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SOLUTION OF BOLTZMAN'S EQUATION FOR MONOENERGETIC NEUTRONS IN AN INFINITE HOMOGENEOUS MEDIUM

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Abstract

The Boltzman's equation is solved in the case of monoenergetic neutrons created by a plane or point source in an infinite medium which has spherically symmetric scattering. The customary solution of the diffusion equation appears to be multiplied by a constant factor which is smaller than 1. In addition to this term the total neutron density contains another term which is important in the neighborhood of the source. It goes with $1/r^2$ in the neighborhood of a point source.

SOLUTION OF BOLTZMAN'S EQUATION FOR MONOENERGETIC NEUTRONS IN AN INFINITE HOMOGENEOUS MEDIUM

E. P. Wigner

1. The following represents a solution of Boltzman's equation for neutrons which do not change their energy during their diffusion in a material. The solution can be easily obtained on the basis of Placzek's work, which is also reproduced in A-21. The contents of the first part of the following note were also published by Bothe in the Zs. f. Physik.

We first consider a plane source of neutrons at $x = 0$. Evidently, the angular distribution of neutrons will be independent of y and z and have at every point axial symmetry with respect to the X axis. The number of neutrons per unit volume for which the velocity component in the X direction v_x lies between μv and $(\mu + d\mu)v$ will be denoted by

$$f(x, \mu)\, d\mu \quad . \tag{1}$$

The density of neutrons at x is obtained by integration over the direction cosine μ

$$n = \int_{-1}^{1} f(x, \mu)\, d\mu = 2\,\bar{f} \tag{1a}$$

where $\bar{f}$ is the average value of f over all directions of the velocity.

The Boltzman equation for spherically symmetric scattering is

$$\left(-\frac{v_x}{v}\frac{\partial}{\partial x} - \frac{v_y}{v}\frac{\partial}{\partial y} - \frac{v_z}{v}\frac{\partial}{\partial z} - \sigma \right) f + \sigma_s \bar{f} + P/v = 0 \tag{2}$$

where σ, σ_s and $\sigma_a = \sigma - \sigma_s$ are total, scattering and absorption cross sections per unit volume of the medium in which the diffusion occurs; P is the production of neutrons per unit volume and unit velocity range. For an f of the form (1) one can write instead of (2)

$$\left(-\mu\frac{\partial}{\partial x} - \sigma\right) f + \sigma_s \bar{f} + \frac{1}{2v}\delta(x) = 0 \ . \qquad (2a)$$

The production is assumed in (2a) to be one neutron per unit area of the $x = 0$ plane, with equal probability for every direction: there are $\frac{1}{2}d\mu$ neutrons produced per unit area with direction cosines between μ and $\mu + d\mu$.

In order to solve (2a) we write

$$\delta(x) = 1/2\pi \int_{-\infty}^{\infty} e^{i\nu x}\, d\nu \ ; \quad f(x,\mu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f_\nu\,(x,\mu)\, d\nu \qquad (3)$$

Introducing this into (2a) we obtain

$$\left(-\mu\frac{\partial}{\partial x} - \sigma\right) f_\nu + \sigma_s \bar{f}_\nu + \frac{1}{2v}\, e^{i\nu x} = 0 \qquad (4)$$

$$\bar{f}_\nu = \frac{1}{2}\int_{-1}^{1} f_\nu\,(x,\mu)\, d\mu$$

We try the assumption

$$f_\nu = a(\mu)\, e^{i\nu x} \qquad (5)$$

Herein, μ is the cosine of the angle between the velocity of the neutron and the x direction. This gives us for a the equation

$$- i\mu\nu a(\mu) - \sigma a(\mu) + \sigma_s \bar{a} + 1/2v = 0 \qquad (5a)$$

whence we obtain

$$a(\mu) = \frac{1/2v + \sigma_s \bar{a}}{\sigma + i\mu\nu} \qquad (5b)$$

where

$$\bar{a} = \frac{1}{2}\int_{-1}^{1} a(\mu)\, d\mu = \frac{1}{4v}\int_{-1}^{1} \frac{1 + 2v\sigma_s \bar{a}}{\sigma + i\mu\nu}\, d\mu = \frac{1 + 2v\sigma_s\bar{a}}{2v\nu}\,\mathrm{arctg}\,\frac{\nu}{\sigma} \qquad (5c)$$

In order that the above solution be self-consistent, we must have, therefore,

$$2v\bar{a} = \frac{\mathrm{arctg}\,\nu/\sigma}{\nu - \sigma_s \mathrm{arctg}\,\nu/\sigma} = \frac{\ln\frac{\sigma + i\nu}{\sigma - i\nu}}{2i\nu - \sigma_s \ln\frac{\sigma + i\nu}{\sigma - i\nu}} \qquad (6)$$

