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FRACTIONAL QUANTIZATION OF THE HALL EFFECT

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Fractional Quantization of the Hall Effect *

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The Fractional Quantum Hall Effect is caused by the condensation of a two-dimensional electron gas in a strong magnetic field into a new type of macroscopic ground state, the elementary excitations of which are fermions of charge 1/m, where m is an odd integer.

1 Preliminary Considerations

We consider a two-dimensional metal in the x-y plane subject to a magnetic field H_0 in the z-direction. The many-body Hamiltonian is

$$H = \sum_j \left(\frac{1}{2m} \left| \frac{\hbar}{i} \vec{\nabla} - \frac{e}{c} \vec{A} \right|^2 + V(z_j) \right) + \sum_{j<k} \frac{e^2}{|z_j - z_k|} \quad , \tag{1}$$

where $z_j = x_j - iy_j$ is a complex number locating the j^{th} electron, $V(z_j)$ is the potential generated by a uniform neutralizing background of density σ

$$V(z) = -\sigma e^2 \int \frac{d^2 z'}{|z - z'|} \quad , \tag{2}$$

and $\vec{A} = \frac{H_0}{2} (y\hat{x} - x\hat{y})$ is the symmetric gauge vector potential. We restrict our attention to the lowest Landau level, for which the single-body wave-functions are

$$|n\rangle = \frac{1}{\sqrt{2^{n+1} \pi n!}} \; z^n \; e^{-\frac{1}{4}|z|^2} \quad , \tag{3}$$

with the magnetic length $a_0 = (\hbar c/eH_0)^{1/2}$ set to 1. These states are degenerate at energy $\hbar\omega_c / 2$, with $\omega_c = eH_0/mc$ the cyclotron frequency. We assume $\hbar\omega_c > e^2/a_0$.

2 Ground State

By analogy with liquid Helium, we propose a variational wavefunction for this system of the Jastrow form

$$\psi = \left(\prod_{j<k} f(z_j - z_k) \right) e^{-\frac{1}{4} \sum_\ell |z_\ell|^2} \quad , \tag{4}$$

as such wavefunctions are efficient as keeping the particle apart. Restriction to the lowest Landau level requires f to be a polynomial, the Pauli principle requires f to be odd, and conservation of angular momentum by H requires f to be homogeneous. Thus the only allowed wavefunctions of the Jastrow form are

$$|m\rangle \equiv \psi_m = \prod_{j<k} (z_j - z_k)^m \, e^{-\frac{1}{4}\sum_\ell |z_\ell|^2} \quad , \tag{5}$$

with m an odd integer. The nature of this state is understood by interpreting its square as the probability distribution function of a classical plasma, in the manner

$$|\psi_m|^2 = e^{-\beta\Phi} \quad , \tag{6}$$

with $\beta = 1/m$ and

$$\Phi = -2m^2 \sum_{j<k} \ln|z_j - z_k| + \frac{m}{2}\sum_\ell |z_\ell|^2 \quad . \tag{7}$$

Φ describes particles of "charge" m repelling one another logarithmically and being attracted logarithmically to a uniform background of "charge" density $\sigma_1 = 1/2\pi$. Local neutrality of this "charge" requires that the electrons be spread out to a density $\sigma_m = \sigma_1/m$. The Fractional Quantum Hall effect occurs when $\sigma = \sigma_m$.

