force (like magnetic field) will do it, which breaks the isotopy (rotational symmetry) of the system. In superconductivity, the phase α corresponds to the phase of the macroscopically occupied Cooper pair states. This phase is unobservable in a single superconductor. But disconnected super-conducting regions (different "worlds") may have their own phases, which show up when communication is established between them, like in the Josephson effect.

The degeneracy of vacuum is accompanied by the existence of acoustic-type (massless) excitations, or zerons.[16,17,18] This statement is the context of the so-called Goldstone Theorem. It corresponds to assigning a phase α(x) in the mass term which changes from place to place instead of being a constant. The corresponding ground vacuum state will now be modified (excited) by virtual creation and annihilation of pairs due to the oscillating part of α. Choosing $\langle \alpha \rangle = 0$, we have, according to (43), the effective mass term in the Lagrangian becomes

$$m\bar{q}q + 2i\bar{q}\gamma_5 q\alpha(x)$$

for small α(x). From this we see that the excitation coupled to α(x) is a pseudoscalar. Also we can argue that the excitation must be massless from the fact that in the long wavelength limit, α(x) → constant, the excitation energy becomes zero because of the degeneracy.

This massless excitation is the analog of the "backflow" introduced by Feynman[19] in discussing the motion of excitations in a superfluid. The backflow re-establishes the overal continuity of matter when an excitation is set in motion. In a similar way, when a massive quark is accelerated, it emits a massless pseudoscalar excitation ("pion") to re-establish the conservation of chirality so that the continuity equation $\partial_\mu a_\mu = 0$ is maintained, even if a massive quark alone is not γ_5 invariant. It is the quark plus the surrounding medium that has to honor the symmetry of dynamical equations.

5) Derivative coupling model. Take the Lagrangian

$$- \mathcal{L} = \bar{q}\gamma_\mu \partial_\mu q + m_0 \bar{q}q + if\bar{q}\gamma_\mu\gamma_5\lambda^i q \, \partial_\mu \phi^i$$
$$+ \frac{1}{2} \partial_\mu \phi^i \partial_\mu \phi^i \tag{44}$$

The mesons are massless, and have derivative coupling to the quarks, or for that matter, to the baryons too. $\mathcal{L}$ is invariant under $\phi^i \to \phi^i +$ const., so there is a conserved current

$$a_\mu^{\;i} = i\bar{q}\gamma_\mu\gamma_5\lambda^i q + \frac{1}{f} \partial_\mu \phi^i \tag{45}$$

the divergence of which just amounts to the meson equation:

$$f \, \partial_\mu a_\mu^{\;i} = \Box \phi^i + \partial_\mu(if\bar{q}\gamma_\mu\gamma_5 q) = 0 \tag{46}$$

Interestingly, the bare mass of the fermions need not be zero since they do not take part in the symmetry operation. From this we realize that the symmetry group is not $SU(3)_L \times SU(3)_R$. [It is a semi-direct product of $SU(3)$ and a displacement

group.] Since the correct group structure appears to be $SU(3)_L \times SU(3)_R$ from the test of current commutation relations, this model may not be of interest. But Eq. (45) is a convenient way of effectively describing the content of PCAC from the second viewpoint of Section II when only matrix elements of a single current operator $a_\mu{}^i$ are considered (but not products of currents). In this case one interprets (45) as being expressed in terms of renormalized operators. Thus, in dealing with nucleons and pions, we replace it by

$$a_\mu{}^i = G_1(0) \, \overline{\psi}_r \gamma_\mu \gamma_5 \tau^i \psi_r - \frac{1}{f} \partial_\mu \phi_r{}^i \tag{47}$$

(ψ_r, etc. stands for renormalized operators.) f is equal to $f_{\pi NN}(0)/G_1(0)$, and clearly the first and second terms are responsible for the primary and induced pseudoscalar terms of the nucleon current vertex G_μ, Eq. (9). In addition (47) contains an asymptotic meson part $-(1/f) \partial_\mu \phi_r{}^i$ which determines the π decay amplitude $g_\pi = G_0/f$.

IV. CONSEQUENCES OF PCAC RELATIONS--FORMULA FOR SOFT PION EMISSION

First we will discuss the consequences of the PCAC hypothesis in problems where only one pion (pseudoscalar meson) is involved. As we have seen, there are two interpretations of PCAC, and correspondingly two derivations of the result, but they are not really different. The one version assumes

essentially the chirality conservation to be exact ($m_\pi = 0$) apart from specific violation terms, while the other relies on the relation $\partial_\mu a_\mu{}^i = C\phi^i$. Let us try both of them.

a)[20,21] We write the total Hamiltonian as

$$H = H_0 + H'$$

where

$$[\chi^i, H_0] = 0, \quad [\chi^i, H'] \neq 0$$

Then

$$i\dot{\chi}^i = [\chi^i, H']$$

so that

$$\chi^{i\,out} = \chi^{i\,in} - i \int_{-\infty}^{\infty} [\chi^i(t), H'(t)]dt = S^{-1}\chi^{in}S$$

or

$$\chi^{i\,in}S - S\chi^{i\,in} = -iS \int_{-\infty}^{\infty} [\chi^i(t), H'(t)]dt \tag{48}$$

where S is the S matrix. χ^{in} and χ^{out} contain, among other things, baryon and meson parts:

$$\chi^{in} = \chi_B{}^{in} + \chi_M{}^{in},$$

and

$$\chi_B{}^{i,in} = \hat{g}_1(0) \quad \bar{\psi}_B{}^{in}\gamma_4\gamma_5\tau^i\psi_B{}^{in}d^3x$$

$$\chi_M{}^{i,in} = \frac{1}{f}\int \phi^{i,in}d^3x \tag{49}$$

according to Eq. (47).

The meson part $\chi_M{}^{i,in}$ corresponds to a process in which a soft meson ($\underset{\sim}{k} = 0$, $k_0 \to 0$) is emitted or absorbed. Thus Eq. (48)

leads to a relation between an arbitrary process $A \to B$ and a radiative process $A \to B + \pi$ of a soft meson:

$$\lim_{\underset{\sim}{k}=0, \; k_0 \to 0} \frac{i}{f} \int \mathcal{X}^i_{rad} = [\; \mathcal{X}_B^{i,in}, \mathcal{m} \;] + S[\mathcal{X}^i, H'] \tag{50}$$

where $\mathcal{m}$ and $\mathcal{m}^i_{rad}$ are invariant amplitudes for non-radiative and radiative processes. The limiting procedure is for avoiding the vanishing of $\mathcal{X}_M(k) \sim \sqrt{k_0}$. This factor is actually cancelled by the normalization $\sim 1/\sqrt{k_0}$ of the meson wave function. On the right-hand side, the first term is computed by inserting only real intermediate states (energy shell), whereas in the second term we have run over a complete set of intermediate states. The latter can also be computed directly if we know the transformation property of H' under $\mathcal{X}^i$.