From this and (5) we obtain

$$2v\bar{f}_\nu = 2v\bar{a}e^{i\nu x} = \frac{e^{i\nu x}}{2i\nu - \sigma_s \ln\frac{\sigma + i\nu}{\sigma - i\nu}}\ln\frac{\sigma + i\nu}{\sigma - i\nu} \qquad (6a)$$

and (5c), (6) and (5) give

$$f_\nu = \frac{1}{2v}\,\frac{e^{i\nu x}}{\sigma + i\mu\nu}\,\frac{2i\nu}{2i\nu - \sigma_s \ln\frac{\sigma + i\nu}{\sigma - i\nu}} \qquad (6b)$$

This finally gives for f in the case of a plane source because of (3)

$$f(x,\mu) = \frac{1}{4\pi v}\int_{-\infty}^{\infty} d\nu\, \frac{e^{i\nu x}}{\sigma + i\mu\nu}\,\frac{1}{1 + \frac{i\sigma_s}{2\nu}\ln\frac{\sigma + i\nu}{\sigma - i\nu}} \qquad (6c)$$

and for the density of neutrons we have from (6a)

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$$n = 2\bar{f} = \frac{1}{2\pi v} \int_{-\infty}^{\infty} d\nu \frac{e^{i\nu x}}{2i\nu - \sigma_s \ln \frac{\sigma + i\nu}{\sigma - i\nu}} \ln \frac{\sigma + i\nu}{\sigma - i\nu} . \qquad (6d)$$

Of course, one can write 2i arctg ν/σ for $\ln \frac{\sigma + i\nu}{\sigma - i\nu}$. The analysis so far closely follows Placzek's work.

This is Bothe's result. In order to bring it into a more suitable form which also shows the asymptotic behavior, the derivation of which was the subject of A-21, we push the path of integration path toward the upper part of the positive imaginary axis. Evidently if the imaginary part of ν is infinite, the integrand vanishes for positive x. However, the integrand has two singularities in the upper half plane, one where the denominator under $e^{i\nu x}$ vanishes and the other one at $\nu = i\sigma$. As a result of the deformation the path of integration will become as indicated in Figure 1. The first singularity (at $i\varkappa$) is a pole and corresponds to the value of ν, which is i times the macroscopic absorption coefficient given in A-21. The other one is an essential singularity. The contribution to the first part can be easily calculated and expressed in a closed form. It is for $x > 0$

$$n_1 = I \frac{\varkappa}{2\sigma_a v} e^{-\varkappa x} \quad ; \; I = \frac{2\sigma_a}{\sigma_s} \frac{\sigma^2 - \varkappa^2}{\varkappa^2 - \sigma_a \sigma} \qquad (7)$$

In this $\varkappa$ is the value of ν/i for which the denominator vanishes. It is, therefore, given by the equation

$$2\varkappa = \sigma_s \ln(\sigma + \varkappa)/(\sigma - \varkappa) \qquad (7a)$$

According to A-21, it is closely approximated by the expression

$$\varkappa = \sqrt{3\sigma\sigma_a}\,(1 - 2\sigma_a/5\sigma) \; ; \quad \sigma_a = \sigma - \sigma_s . \qquad (7b)$$

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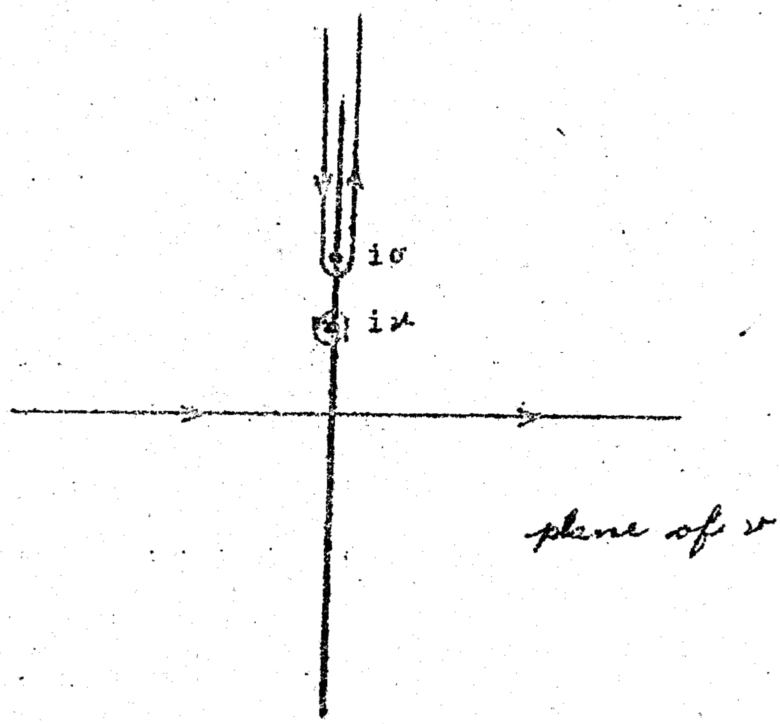