We calculate $\langle m|m\rangle$ and $\langle m|H|m\rangle$ using the hypernetted chain approximation for the radial distribution function g(r) of the plasma. If we let $x = r/\sqrt{2m}$ and define fourier transforms in the manner

$$\hat{h}(k) = \int_0^\infty h(x)\, J_0(kx)\, x dx \quad , \tag{8}$$

where J_0 is an ordinary Bessel function of the first kind, then the equations we solve are [1,2]

$$g(x) = \exp\{\, h(x) - c_s(x) - 2mK_0(Qx)\,\} \quad , \tag{9}$$

where K_0 is a modified Bessel function of the second kind, Q is an arbitrary cutoff parameter, and

$$\hat{h}(k) = \hat{c}(k) + 2\hat{c}(k)\hat{h}(k) \quad , \tag{10}$$

with

$$\hat{c}_s(k) = \hat{c}(k) + \frac{2mQ^2}{k^2(k^2+Q^2)} \quad , \tag{11}$$

and $h(x) = g(x) - 1$. The numerical solution to these equations for m=3 is displayed in Figs. 1 and 2. The absence of structure in g(x) beyond x=4 reflects the liquid nature of the state. In terms of g(x), the total energy per electron is

$$U_{total} \equiv \frac{\langle m|H|m\rangle}{\langle m|m\rangle}/N - \frac{1}{2}\hbar\omega_c = \frac{1}{\sqrt{2m}}\int_0^\infty h(x)\, dx \quad , \tag{12}$$