The first term of Eq. (50) can be simply written

$$[\; \mathcal{X}_B^{i,in}, \mathcal{m} \;] = (\mathcal{X}_B^i)_f \, \mathcal{m} - \mathcal{m} \, (\mathcal{X}_B')_i \tag{51}$$

where $(\mathcal{X}_B^i)_{i,f}$ is the chirality $-h\sigma\lambda^i$ for the initial or final baryons if only one baryon is involved. So the soft pion is emitted when the baryon chirality changes (the first term, which may be called surface term), or when the chirality conservation is manifestly disturbed by H' (the second term). If $\mathcal{m}$ itself is induced by H' alone, Eq. (50) becomes

$$\frac{i}{f} \mathcal{m}_{rad} = -[\mathcal{X}_B^{in}, S_0 H'] + S_0[\mathcal{X}, H'] \tag{52}$$

to the first order in H'. So is the S matrix due to H_0 alone, and symbolizes the final state interaction.

b) Now the second derivation.[2,22] The amplitude $\mathcal{M}_{rad}$ is given by

$$\mathcal{M}^i_{rad} = i \int d^4x \, e^{+ikx}(m_\pi^2 - \Box)<B|\phi^i|A>$$

$$= i \int d^4x \, e^{ikx}(m_\pi^2 - \Box)<B|\frac{1}{C}\partial_\mu a_\mu^i|A>$$

$$= -i \int d^3x \, e^{ikx} \frac{m_\pi^2 - \Box}{C} <B|a_0^i|A> \Big|_{t=-\infty}^{t=\infty}$$

$$+ \int d^4x \, e^{ikx} \frac{m_\pi^2 - \Box^2}{C} k_\mu <B|a_\mu^i|A> \tag{53}$$

By taking the limit $k_\mu \to 0$, we have the first term left, which is $-i(m_\pi^2/C)(\chi^{i,out} - \chi^{i,in})$. This is the same formula as (50) if we evaluate it from (48). Since we have taken $m_\pi^2 \neq 0$, and $k \to 0$, it is not an amplitude for real meson emission. So there is an underlying assumption that $\mathcal{M}_{rad}$ is a slowly varying function of k^2. In contrast, the first derivation assumes smooth behavior on k^2 and m_π^2, maintaining $k^2 + m_\pi^2 = 0$.

In case H' is the sole agent of the reaction, we can explicitly write

$$\mathcal{M} = -<B^{out}|H'(0)|A^{in}>$$

$$\mathcal{M}_{rad} = -i \int d^4x \, e^{ikx}(m_\pi^2 - \Box^2)<B^{out}|[\phi^i(x), H'(0)]|A^{in}>$$

$$\times \theta(-x_0), \tag{54}$$

replace ϕ^i by $(1/C) \partial_\mu a_\mu{}^i$, and carefully integrate by parts, watching the factor $\theta(-x_0)$. We arrive at the same formula (52), in which the surface term arises from the upper limit of time integration.

We can analyze the meaning of Eq. (52) in terms of Feynman diagrams. The first term involves the operator $\chi_B{}^i$ for initial and final single baryons. Thus it corresponds to meson lines attached to those single baryon lines, or the so-called baryon pole diagrams. The meson is coupled to the baryon via derivative coupling, as in Eq. (43). Such a term has an energy denominator $E - E_n \sim \sqrt{m_\pi{}^2 + k^2} \sim m_\pi$ which vanishes in the limit $k \to 0$, $m_\pi \to 0$. No other graphs will produce this type of singularity. Actually the derivative coupling is proportional to k_μ, so in the above limit, numerators and denominators cancel to give a finite value $\sim \chi_B{}^i \sim$ hv which however depends on the coordinate system in which the limit is taken.

The situation is very similar to the infrared problem, where the infrared singularity comes from photons emitted from initial or final particle lines.

The difference is that in the present case the amplitude at the singularity is finite but non-unique.

The second term of Eq. (52), on the other hand, represents contributions from non-pole diagrams in the limit k, $m_\pi = 0$. Taking a specific model Hamiltonian, we can see how various contributions cancel each other and the final result depends only on the presence of the symmetry violating term H'.

The above analysis is similar to that of the infrared problem due to F. Low.[24] Both are based on the symmetry property (γ_5 and gauge invariance). Thus it is not surprising that it is also possible to derive them by means of the Ward identity technique.[25]

As a final remark, we point out that taking the limit, $\underset{\sim}{k} \to 0$ and $m_\pi \to 0$, amounts to running away with the meson at light velocity. In this reference frame all the rest of the (massive) particles participating in the reaction will also be running with light velocity. This is an interpretation of the prescription $v \to c$ (or $p \to \infty$) in the general formalism of Fubini and Furlan.[26] It is related to the simple fact that Dirac particle, whether free or quasi-free (like in a loosely bound system or in the static quark model of hadrons), become diagonal in chirality (or helicity) because the Lorentz transformation enhances one helicity component and suppresses the other.

V. APPLICATION TO WEAK AND ELECTROMAGNETIC PROCESSES

The formula (50) or (52) should apply to all cases in general, but so far interesting results have come out mostly in weak and electromagnetic transitions where H' is theoretically well defined.

1) Photo-pion and electro-pion production.[21,23,25] We relate the two processes $\gamma + N \to N$ and $\gamma + N \to N + \pi$, or $e + N \to e + N$ and $e + N \to e + N + \pi$. Since the first reaction $\gamma + N \to N$ is possible only for a virtual γ, actually we regard it as a limiting case of the electroproduction via a virtual photon field $A_\mu(q)$. The Hamiltonian H' is in this case

$$H' = -ie\bar{q}\gamma_\mu\lambda_Q qA_\mu, \quad \lambda_Q = \frac{1}{2}(\lambda_3 + \frac{1}{\sqrt{3}}\lambda_8) \qquad (55)$$

and its matrix element $\mathcal{M}$ between nucleons is

$$\mathcal{M} = e\, J_\mu A_\mu, \qquad (56)$$

where J_μ is the well-known electromagnetic current vertex equipped with electric and magnetic form factors:

$$J_\mu = i\gamma_\mu F_1(q^2) - i\frac{\sigma_{\mu\upsilon}q_\upsilon}{2m_N}F_2(q^2) \qquad (57)$$

On the other hand, $[\chi^i, H']$ corresponds to a chiral transformation on H', changing a vector into an axial vector:

$$[\chi^i, H'] = -ie\bar{q}\gamma_\mu\gamma_5[\lambda^i, \lambda_Q]q\, A_\mu$$

$$= ie\epsilon_{3ik}a_\mu^k A_\mu \quad \text{for} \quad i = 1, 2, 3. \qquad (58)$$

Its nucleon matrix element is given by Eq. (9) (with the replacement $q^2 \to q^2 + m_\mu^2$ of course). In this way we obtain the formula

$$i\mathcal{M}_{rad}^i = e\, f\, [\chi_N^{i, in}, J_\mu(q)A_\mu(q)]$$

$$+ ie\, f\epsilon_{3ik}G_\mu(q)A_\mu(q) \qquad (59)$$

The first term represents pion emission from the nucleon via derivative coupling, the second term is the sum of the induced contact term ($\partial_\mu \to \partial_\mu - ieA_\mu$ in the derivative coupling) and the photomesic emission term. Note that only the contact term survives in the limit $q \to 0$ and pion four-momentum $k \to 0$, and lead to the Kroll-Ruderman theorem. In fact, its coefficient is $efF_1(0) = ef_{\pi NN}(0)$. (Eq. (59) does not satisfy gauge invariance unless the axial current G_μ is strictly conserved. For an ad hoc prescription to avoid it, see Reference 21. However, the best way to utilize Eq. (59) will be as a boundary condition on $\mathcal{M}_{rad}$ at the unphysical meson momentum $k_\mu = 0$.)