Figure 1

The contribution of the second part cannot be expressed in a closed form because the singularity is essential. However, it can be expressed as a real integral if one introduces a variable along the imaginary axis. It is with $\nu = i\,\sigma\,(1+\eta)$

$$n_2 = \frac{1}{2\pi v}\int i\,\sigma\,d\eta\;\frac{e^{-\sigma(1+\eta)x}}{-2\,\sigma(1+\eta)-\sigma_s\ln\frac{-\eta}{2+\eta}}\ln\frac{-\eta}{2+\eta} \qquad (8)$$

where the path of integration corresponds to the loop around $i\,\sigma$ in Figure 1. It goes, for η, from $\eta = \infty$ around $\eta = 0$ back to $\eta = \infty$. On the first part of the path, $\ln\frac{-\eta}{2+\eta}$ must be replaced by

$$\ln\frac{\eta}{2+\eta} - i\pi = -\ln(1+2/\eta) - i\,\pi \quad ,$$

on the second part by

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$$\ln \frac{\eta}{2+\eta} + i\pi = -\ln(1 + 2/\eta) + i\pi \quad .$$

We have, altogether

$$n_2 = \frac{i\sigma}{2\pi v} \int_0^\infty e^{-\sigma(1+\eta)x}\, d\eta \left[\frac{-\ln(1+2/\eta) - i\pi}{2\sigma(1+\eta) - \sigma_s \ln(1+2/\eta) - \sigma_s i\pi} - \frac{-\ln(1+2/\eta) + i\pi}{2\sigma(1+\eta) - \sigma_s \ln(1+2/\eta) + \sigma_s i\pi} \right]$$

$$= \frac{2\sigma^2}{v} \int_0^\infty \frac{(1+\eta)\, e^{-(1+\eta)\sigma x}\, d\eta}{\left[2\sigma(1+\eta) - \sigma_s \ln(1+2/\eta)\right]^2 + \pi^2 \sigma_s^2} \qquad (8a)$$

The first part of $n = n_1 + n_2$, i.e., n_1 is the only one that is important at distances from the source which are larger than a mean free path. Its functional dependence on x is the same as given by the diffusion equation. However, its absolute value is everywhere smaller by a factor

$$I = \frac{2\sigma_a}{\sigma_s} \frac{\sigma^2 - \varkappa^2}{\varkappa^2 - \sigma_a \sigma} \qquad (8b)$$

The reason for this "initial absorption" is that the density in the neighborhood of the source is greater (by n_2) than given by the diffusion equation so that the absorption in the neighborhood of the source is also greater. As a result, fewer neutrons get away from the source than the diffusion equation would indicate. The initial absorption is plotted in Figure 2 as a function of $1 - \sigma_s/\sigma = \sigma_a/\sigma$.

In order to calculate the second part of n, i.e., n_2, it is advisable first to go over from the density distribution around a plane source n to the density distribution N around a point source. This can be done by the well-known equation

$$N(r) = -\frac{1}{2\pi r}\frac{dn(r)}{dr} \tag{9}$$

If this transformation is applied to n_1, etc., it only gives the customary solution of the diffusion equation multiplied by I, i.e.,

$$N(r) = N_1(r) + N_2(r)\ ; \qquad N_1(r) = \frac{I}{4\pi}\frac{\varkappa^2}{\sigma_a vr} e^{-\varkappa r} \quad . \tag{10}$$

The second part becomes

$$N_2(r) = \frac{\sigma^3}{\pi rv}\int_0^\infty \frac{(1+\eta)^2\, d\eta\, e^{-(1+\eta)\sigma r}}{\left[2\sigma(1+\eta) - \sigma_s \ln(1+2/\eta)\right]^2 + \pi^2\sigma_s^2} \tag{10a}$$

This expression, as well as (8a), shows that the second part of the density (n_2 and N_2) is a sum of exponentials, all of them with positive coefficients. However, the relaxation lengths of all exponentials are equal to or shorter than the mean free path, $1/\sigma$. It does not seem to be possible to express n_2 or N_2 in a closed form. The coefficients of the different exponentials

$$\frac{(1+\eta)^2}{4\pi rv} e^{-\sigma(1+\eta)r}$$

(normalized so that the integral over all space be σ) are given in Fig. 3 as function of $1+\eta$ for various $\sigma_s/\sigma = 1 - \sigma_a/\sigma$.

2. It appears reasonable to approximate N_2 by a single exponential which has the right singularity at $r = 0$

$$N_2 \approx \frac{1}{4\pi r^2 v} e^{-\sigma' r} \quad . \tag{11}$$

Such an approximation cannot be very accurate since N_2 contains not one but an infinite range of exponentials. For σ' one can choose the value of $(1+\eta)\sigma$ for which the normalized exponential contributes most to (10a), i.e. the point where the curves of Fig. 3 have their maxima. The corresponding $1 + \eta$ is given by the transcendental equation $2\sigma(1 + \eta) = \sigma_s \ln(1 + 2/\eta)$ which gives for σ'

$$2\sigma' = \sigma_s \ln \frac{\sigma' + \sigma}{\sigma' - \sigma} \qquad (11a)$$

The formal similarity between this equation and that determining $\varkappa$ (7a) is remarkable. The resulting σ'/σ is given in Fig. 4 as function of σ_a/σ (curve a).