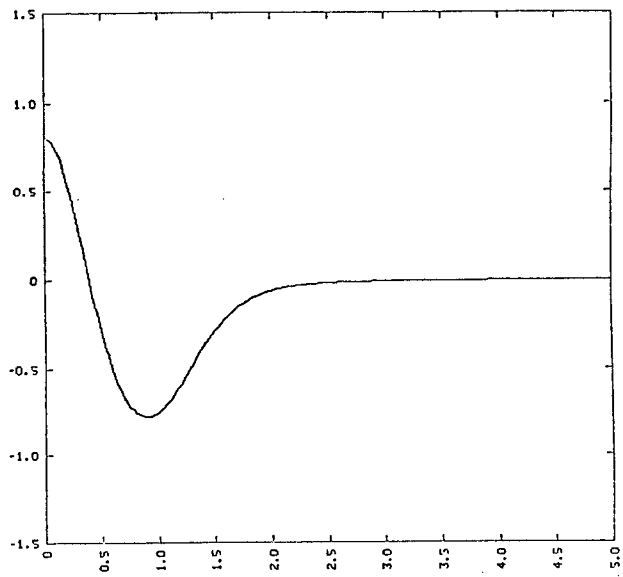

Figure 1: $c_s(x)$ versus x for m=3 and Q=2

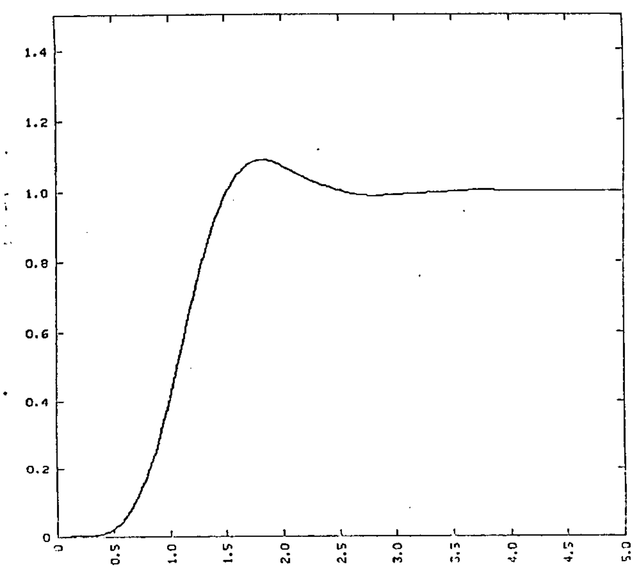

Figure 2: g(x) versus x for m=3

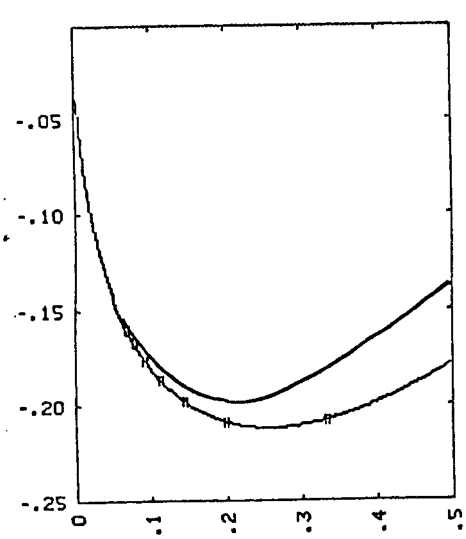

Figure 3: Cohesive energy per electron in units of e^2/a_0 versus filling factor $\nu = 1/m$. Top curve is charge density wave value from [3]. Bottom curve is (13).

in units of e^2/a_0. N is the number of electrons. We have fit a sequence of such calculations to the semiemperical formula

$$U_{total}(m) = \frac{0.814}{\sqrt{m}} \left(\frac{0.23}{m^{0.64}} - 1 \right) . \tag{13}$$

The cohesive energy per electron, defined by

$$U_{coh} = U_{total} - \sqrt{\frac{\pi}{8}} \; \frac{1}{m} , \tag{14}$$

is compared with that calculated by YOSHIOKA and FUKUYAMA [3] for a charge density wave in Fig. 3. The normalization integral $\langle m|m\rangle$ is the plasma partition function, and is given by

$$\frac{1}{N} \ln(\langle m|m\rangle) = mN \left(\frac{1}{2}\ln(2mN) - \frac{3}{4} \right) + \ln(2mN) - \frac{m}{2}\ln(2m)$$

$$- 2mf(2m) + O\left[\frac{\ln(N)}{N}\right] , \tag{15}$$

where f is a slowly varying function of order 1 fit from monte carlo experiments [4] to the formula

$$f(\Gamma) = A + \frac{B}{\Gamma^{\alpha}} + \frac{C}{\Gamma^{\gamma}} + \frac{D}{\Gamma} , \tag{16}$$

with $\Gamma = 2m$, valid in the range of interest. The parameters are listed in Table 1. The function f is the excess free energy of the plasma, while the remaining terms are "electrostatic" in nature, except for $\ln(2mN)$, which is just the log of the volume.

Table 1

A = -0.3755	D = -1.2862
B = 1.6922	α = 0.74
C = 0.1494	γ = 1.70

3 Quasiparticles

The elementary excitations of ψ_m are made with a thought experiment in which the exact ground state is pierced at location z_0 with an infinitely thin magnetic solenoid through which is passed adiabatically a flux quantum hc/e . The solenoid may then be removed by a gauge transformation, leaving behind an exact excited state of the many-body Hamiltonian. Operators which approximate the effect of this procedure are

$$S_{z_0} = \prod_i (a_i^{\dagger} - z_0) , \tag{17}$$

and its hermitean adjoint $S_{z_0}^{\dagger}$, where a_j is the ladder operator

$$a_j = \frac{x_j + iy_j}{2} + \left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right) . \tag{18}$$

That they do so may be seen from the fact that the thought experiment maps the single-body states (3) in the manner $|n\rangle \to |n\pm 1\rangle$, whereas

$$a|n\rangle = \sqrt{2n}\ |n-1\rangle \tag{19}$$

and

$$a^{\dagger}|n\rangle = \sqrt{2(n+1)}\ |n+1\rangle . \tag{20}$$

The operator a annihilates $|0\rangle$, consistent with the thought experiment's mapping it to the next Landau level. Note that S_{z_0} and $S_{z_0}^{\dagger}$ are exact for non-interacting electrons when they are described by a single Slater determinant of the single-body functions $|n\rangle$.