2) Neutrino-pion production.[21,22,23,25] In the same fashion as above, we can discuss the processes

$$\nu + N \to e(\mu) + N$$
$$\nu + N \to e(\mu) + N + \pi$$

We use the basic weak current-current interaction (1) instead of the electromagnetic current-current interaction. We expect therefore to obtain a formula similar to (59). Adler and co-workers have made a detailed analysis of the electron and neutrino-induced reactions following the PCAC. We refer to their papers for the details.

3) Leptonic decays of baryons and mesons. In this category we have processes like

$$\mathcal{M} \quad : \quad B \rightarrow B' + e + \nu$$

$$\mathcal{M}_{rad}: \quad B \rightarrow B' + e + \nu + \pi$$

for the baryons, and

$$\mathcal{M} \quad : \quad M \rightarrow e(\mu) + \nu \qquad\qquad\qquad (60a)$$

$$\mathcal{M}_{rad}: \quad M \rightarrow e(\mu) + \nu + \pi \qquad\qquad\qquad (60b)$$

for the mesons ($M = \pi$ or K). [Strictly speaking, the last one involves two mesons but we treat it here.] The soft pion emission from hyperons is energetically possible, but the branching ratio is too small ($\lesssim 10^{-4}$) to be of interest. (Some investigations were done by L. Clavelli.) The meson processes ($K_{\ell 3}$) have been treated by Callan and Treiman.[27]

Separating out the leptonic part, it is sufficient to consider a matrix element of the type

$$\langle b | J_\mu | a \rangle$$

where J_μ is the relevant hadronic weak current. The state a is K or π, and b is the vacuum or π for the two reactions (60a) and (60b). The first process is via the axial vector, and second is via the vector part of J_μ. So the latter is simply of the form

$$2\sqrt{E_p E_p'} \, \langle p' | J_\mu | p \rangle = G_V [(p+p')_\mu f_+^{(K \text{ or } \pi)} + (p-p')_\mu$$

$$\times f_-^{(K \text{ or } \pi)}] \times \begin{cases} \sin\theta & (K \rightarrow \pi) \\ \cos\theta & (\pi \rightarrow \pi) \end{cases} \qquad\qquad (61)$$

where f_+ and f_- are form factors being functions of $q^2 = (p'-p)^2$. In the limit of strict vector current conservation $m_a = m_b$,

we expect $G_V = G_0$, $f_+(0) = 1$, $f_-(0) = 0$, but we do not have a theory to compute f_+ and f_- in general. For simplicity, we have suppressed the SU(3) indices in (61). For the process (60a), on the other hand, we may replace J_μ by

$$(G_0/\sqrt{2})(-1/f_\pi)\, \partial_\mu \phi_\pi \cos\theta \quad \text{or} \quad (G_0/\sqrt{2})(-1/f_K)\, \partial_\mu \phi_K \sin\theta \quad (62)$$

in accordance with Eq. (47) et seq. So

$$\sqrt{2E_\pi}\langle 0 | J_\mu | \pi(q)\rangle = -(G_0/\sqrt{2}f_\pi)\cos\theta\, q_\mu = g_\pi q_\mu$$

$$\sqrt{2E_K}\langle 0 | J_\mu | K(q)\rangle = -(G_0/\sqrt{2}f_K)\sin\theta\, q_\mu = g_K q_\mu \quad (63)$$

which should be correct in the limit $m_\pi \to 0$. In the SU(3) limit $(m_\pi = m_K)$, we expect of course $f_\pi = f_K = f$. Since $G_0 \cos\theta/f_\pi = G_0 G_1(0) \cos\theta/f_{\pi NN}(0) = -G_A/f_{\pi NN}(0)$, (62) is nothing but the Goldberger-Treiman relations applied to π and K, appropriately corrected for the Cabibbo angle.

Essentially the same results can be derived from the soft-pion formula (52). Since no baryons are involved, only the second term is relevant, where $H' = J_\mu$. Now J_μ involves only the q_L, for which $\gamma_5 = +1$. Thus a chiral transformation is equivalent to an ordinary SU(3) transformation. This is a special feature of the weak currents. $[\chi^i, J_\mu]$ is simply a different isospin component of the same J_μ octet. In this way we can obtain the connection between (60a) and (60b), with the final meson momentum p' put equal to zero in Eq. (61) as the soft pion limit:

$$G_V(f_+^{(K)} + f_-^{(K)})\sin\theta = g_K f_\pi = g_K f_{\pi NN}(0)/G_1(0) \quad (64)$$

The form factors are to be evaluated at the (unphysical) point

$m_\pi = 0$, $q^2 = m_K^2$. Taking straight experimental values, this equation is actually well satisfied, although there is a large experimental uncertainty on the left-hand side (i.e., the ratio f_-/f_+). There is a slight difference between the above two approaches. In the second one, we did not use the K-current in Eq. (62) to compute $K \rightarrow e(\mu) + \nu$ amplitude explicitly, but only applied the soft pion formula to relate the two observed amplitudes. Hence the appearance of the form factors on the left-hand side of (64). One would expect that the large m_K, violating both SU(3) ($m_K \neq m_\pi$) and γ_5 SU(3) ($m_K \neq 0$), would make the PCAC relation for the K currents unreliable. It is somewhat surprising that we have still $f_\pi \approx f_K$ in spite of the K π mass difference ($f_K/f_\pi \sim 1.28$ if $\sin\theta_A = \sin\theta_V = 0.21$).

4) Non-leptonic decays of baryons. Non-leptonic processes are a more significant test of both the current-current weak Hamiltonian and the soft-pion formula. Here the basic processes are

$$\mathcal{M} \ : \ B \rightarrow$$

$$\mathcal{M}_{rad} \ : \ B \rightarrow B' + \pi$$

which are caused by the $\left| \Delta S \right| = 1$ part

$$H' = \frac{G_0}{2\sqrt{2}} \{ j_\mu(\Delta S = 1), \ j_\mu^+(\Delta S = 0) \} + h.c.$$

$$= -\frac{G_0}{2\sqrt{2}} \cos\theta \sin\theta \{ \bar{q}_1 \gamma_\mu (1+\gamma_5) q_3, \ \bar{q}_2 \gamma_\mu (1+\gamma_5) q_1 \} + h.c.$$

$$(65)$$

of the hadronic current-current interaction. Some general consequences of this assumption are well known. In particular, a) it is CP invariant; b) it is invariant under the reflection $q_2 \leftrightarrow q_3$; c) it consists of SU(3) spurions

$$8(|\Delta I| = 1/2) + 27(|\Delta I| = 1/2) + 27(|\Delta I| = 3/2)$$

if the quarks obey Fermi statistics. (For Bose statistics, we have only $8(|\Delta I| = 1/2!)^*$

If we assume only the octet part to be present, we obtain the $|\Delta I| = 1/2$ rule within each isotopic multiplet, and the Lee-Sugawara sum rule linking different multiplets for the parity violating amplitudes (S wave pion emission). These are well satisfied experimentally.