An alternative procedure is to choose σ' in such a way that the integral of (11) over all space have the right value. Since the production of neutrons is 1 per unit time, the absorption of neutrons also is 1

$$\sigma_a v \int (N_1 + N_2) 4\pi r^2 dr = 1$$

From this and (11) $I + \sigma_a/\sigma' = 1$ and

$$\sigma' = \frac{\sigma_a}{1 - I} \qquad (11b)$$

follows. The σ'/σ corresponding to (11b) is the curve marked (b) in Fig. 4. For small σ_a/σ the two curves in Fig. 4 are close to each other indicating that (11) is a reasonably good approximation in this case. For large σ_a/σ, the σ' of (11b) becomes smaller than σ which shows that the approximation is poor in this case.

The above formulae give a representation of the whole neutron density due to a point source. The first part of the density N, i.e. N_1, is dominant at large distances from the source (except if σ_a/σ is nearly one). The second part, N_2, dominates in the neighborhood of the source. As σ_a/σ approaches 1, the most probable exponential in N_2 becomes the one with $\sigma' \approx \sigma \approx \sigma_a$. At the same time, the exponent $\varkappa$ of N_1 also approaches $\sigma \approx \sigma_a$. In this limiting case N_1 and N_2 have the same exponential. However, N_1 contains the factor I/r, N_2 the factor $1/r^2$. Since I goes, at the same time, to zero, the total N becomes in this limiting case $(4\pi r^2 v)^{-1} e^{-\sigma_a r}$ - as evident from physical considerations.

A third way of writing (10a)

$$N_2(r) = \int_\sigma^\infty \frac{e^{-\xi r}}{4\pi r v} \frac{\xi^2 d\xi}{\left(\xi - \frac{\sigma_s}{2} \ln \frac{\xi + \sigma}{\xi - \sigma}\right)^2 + \frac{\pi^2 \sigma_s^2}{4}} \tag{10b}$$

$[\xi = (1 + \eta)\sigma]$ can be obtained from (10b) by partial integration. This gives

$$N_2(r) = \int_\sigma^\infty \frac{e^{-\xi r}}{4\pi r^2 v} \frac{d}{d\xi} \frac{\xi^2}{\left(\xi - \frac{\sigma_s}{2} \ln \frac{\xi + \sigma}{\xi - \sigma}\right)^2 + \frac{\pi^2 \sigma_s^2}{4}} d\xi . \tag{12}$$

This integral, incidentally, converges for all r, including r = 0 and thus shows the behavior of $N_2(r)$ for small r. It gives $N_2(r)$ as a sum of functions $e^{-\xi r}/4\pi r^2 v$ with $\xi \geqq \sigma$. The coefficient is infinite (but integrable) for $\xi = \sigma$, drops hence and goes to zero as ξ^{-3} for large ξ. It stays positive for all ξ if $\sigma_s/\sigma > 8/\pi^2$, otherwise it becomes negative and approaches 0 for large ξ from below. Since the second factor has a relatively sharp maximum in the neighborhood of $\xi = \sigma$, it seems reasonable to replace (11c) by a single $e^{-\sigma' r}/4\pi r^2 v$ curve, the coefficient of which is equal to the integral of the second factor (i.e. equal to 1) and the σ' of which corresponds to the center of mass of the same factor, i.e. is equal to

$$\sigma' = \int_\sigma^\infty \xi \frac{d}{d\xi} \frac{\xi^2}{\left(\xi - \frac{\sigma_s}{2} \ln \frac{\xi + \sigma}{\xi - \sigma}\right)^2 + \frac{\pi^2 \sigma_s^2}{4}} d\xi . \tag{11c}$$

One obtains from this a σ' rather similar to that of (11a).

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It may be, in many cases, just as simple to use the accurate formula (10b) than any approximate expression. The functions $f_\xi = e^{-\xi^2 r/4\pi r v}$ obey a simple differential equation $\Delta f_\xi = \xi^2 f_\xi$ which makes it often possible to obtain the combined effect of a distribution of sources. If the source density is ρ, the equation $\Delta f_\xi - \xi^2 f_\xi + \rho/v = 0$ gives f_ξ. This integrated over ξ with the weight factor appearing in (10b) gives the "second part" of the neutron density created by sources of the density ρ. Similarly, the "first part" of the neutron density is in most cases most easily obtained by solving the equation $\Delta f - \varkappa^2 f + (I\varkappa^2/\sigma_a v)\rho = 0$. On the whole, one can say that the assumption of a single energy value makes it almost as easy to obtain the neutron density in an infinite homogeneous medium as it is to obtain the electric potential if the charges are given. It is much more difficult, however, to take into account the variation of energy as is also to consider a problem with two media.