We calculate quasiparticle properties with the hypernetted chain. For the quasihole wavefunction

$$S_{z_0}|m\rangle \equiv \psi_m^{+z_0} = e^{-\frac{1}{4}\sum_{\ell}|z_\ell|^2} \prod_i (z_i - z_0) \prod_{j<k} (z_j - z_k)^m , \tag{21}$$

we write $|\psi_m^{+z_0}|^2 = e^{-\beta\Phi'}$, with $\beta = 1/m$ and

$$\Phi' = \Phi - 2m \sum_i \ln|z_i - z_0| . \tag{22}$$

This is a plasma with two components, N particles of "charge" m and one particle of "charge" 1. The two-component hypernetted chain equations are

$$g_{ij}(x) = \exp\{ -\beta v_{ij}(x) + h_{ij}(x) - c_{ij}(x) \} , \tag{23}$$

and

$$\hat{h}_{ij}(k) = \hat{c}_{ij}(k) + 2 \sum_{\ell} \hat{h}_{i\ell}(k)\, \rho_\ell\, \hat{c}_{\ell j}(k) , \tag{24}$$

where the indices run over the two kinds of particle. With x defined as before, the densities are $\rho_1 = 1$ and $\rho_2 = 1/N$. To solve the problem, we do perturbation theory in ρ_2: The zero-order solution to g_{11} is given by (9) through (11) . For $g_{12}(x)$ we have

$$\hat{h}_{12}(k) = \{ 1 + 2\hat{h}_{11}(k) \} \hat{c}_{12}(k) , \tag{25}$$

$$\hat{c}_{12_s}(k) = \hat{c}_{12}(k) + \frac{2Q^2}{k^2(k^2+Q^2)} , \tag{26}$$

and

$$g_{12}(x) = \exp\{ h_{12}(x) - c_{12_s}(x) - 2K_0(Qx) \} . \tag{27}$$

The numerical solution of these equations for m=3 is shown in Figs. 4 and 5. Note that the divergence of (26) as $k \to 0$ requires the total excess charge accumulated around z_0 to be exactly -1/m of an electron. Using $g_{12}(x)$, we construct the change to $g_{11}(x)$ resulting from the presence of the quasihole.