The application of the PCAC formula was first done by Suzuki[28] and by Sugawara.[29] The amplitude $\mathcal{M}$ is a SU(3) tensor spurion

$$\langle B_i^{\ i} | H' | B_k^{\ \ell} \rangle = \bar{u}_i^{\ j} \mathcal{M}_{j\ell}^{\ ik} u_k^{\ \ell} \qquad (66)$$

which behaves as $|\Delta S| = 1$, $|\Delta I| = 1/2$ or $3/2$, $\Delta Q = 0$ members of 8_s, 8_a and 27. $\mathcal{M}$ is further divided into scalar (parity-conserving) and pseudoscalar (parity-violating) parts, so that

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*This possibility is being examined

$\mathcal{M} = S + \gamma_5 P$. We can write

$$S_{j\ell}^{ik} = S(8_{\mathbf{s}})[(\delta_\ell{}^i \delta_j{}^2 \delta_3{}^k + \delta_\ell{}^i \delta_j{}^3 \delta_2{}^k)$$

$$+ (\delta_j{}^k \delta_2{}^i \delta_\ell{}^3 + \delta_j{}^k \delta_3{}^i \delta_\ell{}^2)]$$

$$+ S(8_{\mathbf{a}})[(\delta_\ell{}^i \delta_j{}^2 \delta_3{}^k + \delta_\ell{}^i \delta_j{}^3 \delta_2{}^k)$$

$$- (\delta_j{}^k \delta_2{}^i \delta_\ell{}^3 + \delta_j{}^k \delta_3{}^k \delta_\ell{}^2)]$$

$$+ S(27)[(\delta_1{}^i \delta_j{}^1 - \tfrac{1}{3}\delta_j{}^i)(\delta_\ell{}^2 \delta_3{}^k + \delta_\ell{}^3 \delta_2{}^k)$$

$$+ \text{perm } (j\ell),\ (ik)]$$

$$P_{j\ell}^{ik} = P(8_{\mathbf{s}})[(\delta_\ell{}^i \delta_j{}^2 \delta_3{}^k - \delta_\ell{}^i \delta_j{}^3 \delta_2{}^k)$$

$$+ (\delta_j{}^k \delta_3{}^i \delta_\ell{}^2 - \delta_j{}^k \delta_2{}^i \delta_\ell{}^3)]$$

$$+ P(8_{\mathbf{a}})[(\delta_\ell{}^i \delta_j{}^2 \delta_3{}^k - \delta_\ell{}^i \delta_j{}^3 \delta_2{}^k)$$

$$- (\delta_j{}^k \delta_3{}^i \delta_\ell{}^2 - \delta_j{}^k \delta_2{}^i \delta_\ell{}^3)]$$

$$+ P(27)[(\delta_1{}^i \delta_j{}^1 - \tfrac{1}{3}\delta_j{}^i)(\delta_\ell{}^2 \delta_3{}^k - \delta_\ell{}^3 \delta_2{}^k)$$

$$+ \text{perm } (j\ell),\ (ik)]$$

$$(67)$$

introducing six real parameters. The P part, however, does not
satisfy the property b) of H', so that $P = 0$ (in the SU(3) limit
in which the strong interactions also preserve this property).

Applying now the formula (52), the second term $[\chi^i, H']$ gives back spurions $\mathcal{m}^i$ rotated from $\mathcal{m}$ by isotopic transformation. Thus $\mathcal{m}(8) \to \mathcal{m}(8)^i$, $\mathcal{m}(27) \to \mathcal{m}(27)^i$ which are again scalars. On the other hand, the first term must be calculated by inserting baryon intermediate states. It will be pseudoscalar, and besides of order υ since $\chi^i \sim h\upsilon$. Thus the first and second terms provide p wave and S wave parts of the amplitude.[30,31,32] There are seven processes to be considered.

$$\Lambda^0 \to p + \pi^-$$

$$\Lambda^0 \to n + \pi^0$$

$$\Sigma^+ \to p + \pi^0$$

$$\Sigma^+ \to n + \pi^+$$

$$\Sigma^- \to n + \pi^-$$

$$\Xi^- \to \Lambda + \pi^-$$

$$\Xi^0 \to \Lambda + \pi^0$$

We write each amplitude as $A(\Lambda_-^0) + \gamma_5 B(\Lambda_-^0)$, etc. The S wave amplitudes A satisfy the $|\Delta I| = 1/2$ rules for Λ and Ξ because only $\mathcal{m}(8_s)^i$, $\mathcal{m}(8_a)^i$ can contribute to them. For the Σ amplitudes, it turns out that the contributions from 8 and 27 conspire to produce a sum rule which is different from the $|\Delta I| = 1/2$ triangle except for a sign, and therefore experimentally indistinguishable. In addition, we get

$$A(\Lambda_-^0) + 2A(\Xi_-^-) = \sqrt{3}\, A(\Sigma_0^+) + \sqrt{3/2}\, A(\Sigma_+^+).$$

This is equal to the Lee-Sugawara relation only if $A(\Sigma_+^+) = 0$, which requires $\mathcal{M}(27) = 0$, and we are back to octet dominance and $|\Delta I| = 1/2$ rule. Although these results are interesting, they are not altogether satisfactory. Especially the p wave part seems to vanish like v_B^τ in the SU(3) limit.

One way to improve the Suzuki-Sugawara results seems to be as follows. We note that the p wave part comes from the baryon pole diagrams which become singular as we switch off the baryon mass difference and the pion mass. Thus the limit is actually not unique, but depends on the exact limiting procedure. We already remarked that $m_\pi \to 0$, means going to a reference frame in which other particles (the baryon in this case) are moving with light velocity. So if we keep baryon mass diffference finite and make $m_\pi \to 0$, v_B^τ tends to C. The p wave contribution as defined in this sense is not zero but of the same order as the S wave. To make the result manifestly covariant, however, we should use the actual pole diagrams to express the p wave amplitudes, where we can insert actual baryon and meson masses.

Rather interesting results follow along this line if we make the simplifying assumptions

a) $\mathcal{M}(27) = 0$,

b) $m_\Lambda = m_\Sigma$, $m_\Lambda - m_N = m_\Xi - m_\Lambda$.

Both S and P wave amplitudes then satisfy $\Delta I = 1/2$ and Lee-Sugawara relations, and can be expressed in terms of four adjustable parameters: the spurions $\mathcal{M}(8_S)$ and $\mathcal{M}(8_a)$, and

the baryon-meson coupling strengths $f(8_a)$ and $d(8_S)$. We can fit the data with $\mathcal{M}(8_S)/\mathcal{M}(8_a) \sim -0.5$, $d/f \sim 2$. The latter is consistent with the value 1.7 obtained from the leptonic decays. There remain, however, various difficulties and objections:

a) The absolute values of d and f necessary to fit the p-wave to s-wave ratios are sensitive to the mass differences in the pole diagrams, but generally tend to be twice as large as the theoretically expected values $d \sim 1$, $f \sim 0.5$. Individual p wave decay amplitudes also fluctuate sensitively with the baryon mass differences and $\mathcal{M}(8_S)/\mathcal{M}(8_a)$. In fact if the spurions causing baryon mass differences and the weak spurions have the same f/d ratio, $B(\Sigma_+^+)$ becomes zero. [*]

b) The parity conserving meson pole diagrams $(B \rightarrow B + K$, $K \rightarrow \pi)$ do not appear in the formula, but they also exhibit a singularity in the limit $m_k - m_\pi \rightarrow 0$. They cannot belong to the commutator term $[\chi^i, H']$ because of the CP properties.)

c) Pole diagrams are generated by spurions acting on baryons and mesons. These can be transformed away by redefining the baryon and meson states. But then we lose also the singularity of pole diagrams which we exploited above in the p wave amplitudes. [Of course the weak Hamiltonian H' cannot be transformed away by redefining quark states.]