3. The approximate expressions given above for the neutron density tempt one to try the following procedure: Take the path before the first collision into account rigorously and take the place where the first collision occurs to be the source of neutrons which are to be treated by a diffusion equation. Such a procedure deals with two kinds of neutrons, those which have not yet suffered any collision and those which have already suffered a collision. The sources of the latter are more extended than the original sources and it appears more justifiable to treat them by means of the diffusion equation. The procedure can be further generalized by treating not only the first

but the first two collisions rigorously etc. Evidently the above method will give more accurate results in the neighborhood of the source than the straight application of the diffusion theory. It will be seen, however, that no matter how many collisions one takes into account rigorously, the asymptotic behavior remains the same as in the straight diffusion theory, i.e. does not contain the factor I of (10). Of course, one can obtain a very good expression for the density at any point by taking into account sufficiently many collisions rigorously, but no matter how many collisions one treats this way, there is always a distance where the result becomes inaccurate. Let us consider the case of a point source of unit strength. The density before the first collision is, evidently

$$N_{b_1} = \frac{1}{4\pi r^2 v} e^{-\sigma r} = \int_{\sigma}^{\infty} \frac{e^{-\xi r}}{4\pi r v}\, d\xi \qquad (13)$$

The density, after the first collision N_{a_1} will be assumed to obey the equation

$$\frac{\sigma_a}{\varkappa^2} \Delta N_{a_1} - \sigma_a N_{a_1} + \frac{\sigma_a}{\sigma} N_{b_1} = 0 \qquad (14)$$

One may be tempted to introduce a factor I into the last term of (14) (which would give the correct asymptotic behavior to N_{a_1}) but this is not justifiable since, evidently, the integral of $\sigma_a N_{a_1}$ must be equal to the integral of the production $(\sigma_a/\sigma) N_{b_1}$. One can solve (14) most easily by using for N_{b_1} the last expression in (13) and writing for N_{a_1}

$$N_{a_1} = \int a(\xi) \frac{e^{-\xi r}}{4\pi r v}\, d\xi + a \frac{e^{-\varkappa r}}{4\pi r v}. \qquad (14a)$$

One obtains in this way

$$N_{a_1} = -\frac{\sigma_s}{\sigma}\int_{\sigma}^{\infty}\frac{\varkappa^2}{\sigma_a}\,\frac{1}{\xi^2-\varkappa^2}\,\frac{e^{-\xi r}}{4\pi r v}\,d\xi + \frac{\varkappa^2}{\sigma_a\sigma}\,\frac{e^{-\varkappa r}}{4\pi r v} \tag{14b}$$

The coefficient of (14a) is determined so that the Laplacian of N_{a_1} shall contain no δ-function at $r = 0$. Hence, N_{a_1} has only a logarithmic singularity at $r = 0$.

The solution of the problem in the present approximation is the sum of (13) and (14b). Had we used the diffusion equation right from the start, the solution would have been only the last term of (14b). One sees that the present procedure gives a much more nearly correct result at small distances. In particular, the density is $\sim 1/4\pi r^2 v$ near the origin, as it is in the accurate solution. However, at large distances the behavior of the sum of (13) and (14b) is the same as that of the solution of the diffusion equation. Comparison with (10) shows that it is too large by the factor $1/I$. Neither will this behavior change if one takes further collisions rigorously into account before going over to the diffusion equation: the neutrons which suffered their first collision at a point P will, even if another collision is taken into account rigorously, give the same density at large distances as in the foregoing treatment in which they were treated, from P on, by the diffusion equation. This holds for all points P and thus for the whole distribution. The reason for this surprising behavior is that the diffusion equation assumes a bias in the velocity distribution of the neutrons in the direction of decreasing density right from the start. In reality, this bias develops only after a distance of about one mean free path.

It may be worthwhile to remark that the calculation of the first two sections could be carried out also in case of a not spherically symmetric scattering. The exponent in N_1, i.e. ν, has already been calculated in A-21 for the case that the differential cross section contains, in addition to a constant term, a term proportional to the cosine of the scattering angle.

We had occasion to derive and use the above results for the calculation of the multiplication constant of a water cooled pile. In this case, some of the neutrons are made thermal in the water and a certain loss by initial absorption was to be expected. σ_a/σ is in this case about .01, and thus much greater than it is in graphite. However, figure 2 shows that the initial absorption remains quite small and amounts to only .8% causing a loss in the total multiplication constant of less than .02%.