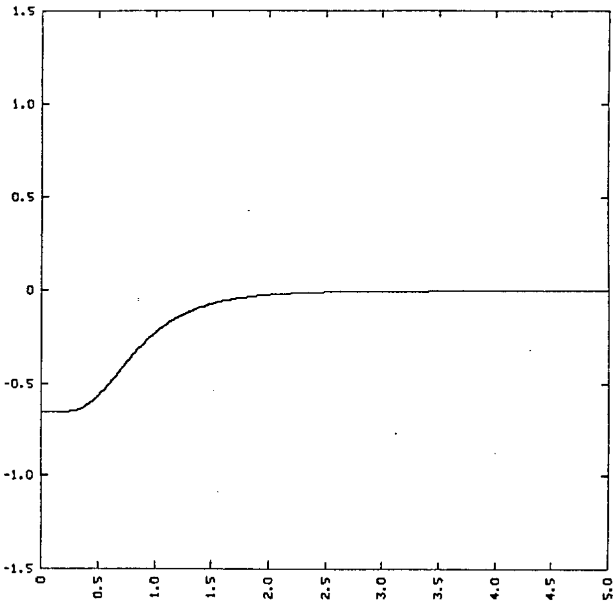

Figure 4: $c_{12_s}(x)$ versus x for m=3 and Q=2

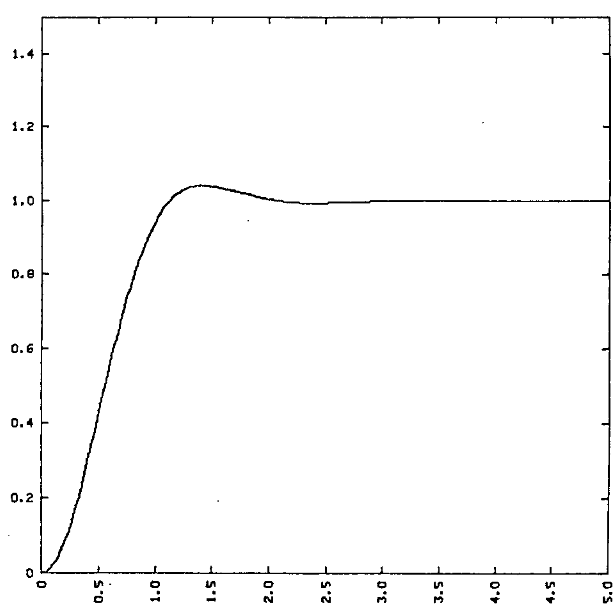

Figure 5: $g_{12}(x)$ versus x for m=3

We have

$$\delta\hat{h}_{11}(k) \simeq \{1 + 2\hat{h}_{11}(k)\}^2 \; \delta\hat{c}_{11}(k) + \frac{2}{N}\hat{h}_{12}(k) \quad , \qquad (28)$$

and

$$\delta c_{11}(x) = \left(\frac{h_{11}(x)}{1 + h_{11}(x)} \right) \delta h_{11}(x) \quad . \qquad (29)$$

The solution $N\delta h_{11}(x)$ to these equations for m=3 is plotted in Fig. 6. The energy to make a quasihole can be calculated from it in the manner

$$\Delta_{\text{Quasihole}} = \frac{N}{\sqrt{2m}} \int_0^\infty \delta h_{11}(x)\, dx \quad , \qquad (30)$$

in units of e^2/a_0. We obtain 0.026, which is considerably lower than the "Debye" estimate of 0.062.

A similar procedure may be used for the quasielectron. We have

$$S_{z_0}^\dagger |m\rangle \equiv \psi_m^{-z_0} = e^{-\frac{1}{4}\sum_\ell |z_\ell|^2} \prod_i \left(2\frac{\partial}{\partial z_i} - z_0^*\right) \prod_{j<k} (z_j - z_k)^m \quad . \qquad (31)$$

Normalizing this wavefunction and calculating its charge density involve integrating over spatial variables, which allows us to integrate by parts and then consider a situation similar to (21) and (22) but with [1]

$$\Phi' = \Phi - 2m \sum_i \ell n\{ |z_i - z_0|^2 - 2 \} \quad . \qquad (32)$$

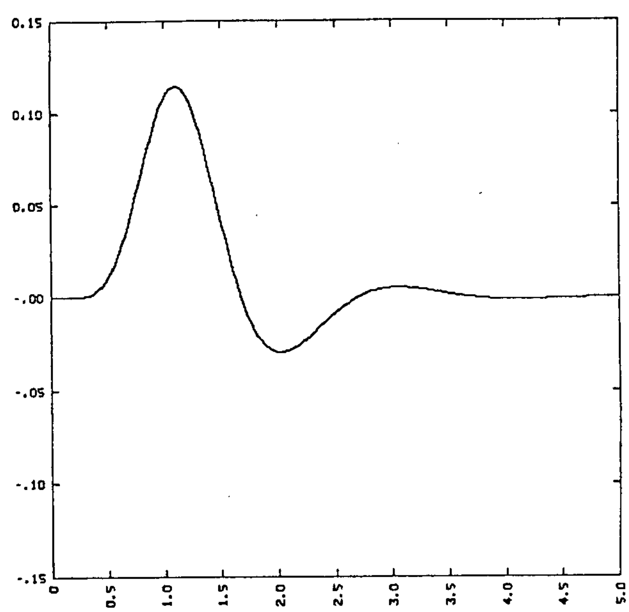

Figure 6: $\delta h_{11}(x)$ versus x for quasihole at m=3.