It appears thus that the problem of p wave amplitudes is not quite settled yet. We also remark that there is a somewhat

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[*]S. L. Adler, private communication

different theory of non-leptonic decays in which the spurion is assumed to be proportional to $\partial_\mu v_\mu$ and $\partial_\mu a_\mu$.[32a]

Calculation of the spurion (self-energy) $\mathcal{M}$, Eq. (66), from the original current-current Hamiltonian H', Eq. (65), was carried out by Chiu and Schechter,[33] and by Hara[34] recently. They estimated it by inserting baryon and decuplet intermediate states between the currents, and using the known electromagnetic form factors to provide a cut-off. Interestingly, their results for $\mathcal{M}$ are of the right order of magnitude, and $\mathcal{M}$ (27) tends to be small compared to $\mathcal{M}$ (8). We must say, however, that the meaning of such a calculation is not very clear, because it is based on two assumptions: 1) strictly local current-current interaction, and 2) picking of only a few intermediate states to achieve convergence in spite of 1).

VI.   MANY-PION PROCESSES

Processes involving more than one soft meson depend on more details of the nature of axial vector currents, e.g., their commutation relations. Therefore they can also give us more information about chiral symmetry.

General formulas for n-meson processes can be written down in terms of time ordered products of n axial vector currents, taking their divergences. This was done explicity by Weinberg,[35]

Complications arise when we take the limit $k \to 0$ for individual currents. In taking multiple limits the result can depend on the order in which we perform it and on the direction of the vectors k. We can see this situation by considering, for example, pole diagrams in which N mesons are successively emitted from a baryon line at the end or beginning of a diagram. We have a product of energy denominators

$$\prod_{n=1}^{N} \left[(p + \sum_{i=1}^{n} k_i)^2 + m^2 \right] = \Pi[2p \cdot \Sigma \, k_i + (\Sigma \, k_i)^2]$$

This becomes singular as any partial sum

$$\sum_{i=1}^{n} k_i \to 0,$$

although this will be compensated for by the numerator so that no infinities will result. The numerator also contains products of vertices which do not commute because of the SU(2) or SU(3) spins. Turning things around, when we are given various multiple limits of an amplitude, we have to find an interpolation formula between these limits showing the dependence on k_i explicitly. No general way to do this is known, and it will probably depend on detailed dynamics.

Two examples will be discussed here. One is the low energy π - N scattering according to a version due to Tomozawa,[36] Raman and Sudarshan,[25] and Weinberg.[37] The other is the non-leptonic decays of K meson.

1) $\pi - N$ scattering

Consider the amplitude

$$\bar{u}_b \mathcal{M}_{\mu\nu}^{ij} u_a = \langle b | (a_{\mu}^{i}(x), a_{\nu}^{j}(x')_{+} | a \rangle ,$$

$$(i,j = 1,2,3) \tag{68}$$

where a and b are nucleon states. Taking one divergence, we get

$$\bar{u}_b \partial_{\mu} \mathcal{M}_{\mu\nu}^{ij} u_a = \langle b (\partial_{\mu} a_{\mu}^{i}(x), a_{\nu}^{j}(x'))_{+} a \rangle$$

$$- \delta(x_0 - x'_0) \langle b [a_0^{i}(x), a_{\nu}^{j}(x')] a \rangle$$

At this stage we will use the commutation relations between current densities according to the quark model. Thus

$$[a_0^{i}(x), a_{\nu}^{i}(x')] = 2i\epsilon_{ijk} v_{\nu}^{k}(x) (x_{\mu} - x'_{\mu}) \tag{69}$$

(We ignore the problem of Schwinger terms.)

Taking another divergence, then,

$$\bar{u}_b \partial_{\mu} \partial'_{\nu} \mathcal{M}_{\mu\nu}^{ij} u_a = \langle b | (\partial_{\mu} a_{\mu}^{i}(x), \partial_{\nu} a_{\nu}^{i}(x'))_{+} | a \rangle$$

$$- \delta(x_0 - x'_0) \langle b | [\partial_{\mu} a_{\mu}^{i}(x), a_0^{j}(x')] | a \rangle$$

$$- \delta^{4}(x-x')2i\epsilon_{ijk} \langle b | \partial_{\nu} v_{\nu}^{k}(x) | a \rangle$$

$$- \partial'_{\nu} \delta^{4}(x-x')2i\epsilon_{ijk} \langle b | v_{\nu}^{k}(x) | a \rangle \tag{70}$$

The first term is directly related to the invariant amplitude $\mathcal{M}^{ij}(k,k')$ for the process $\pi^{j}(k') + N_a \rightarrow \pi^{i}(k) + N_b$ via the PCAC formula. The second term $\sim [\phi^{i}, a_0^{i}]$ depends on what happens

to ϕ^i under a chiral transformation, which cannot be fixed by the current algebra. If we borrow the language of the σ model, we may write

$$\delta(x_0 - x'_0)[\phi^i(x), a_0^j(x')] = -2i\delta_{ij}\sigma(x)\,\delta^4(x-x') \quad (71)$$

where the set $\{\phi^i, \sigma\}$ belong to a quartet $(2,2)$ of SU(2) x SU(2). [The sextet $(3,1) + (1,3)$ is not possible because it gives the antisymmetric form $-2i\epsilon_{ijk}\sigma^k(x)\delta^4(x-x')$ for the commutation $(6,4)$, which violates the crossing symmetry.] Going to the momentum space (k'^{in}, k^{out}), we have

$$\overline{u}_b k_\mu k'_\nu \, \mathcal{M}_{\mu\nu}^{ij}(k,k')u_a$$

$$= \frac{-ic^2}{(k^2 + m_\pi^2)(k'^2 + m_\pi^2)} \, \overline{u}_b \, \mathcal{M}^{ij}(k,k')u_a$$

$$+ 2i\delta_{ij}c\langle b|\sigma(k-k')|a\rangle$$

$$- (k+k')_\nu \epsilon_{ijk}\langle b|\mathcal{V}_\nu^k(k-k')|a\rangle \quad (72)$$

Now $\mathcal{M}^{ij}(k,k')$ has the general form

$$\mathcal{M}^{ij}(k,k') = A_+\delta_{ij} + A_-\,i\epsilon_{ijk}\tau_k$$

$$+ i\gamma\cdot(k+k')(B_+\,i\epsilon_{ijk}\tau_k + B_-\delta_{ij}) \quad (73)$$

and

$$\langle\sigma(k-k')\rangle = \overline{u}_b u_a F_S(k-k')$$

$$(k_\nu + k'_\nu)\langle\mathcal{V}_\nu^k(k-k')\rangle \rightarrow i\,\overline{u}_b\tau^k[\gamma\cdot(k+k')F_1(k-k')$$

$$+ \sigma_{\mu\nu}(k+k')_\mu(k-k')_\nu F_2(k-k')]u_a \quad (74)$$

$A_\pm$ and $B_\pm$ are functions of $\nu = (p+p')\cdot(k+k')/4$, k^2, k'^2, and $k\cdot k'$, and the $\pm$ sign designates signature under crossing: $\nu \to -\nu$, $k^2 \leftrightarrow k'^2$. If we assume that $\mathcal{M}$ can be expanded in these variables around zero, i.e., $k = k' = 0$, we may write

$$A_+ = a_+$$

$$A_- = a_- + \nu a'_-$$

$$B_+ = b_+$$

$$B_- = b_- + \nu b'_- \qquad (75)$$

up to linear terms in k and k'.