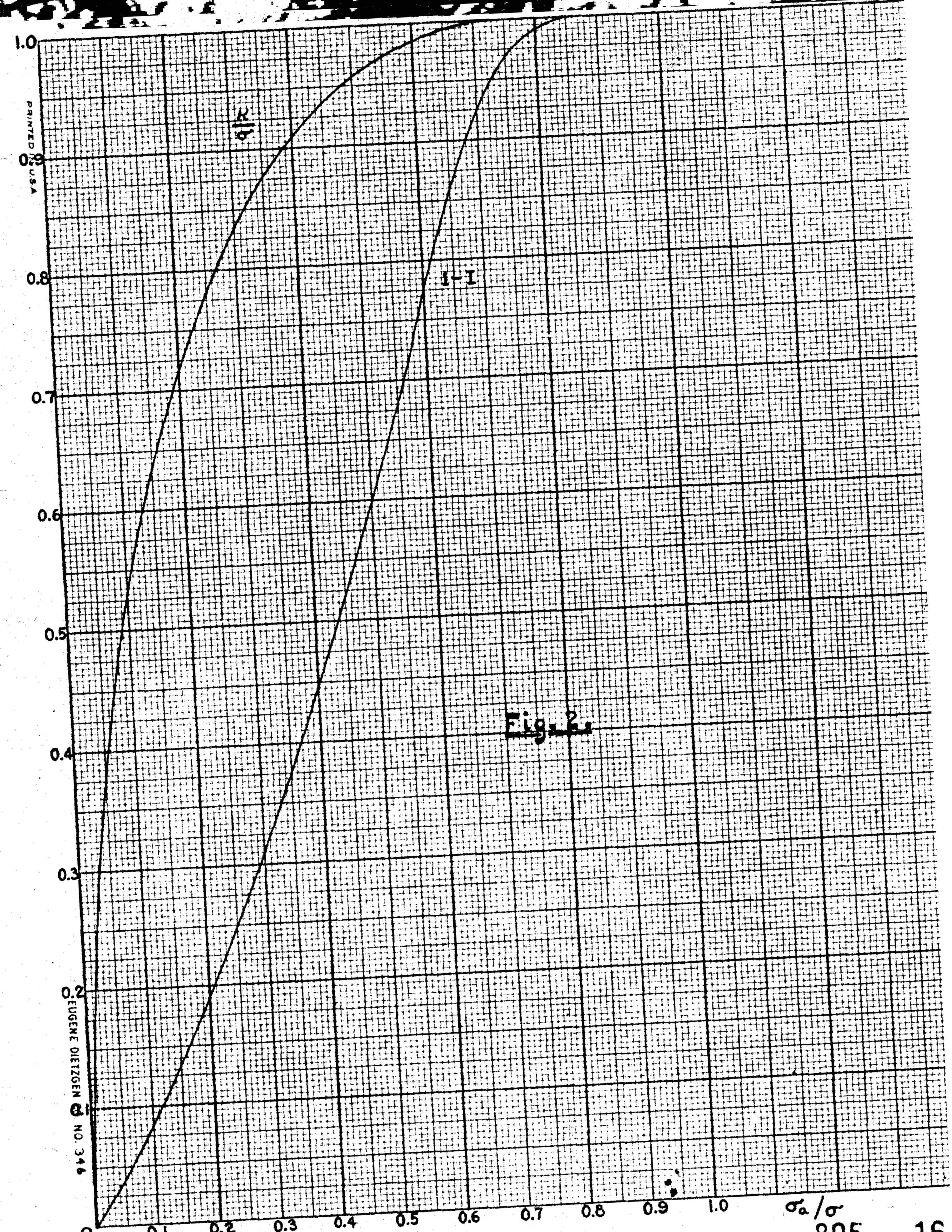
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
σ_a / σ
$\frac{k}{a}$
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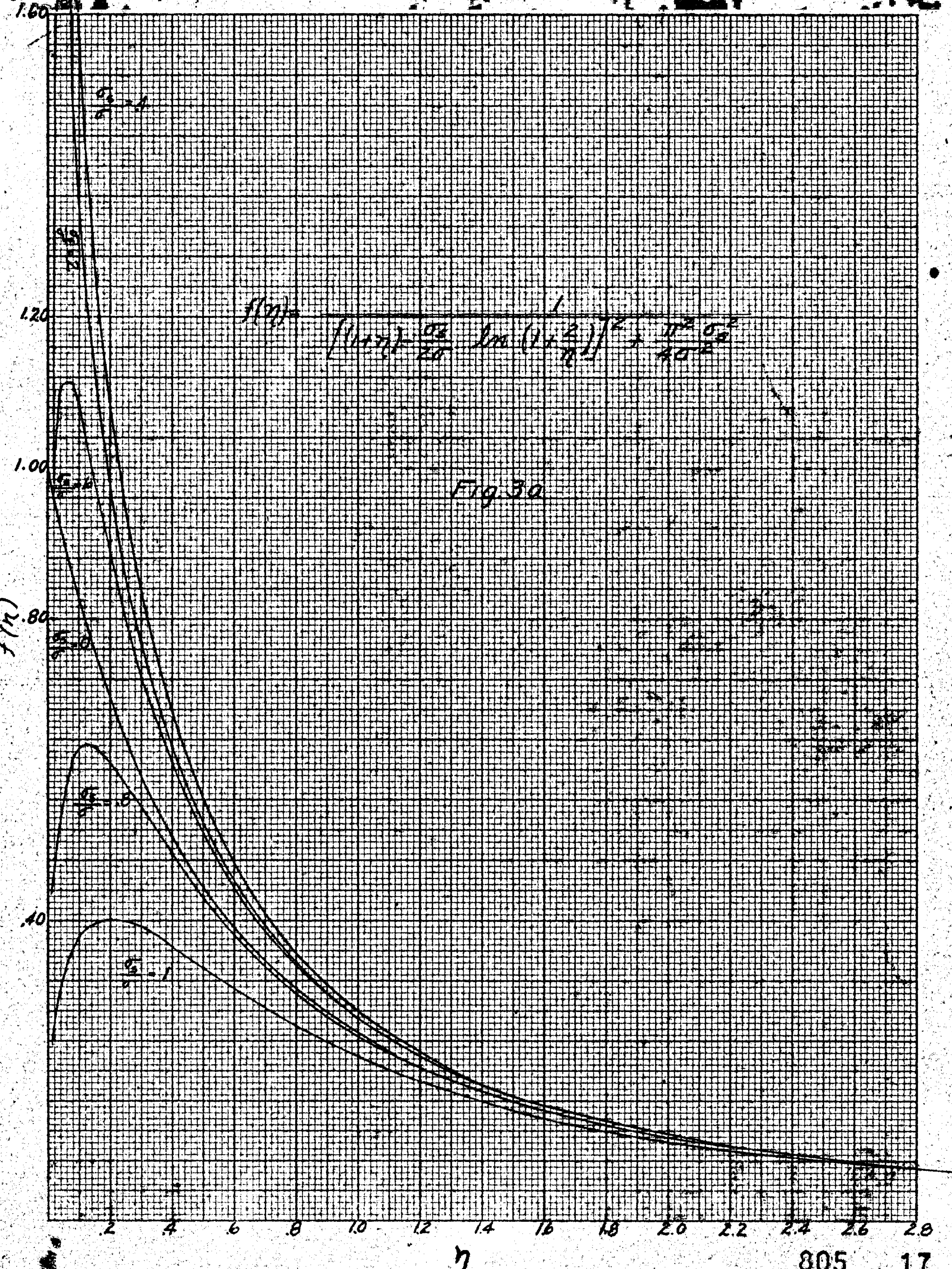
Fig. 2.