For this problem, we obtain an "integrated by parts" $\tilde{g}_{12}(x)$ and $\tilde{c}_{12_s}(x)$ satisfying (25) and (26), but with

$$\tilde{g}_{12}(x) = \left(\frac{x^2-2}{x^2} \right) \exp\{ \tilde{h}_{12}(x) - \tilde{c}_{12}(x) - 2K_0(Qx) \} . \qquad (33)$$

The numerical solution of these equations with m=3 is shown in Figs. 7 and 8. As with the quasihole, the Ornstein-Zernicke relation (25) forces the total charge accumulated around z_0 to be -1/m electrons. However, the *actual* $g_{12}(x)$, given by

$$g_{12}(x) = \left\{ \frac{1}{2m} \left(\frac{\partial^2}{\partial x^2} + \frac{1}{x}\frac{\partial}{\partial x}\right) + 2x\frac{\partial}{\partial x} + 2mx^2 + 2 \right\} \left(\frac{\tilde{g}_{12}(x)}{2mx^2-2} \right) \qquad (34)$$

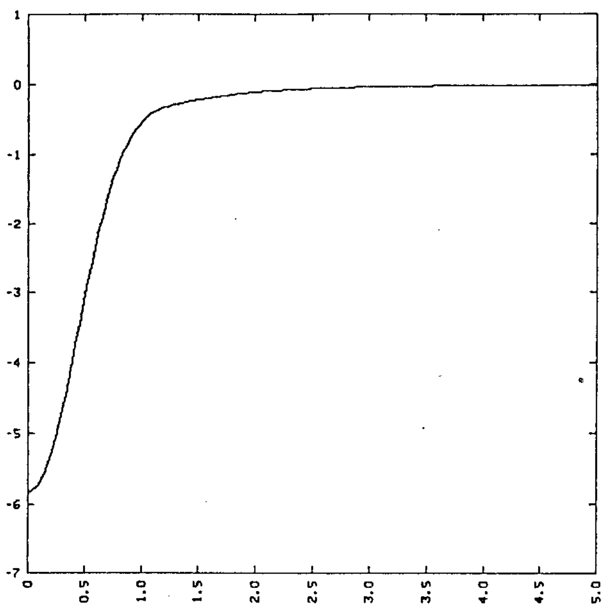

Figure 7: $\tilde{c}_{12_s}(x)$ versus x for m=3 and Q=2

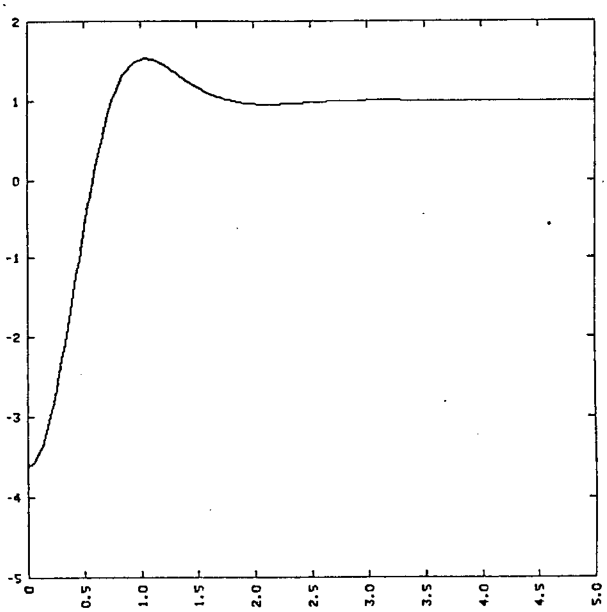

Figure 8: $\tilde{g}_{12}(x)$ versus x for m=3

correctly accumulates +1/m of an electron. $g_{12}(x)$ is shown in Fig. 9. To calculate the quasielectron creation energy, we employ the somewhat uncontrolled approximation of assuming the existence of a "pseudopotential" which when used as $v_{12}(x)$ in (23) and (24) reproduces $g_{12}(x)$. To the extent such a potential is physical, we can calculate $\delta h_{11}(x)$ using (28) and (29), and then calculate the quasielectron creation energy using (30). In Fig. 10, we show the $\delta h_{11}(x)$ obtained using this procedure. Note the similarity to Fig. 6. The quasielectron creation energy we obtain using this $\delta h_{11}(x)$ is 0.030 in units of e^2/a_0.