The left-hand side of Eq. (72) is proportional to both k and k'. So terms of linear order on the right-hand side would have to cancel each other. We must be careful, however, since $\mathcal{M}_{\mu\nu}$ contains pole diagrams, so that the left-hand side can be finite in the limit $k, k' \to 0$. Explicit evaluation gives the result

$$k_\mu k'_\nu \mathcal{M}_{\mu\nu}{}^{ij}(-k,k') \to 0.$$

Thus, comparing the two sides, and noting

$$\nu \sim -m(k_0+k'_0)/2 \sim m\, i\gamma\cdot(k+k')/2,$$

we obtain the relations

$$a_- = 0, \qquad b_- = 0$$

$$a_+ = \frac{2m_\pi^4}{c^2} F_S(0)$$

$$(ma'_- + b_+) = -\frac{m_\pi^4}{c^2} \qquad (76)$$

Eq. (77) predicts the π - N scattering lengths at $k = k' = 0$

$$\frac{1}{3} a_{1/2} + \frac{1}{3} a_{3/2} = \frac{2}{4\pi} \left(\frac{f_{\pi NN}(0)}{1.18}\right)^2 \frac{1}{1 + \frac{m_\pi}{m}} \cdot F_S(0)$$

$$\frac{1}{3} a_{1/2} - \frac{1}{3} a_{3/2} = \frac{2}{4\pi} \left(\frac{f_{\pi NN}(0)}{1.18}\right)^2 \frac{m_\pi}{1 + \frac{m_\pi}{m}}$$

$$= 0.10 \ m_\pi^{-1} \tag{77}$$

Experimental values at threshold: $a_{1/2} = 0.17 \ \mu \ 0.005 \ m_\pi^{-1}$, $a_{3/2} = -0.088 \ \mu \ 0.004 \ m_\pi^{-1}$ agree very well with Eq. (77) if $F_S(0) = 0$. There is no convincing argument to show that this must be so (in the limit $k = k' = 0$). Adler[38] and Tomozawa[36] achieve this by taking the divergence of $\langle N_b + \pi | a_\mu(x) | N_a \rangle$ before, instead of after, reducing it with respect to π. The two results differ by the σ term (which however vanishes on the pion mass shell). Which one to choose is, however, a matter of assumption about analytic structure. We must also bear in mind that the extrapolation from real amplitudes to unphysical limits is a delicate one for multi-pion problems. For example, A_+ equals $g_{\pi NN}(0) \ g_{\pi NN}(m_\pi)/m_N$ (instead of 0) when one pion is on the mass shell ($k^2 = -m_\pi^2$) and the other one is such that $k'^2 = 0$, $p \cdot k = k \cdot k' = 0$.[38]

Eq. (73) is also a relation between scattering lengths and $-G_A/G_V$. If we express the left-hand side of the second equation as a dispersion integral involving $\pi - N$ total cross sections $\sigma_{\pi-} - \sigma_{\pi+}$, we arrive at the Adler-Weisberger relation.[39] (However, they applied the PCAC formula directly to the dispersion integral.)

The same technique may be applied to $\rho \to 2\pi$ (Kawarabayashi and Suzuki),[40] and with more caution, to $\pi - \pi$ scattering (Adler,[38] Weinberg[37]). Kawarabayashi and Suzuki obtained a relation

$$f_{\rho\pi\pi}^{2}/4\pi = 2m_\rho^{2}f^2/4\pi = 2m_\rho^{2}(f_{\pi NN}(0)/1.18)^2/4\pi \simeq 3.$$

(78)

as compared to the experimental value $\simeq 2.5$. The results (74) and (78) are compatible with the ρ-meson dominance model with universal vector meson coupling (Sakurai[41]).

2) $K \to 2\pi$ and 3π decays

The non-leptonic K decays were first treated by Callan and Treiman.[27] Here we follow a more elaborate procedure.[42] The processes

$$\mathcal{M}_1: \quad K \to \pi$$

$$\mathcal{M}_2: \quad K \to \pi_1 + \pi_2$$

$$\mathcal{M}_3: \quad K \to \pi_1 + \pi_2 + \pi_3 \tag{79}$$

are linked by successive applications of the PCAC relation. The main problem is how to relate the mass limits to the real amplitudes, and take account of the non-commutative nature of limiting procedure.

The task is relatively easy if we only compare neighboring reactions in (79). First, $\mathcal{M}_1$ is given by

$$\mathcal{M}_1 = -\langle \pi | H' | K \rangle \sqrt{4 m_K m_\pi} \tag{80}$$

suppressing isospin indices. $\mathcal{M}_2$ is related to this in two limits

$$\lim_{\pi_1 = 0} \mathcal{M}_2 = -i\, f \langle \pi_2 | [\chi_1, H'] | K \rangle$$

$$\lim_{\pi_2 = 0} \mathcal{M}_2 = -i\, f \langle \pi_1 | [\chi_2, H'] | K \rangle \tag{81}$$

We ignored the "surface terms" coming from possible scalar meson poles. Because of the nature of H', Eqs. (81) relate $\mathcal{M}_2$ to $\mathcal{M}_1$ by simple isotopic transformation, as in the earlier examples. If we again assume octet dominance, we get the following relations

$$f \mathcal{M}_1(K^+ \to \pi^+) = - \mathcal{M}_2(K^+ \to \pi^+ \pi^0; \; k_{\pi^+} = 0)$$

$$= \mathcal{M}_2(K^+ \to \pi^+ \pi^0; \; k_{\pi^0} = 0)$$

$$= -i \mathcal{M}_2(K_1^0 \to \pi^+ \pi^-; \; k_{\pi^+} = 0)$$

$$= -i \mathcal{M}_2(K_1^0 \to \pi^+ \pi^-; \; k_{\pi^-} = 0) \tag{82}$$

In addition, if we apply the PCAC to the K meson, we find

$$\mathcal{M}_2(K^+ \to \pi^+ \pi^0; \; k_{K^+} = 0) \sim \langle \pi^+ \pi^0 | [\chi_{K^+}, H'] | 0 \rangle = 0 \tag{83}$$

We can satisfy these requirements by expanding the vertex function $\mathcal{M}_2$ in the three mass variables k_K^2, k_{π_1}, $k_{\pi_2}^2$, up

to linear terms. Actually

$$\mathcal{M}_2(k_K^2, \; k_{\pi_1}^2, \; k_{\pi_2}^2)$$

must vanish at the symmetry point:

$$\mathcal{M}_2(\mu^2, \mu^2, \mu^2) = 0 \tag{84}$$

because of the CP properties mentioned after Eq. (59). The solution to Eqs. (82)-(84) is given by

$$\mathcal{M}_2(K^+ \to \pi^2\pi^0) = A(k_{\pi 0}^2 - k_{\pi+}^2)/\mu^2,$$

$$\mathcal{M}_2(K_1^0 \to \pi^+\pi^-) = -i\, A[2k_K^2 - k_{\pi+}^2 - k_{\pi-}^2]/\mu^2,$$

$$A = f\,\mathcal{M}_1(K^+ \to \pi^+) \tag{85}$$

where μ^2 may be taken to be the average mass of the meson octet, and Eq. (82) is applied to this symmetry limit. The first of Eq. (85) simply expresses the $|\Delta I| = 1/2$ rule: $\mathcal{M}_2(K^+ \to \pi^2\pi^0) = 0$ (to within electromagnetic effects). The second and third equations determine the spurion $\mathcal{M}_1$ from the experimental value for $\mathcal{M}_2(K_1^0 \to \pi^+\pi^-)$: $\mathcal{M}_1 = A/f = 3.9 \times 10^{-4}$ Mev/f.