$f(\eta) = \dfrac{1}{\left[(1+\eta) - \frac{\sigma_2}{2\sigma}\ln\left(1+\frac{2}{\eta}\right)\right]^2 + \frac{\pi^2\sigma_2^2}{4\sigma^2}}$
$\frac{\sigma_2}{\sigma} = 4$
$\frac{\sigma_2}{\sigma} = 2$
$\frac{\sigma_2}{\sigma} = 0$
$\frac{\sigma_2}{\sigma} = 1$
$f(\eta)$
1.60
1.20
1.00
.80
.40
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
2.6
2.8
η

Fig. 3a

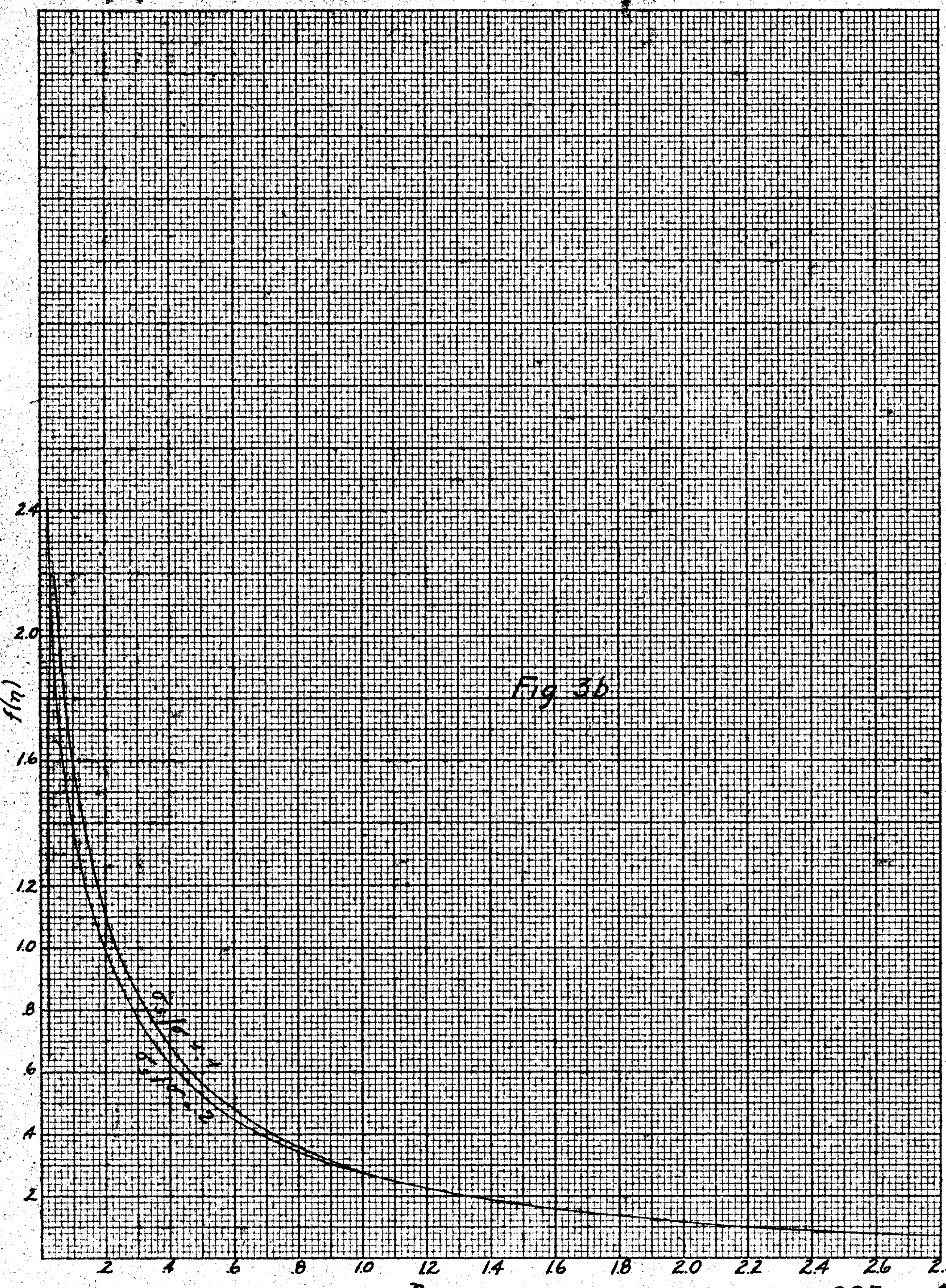


Fig 3b

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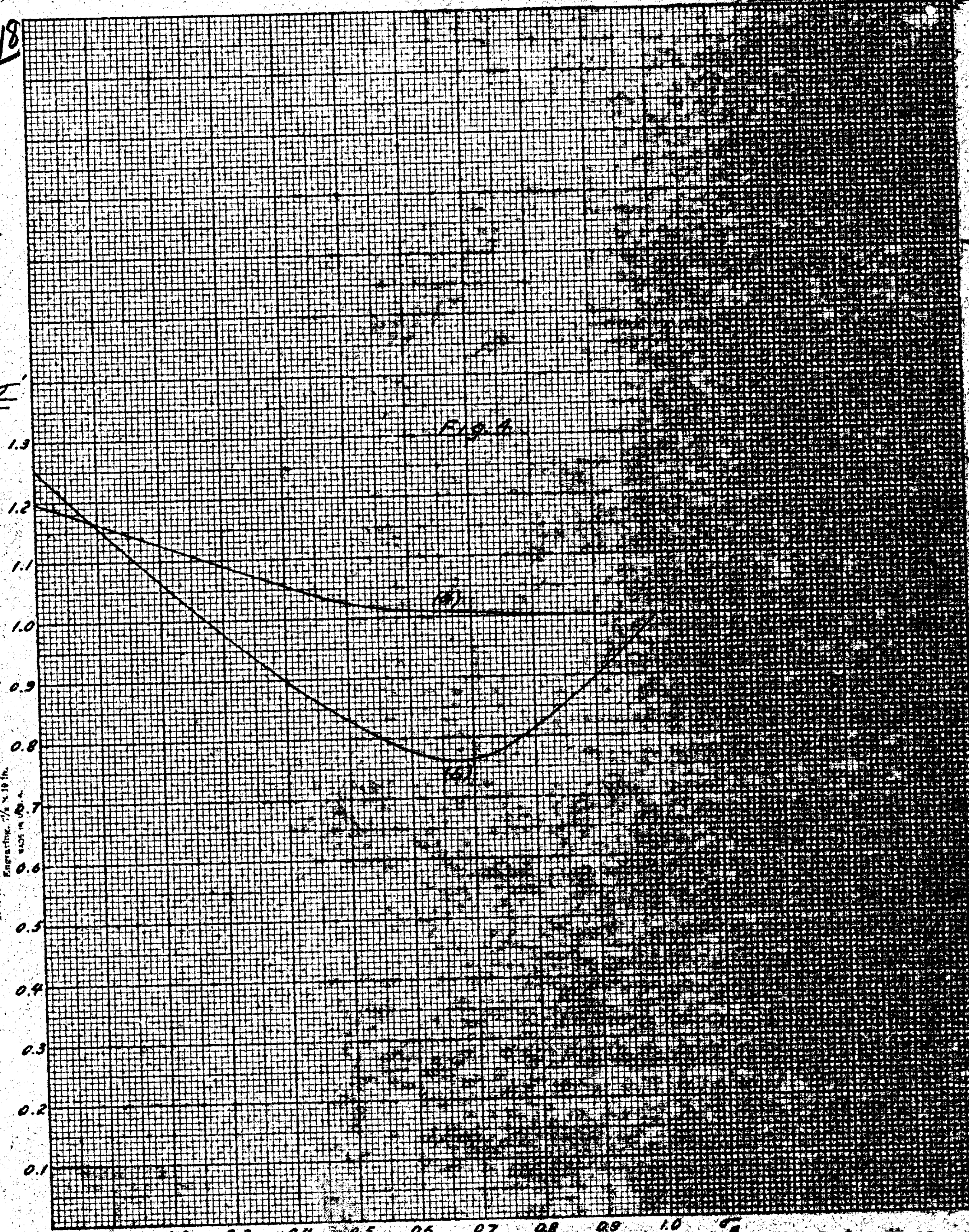

Fig. 4
(B)
(C)
σ/σ'
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
$\frac{\sigma_a}{\sigma}$
KEUFFEL & ESSER CO., N. Y. NO. 358-12
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