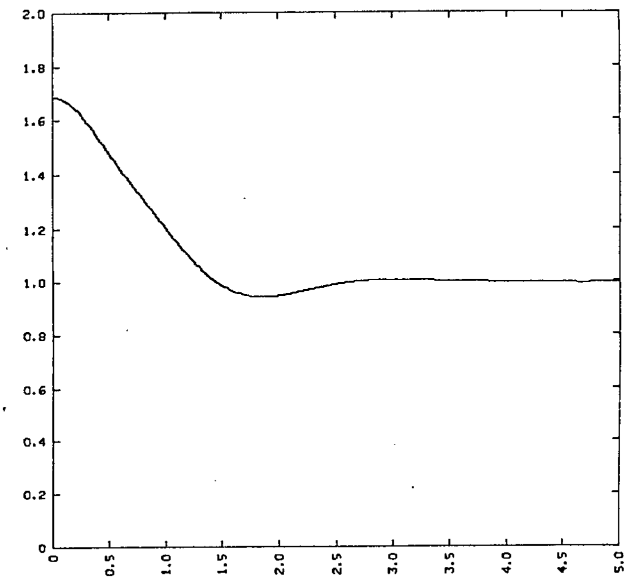

Figure 9: $g_{12}(x)$ versus x for quasielectron at m=3

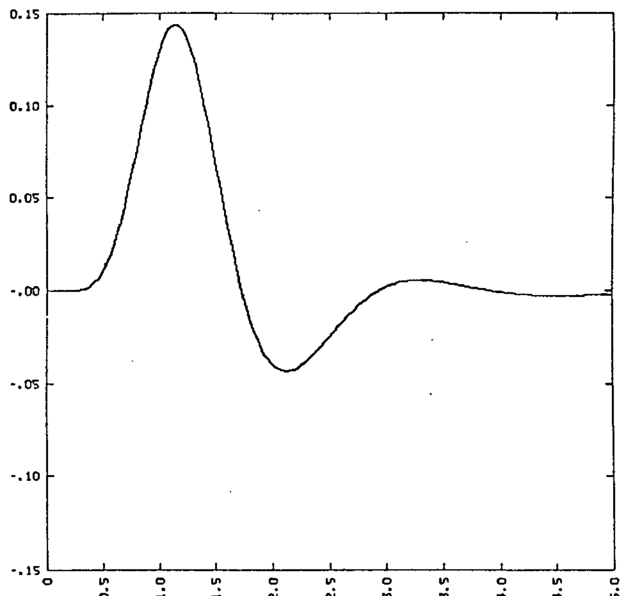

Figure 10: $\delta h_{11}(x)$ versus x for quasielectron at m=3

Operators S_k and $S_k^\dagger$ creating a quasiparticle in an angular momentum state analogous to the single-body state $|n\rangle$ in (3) are the elementary symmetric polynomials [5], defined by the expression

$$S_{z_0} = \sum_k S_k z_0^k \quad . \tag{35}$$

We have explicitly

$$S_0 = z_1 z_2 z_3 \cdots z_N \quad , \tag{36}$$

$$S_1 = -\sum_j z_1 z_2 \cdots \hat{z}_j \cdots z_N \quad , \tag{37}$$

$$\vdots \qquad\qquad \vdots$$

$$S_{N-1} = (-1)^{N-1} (z_1 + \cdots + z_N) \quad , \tag{38}$$

where $\hat{z}_j$ means omit this factor from the product. When m=1, the state $S_k|m\rangle$ is a full Landau level, but for a hole in $|k\rangle$, that is, a hole with orbit radius $\sqrt{2k+2}$. We now show that the quasiparticle behaves *kinematically* as though it has charge e/m: the orbit radius of $S_k|m\rangle$ or $S_k^\dagger|m\rangle$ is exactly $\sqrt{2mk+2}$.