Interestingly, the ratio $\mathcal{M}_1/S$ of the weak spurion and Gell-Mann-Okubo (medium strong) mass spurion turns out to be roughly equal for the meson case and the baryon case (as determined from the baryon non-leptonic decays). It is known also that the electromagnetic mass spurion S_e also satisfies the same universality $S_e/S \sim$ const. So the concept of universal spurion coupling (Coleman and Glashow) seems to be a meaningful one.

Next we come to the relation between $\mathcal{M}_2$ and $\mathcal{M}_3$.

Proceeding in a similar fashion, we obtain the relations

$$\mathcal{M}_3(K^+ \to \pi^-\pi^+\pi^+; k_{\pi^-} = 0) = \mathcal{M}_3(K^+ \to \pi^+\pi_1^0\pi_2^0; k_{\pi^0} = 0)$$

$$= \mathcal{M}_3(K_2^0 \to \pi^0\pi^+\pi^-; k_{\pi^\pm} = 0) = 0 \qquad (86)$$

$$f \mathcal{M}_2(K_1^0 \to \pi^+\pi^-) = -\mathcal{M}_3(K^+ \to \pi^-\pi^+\pi^+; k_{\pi^+} = 0)$$

$$= -\mathcal{M}_3(K^+ \to \pi^+\pi^0\pi^0; k_{\pi^+} = 0)$$

$$= \mathcal{M}_3(K_2^0 \to \pi^0\pi^+\pi^-; k_{\pi^0} = 0)$$

$$= \mathcal{M}_3(K_2^0 \to \pi_1^0\pi_2^0\pi_3^0; k_{\pi_1^0} = 0)$$

$$(87)$$

This time, the off-shell interpolation must be done with respect to four mass variables as well as the two Mandelstan variables. Thus, for example,

$$\mathcal{M}_3(K_2 \to \pi^0\pi^+\pi^-)$$

$$= a + bk_K^2 + c[k_{\pi^+}^2 + k_{\pi^-}^2] + dk_{\pi^0}^2$$

$$+ e\{(k_{\pi^+} - k_K)^2 + (k_{\pi^-} - k_K)^2\}$$

$$+ f(k_{\pi^0} - k_K)^2. \qquad (88)$$

Comparing with Eq. (86) we find

$$a = 0, \quad b + e = 0, \quad c + d + e + f = 0 \qquad (89)$$

since the relations (86) must hold for arbitrary values of k_K^2 and $k_{\pi_+}^2 = k_{\pi_-}^2$ when $k_{\pi^0} = 0$. In the rest frame of K, $\mathcal{M}_3$ therefore reduces to

$$\mathcal{M}_3(K_2^0 \to \pi^0\pi^+\pi^-) = (b+f)\, m_K(m_K - 2E_{\pi 0}) + (c-b)m_\pi^2 \tag{90}$$

The linear expansion (88) amounts to keeping only S waves, which allows only T = 1 final state at the same time. Experimentally, these conditions are well satisfied. Further, if we use the PCAC relation for the K current, we obtain

$$\mathcal{M}_3(K_2^0 \to \pi^0\pi^+\pi^-;\ k_K = 0) = 0 \tag{91}$$

which entails b = c. Complete SU(3) symmetry can be shown to imply, in addition, b = d, so that b + f = 0. These will make Eq. (90) vanish completely, so we cannot adopt the extreme symmetry limit. In reality $m_K^2 \gg m_\pi^2$, and this suggests at any rate that the second term of (90) may be ignored. We then obtain a formula

$$\mathcal{M}_3(K_2^0 \to \pi^0\pi^+\pi^-) = f\, \mathcal{M}_2(K_1^0 \to \pi^+\pi^-)[1 - 2E_{\pi 0}/m_K] \tag{92}$$

which predicts not only the total decay rate, but the energy dependence too. Experimental data are in good agreement on both counts as is shown by

$$\mathcal{M}_3(K_2^0 \to \pi^0\pi^+\pi^-)_{th} = (2.43 \pm 0.03) \times 10^{-6}$$
$$\times\, [1 - (2E_{\pi 0}/500\ \text{Mev})]$$

$$\mathcal{M}_3(K_2^0 \to \pi^0\pi^+\pi^-)_{exp} = (2.67 \pm 0.1) \times 10^{-6}$$
$$\times\, [1 - (2E_{\pi 0}/500\ \text{Mev})(1 \pm 0.1)] \tag{93}$$

Similarly, we find for other decays $K^+ \to \pi^+ \pi^0 \pi^0$, $K^+ \to \pi^- \pi^+ \pi^+$, and $K_2^0 \to \pi^0 \pi^0 \pi^0$ equally satisfactory results.

Complications arise when we apply PCAC relations success- ively as we did for the πN scattering.[43] Thus we obtain

$$\partial_\mu \partial_{\nu'} \langle \pi | (a_\mu^i(x) a_\nu^j(x') H'(0))_+ | K \rangle$$

$$= c^2 \langle \pi | (\phi^i \phi^i H')_+ | K \rangle + c \langle \pi | ([a_0^j, \phi^i] H') | K \rangle \delta(x_0 - x_0')$$

$$+ c \langle \pi | (\phi^i [a_0^j, H'])_+ | K \rangle \, \delta(x_0')$$

$$+ c \langle \pi | (\phi^j [a_0^i, H'])_+ | K \rangle \, \delta(x_0)$$

$$+ 2i \, \epsilon_{ijk} \langle \pi | v_0^k, H'] | K \rangle \, \delta^4(x - x') \delta(x_0)$$

$$+ \langle \pi | [a_0^j, [a_0^i, H']] | K \rangle \, \delta(x_0) \, \delta(x_0')$$

$$+ 2i \, \epsilon_{ijk} \langle \pi | (v_0^k H')_+ | K \rangle \, \partial_\mu' \delta^4(x - x') \qquad (94)$$

The right-hand side must be symmetric under the interchange $i \leftrightarrow j$, $x \leftrightarrow x'$. The first term gives $\mathcal{M}_3$. The rest is related to $\mathcal{M}_2$, $\mathcal{M}_1$ and some new processes.