We first observe that since there are no thermodynamic forces on plasma particles, provided they feel the neutralizing background potential, we have

$$\langle m|S_{z_0}^\dagger S_{z_0}|m\rangle = e^{\frac{1}{2m}|z_0|^2} \langle m|S_0^\dagger S_0|m\rangle \quad . \tag{39}$$

However, we also have

$$\langle m|S_{z_0}^\dagger S_{z_0}|\ \rangle = \sum_{k,k'} (z_0^*)^{k'} (z_0)^k \langle m|S_{k'}^\dagger S_k|m\rangle \ , \tag{40}$$

so that

$$\langle m|S_{k'}^\dagger S_k|m\rangle = \frac{\delta_{kk'}}{(2m)^k k!} \langle m|S_0^\dagger S_0|m\rangle \quad , \tag{41}$$

and similarly for the adjoint. We next observe that from translational invariance of the plasma, matrix elements of the charge density operator $\rho(z)$ may be computed from the relation

$$\langle m|S_{z_0}^\dagger \rho(z) S_{z_0}|m\rangle = \sum_{k,k'} (z_0^*)^{k'} (z_0)^k \langle m|S_{k'}^\dagger \rho(z) S_k|m\rangle$$

$$= \frac{\langle m|S_0^\dagger S_0|m\rangle}{2\pi m} \; e^{\frac{1}{2m}|z_0|^2} \; g_{12}(\,|z-z_0|\,) \quad . \tag{42}$$

Thus

$$\frac{\langle m|S_k^\dagger \rho(z) S_k|m\rangle}{\langle m|S_k^\dagger S_k|m\rangle} = \frac{1}{2\pi m}\left[1 + \frac{(2m)^k}{k!}\left(\frac{\partial}{\partial z_0^*}\frac{\partial}{\partial z_0}\right)^k \{ e^{\frac{1}{2m}|\;\;|^2} \right.$$

$$\left. \times\, h_{12}(\,|z-z_0|\,)\,\}\,\Big|_{z_0=0} \right] \quad . \tag{43}$$

Since $h_{12}(x)$ is short-ranged, the charge density is $(2\pi m)^{-1}$ almost everywhere. Also, since from the charge-neutrality sum rule

$$\frac{1}{2\pi m}\int h_{12}(|z|)\, d^2z = -\frac{1}{m} \quad , \tag{44}$$

we have

$$\int \left(\frac{\langle m|S_k^\dagger \rho(z) S_k|m\rangle}{\langle m|S_k^\dagger S_k|m\rangle} - \frac{1}{2\pi m} \right) d^2z = -\frac{1}{m} \quad . \tag{45}$$

Similarly, the constant-screening sum rule [2]

$$\frac{1}{2\pi m}\int h_{12}(|z|)\, |z|^2\, d^2z = -\frac{2}{m} \quad , \tag{46}$$

implies that

$$\int \left(\frac{\langle m|S_k^\dagger \rho(z) S_k|m\rangle}{\langle m|S_k^\dagger S_k|m\rangle} - \frac{1}{2\pi m} \right) |z|^2\, d^2z$$

$$= -\frac{2}{m} + \frac{1}{2\pi m}\left(\frac{(2m)^k}{k!}\left(\frac{\partial}{\partial z_0^*}\frac{\partial}{\partial z_0}\right)^k \{ e^{\frac{1}{2m}|z_0|^2}\, |z_0|^2 \}\,\Big|_{z_0=0} \right.$$

$$= -\frac{1}{m}\left[2(km+1) \right] \quad . \tag{47}$$

and similarly for quasielectrons.

4 Acknowledgements

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