3) K_{e4} decays

Recently Weinberg[44] and Clavelli[45] treated the K_{e4} decays and were able to reproduce the experimental data quite well. Here we follow Clavelli who combined the vector meson dominance model with PCAC. Consider the decays $K^+ \to \pi^+ + \pi^- + e^+ + \nu$, $\pi^0 + \pi^0 + e^+ + \nu$, etc., whose amplitudes are denoted by

$$\mathcal{M}_\mu^{+-}, \quad \mathcal{M}_\mu^{oo}, \quad \text{etc.},$$

$$\mathcal{M}_\mu^{+-} = \langle \pi^+ \pi^- | J_\mu | K^+ \rangle \, (8 E_{\pi^+} E_{\pi^-} E_{K^+})^{1/2}, \quad \text{etc.}$$

By taking different mesons out via PCAC, we get relations between K_{e4} and K_{e3} amplitudes. In particular, $\lim\limits_{\pi^+ \to 0} \mathcal{M}_\mu^{-+} = 0$. This shows, as in $K \to 3\pi$ decays, the importance of off-mass-shell corrections without which agreement between theory and experiment is not very impressive. Furthermore, if we take two mesons out via PCAC, we find a host of further relations between K_{e4} and K_{e2} amplitudes which depend on the order of two limiting processes. For example

$$\lim\limits_{\pi^- \to 0} \lim\limits_{\pi^+ \to 0} \mathcal{M}_\mu^{-+} = 0,$$

$$\lim\limits_{\pi^+ \to 0} \lim\limits_{\pi^- \to 0} \mathcal{M}_\mu^{-+} = 4 (m_\pi^2 m_K / c)^2 f_K$$

where f_K is the K_{e2} coupling constant. It turns out that these limiting values, not only for soft π but also for soft K limits, can be reproduced by taking the vector meson dominance model. The PCAC relations serve as boundary conditions which fix certain parameters. The diagrams we consider are those containing ρ and K^* poles, as shown in Fig. 1. As we learned in π-N scattering, the ρ meson is related to the commutator of two π meson currents. Similarly, the K^* is related to the commutator of π and K currents. The lepton vertex in these diagrams correspond to the axial current $\langle \rho | a_\mu | K \rangle$ and $\langle \pi | a_\mu | K^* \rangle$ respectively. According to the spirit of PCAC, therefore, they must also contain the

induced pseudoscalar terms (Fig. 1). It is these induced terms that play a key role in satisfying the PCAC relations.

The amplitude $\mathcal{M}_\mu$ has the general form

$$\mathcal{M}_\mu = F_1{}^A(\pi^1+\pi^2)_\mu + F_2{}^A(\pi^1-\pi^2)_\mu + F_3{}^A(K-\pi^1-\pi^2)_\mu$$
$$+ F_4{}^V \epsilon_{\mu\nu\lambda\rho}\, \pi_\nu{}^1\pi_\lambda{}^2 K_\rho$$

where $\pi = k_\pi$ etc., and the F's are functions of q^2 (momentum transfer to leptons) and other variables.

Clavelli shows that this model makes a prediction about a) f_+ and f_- form factors for K_{e3}, and b) $F_1, \ldots, F_3$ form factors for K_{e4}, including their momentum dependence. $F_4 = 0$ in this model, and is probably small anyway.) Agreement with experiment is very good. (He further takes into account the 2π final state interaction in K_{e4} by means of a $\pi\pi$ scattering phase shift of $\sim 1\ m_\pi{}^{-1}$.

4) Many-meson problem in the Gürsey model.

We discuss here a possible unified treatment of many-meson processes on the basis of the Gürsey type model (Section III). The underlying assumptions will be that 1) the scalar mesons do not exist as well-defined resonances (in contrast to the Gell-Mann-Levy type model), and 2) we may expand non-linear functions of meson fields and interpret the latter as effective renormal-ized fields. The choice of the unitary meson matrix (Eq. (35))

is where the dynamics will come in, and the result will depend in general on the choice.

The essential points are as follows. We first take the meson matrices $\mathcal{M}(\phi(x))$ and $\mathcal{M}^+(\phi(x))$ which behave as $(3_R, 3_L{}^*)$ and $(3_L, 3_R{}^*)$. We also have definite expressions for the vector and axial currents $v_\mu{}^i$ and $a_\mu{}^i$. These will be expanded and expressed in terms of ϕ^i and $\partial_\mu \phi^i$. In discussing the leptonic decays of mesons, we replace the basic hadronic current j_μ in H_W by the above meson current. If we regard this as an effective Hamiltonian, we get automatically all possible processes like $K \rightarrow \mu\nu$, $K \rightarrow \mu\nu + \pi$, $K \rightarrow \mu\nu + 2\pi$. In the case of non-leptonic meson decays, one way may be to form a product of meson currents (probably with a phenomenological coupling constant rather than G_0).

Another way is to introduce a spurion matrix S $\sim \lambda_6$, which should belong to $(8_L, 1_R)$ according to the basic current-current Hamiltonian. We then form an invariant with respect to $SU(3)_L \times SU(3)_R$ out of S, $\mathcal{M}$ and $\mathcal{M}^+$. A simplest non-trivial form is

$$H_{eff} = \text{Tr}(\partial_\mu \mathcal{M} \, S \, \partial_\mu \mathcal{M}^+)$$

The derivatives are necessary since otherwise $\mathcal{M}\mathcal{M}^+ = 1$. By expanding $\mathcal{M}$ and $\mathcal{M}^+$ we can reproduce the relations obtained earlier about $K \rightarrow \pi$, 2π, 3π decays. In particular, the energy and mass dependence of the amplitudes arises from the derivatives in the above formula.

Probably this procedure can be generalized to all other processes we have considered. The basic point will be to write down an effective chiral invariant local Hamiltonian (to be regarded as an approximate S-matrix at low energy) in terms of $\mathcal{M}(x)$, $\mathcal{M}^+(x)$ and the phenomenological fields for other particles. Work along these lines is being pursued by J. Cronin.
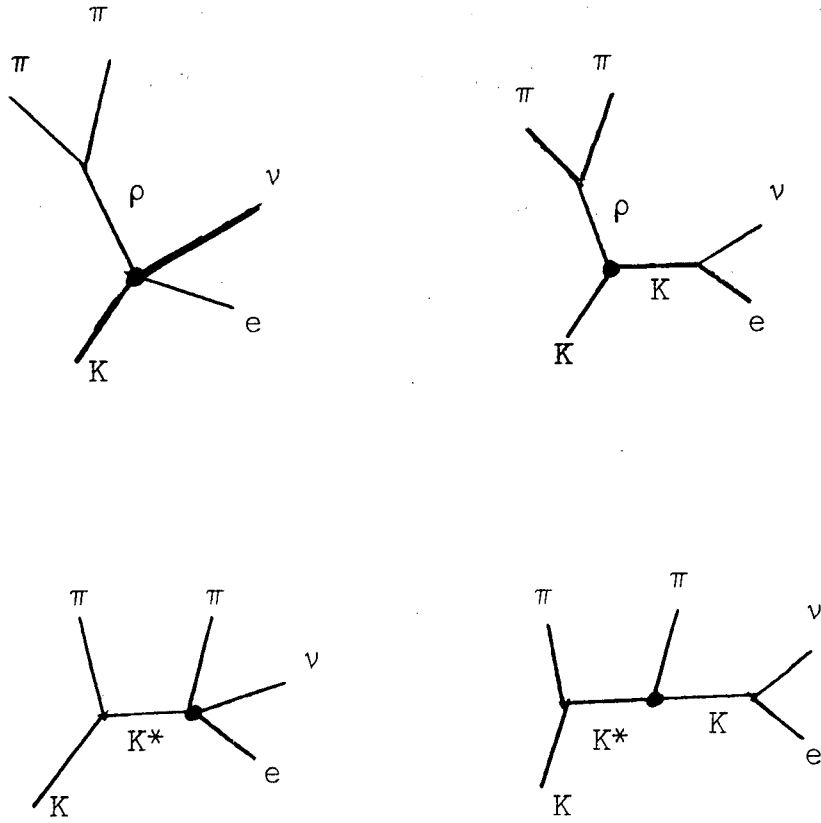
Figure 1. K_{e4} Diagrams

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