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Relativistic Quark Model as Representation of Current Algebra[*]

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ERRATA for

"Relativistic Quark Model as Representation of Current Algebra"

by MURRAY GELL-MANN

(CALT-68-103)

Page 9: 2nd line of   instead of $+ e^{iS_2} \dfrac{\lambda_i^{(2)}}{2} \ldots \ldots$
         Eq. (2.5):

                   read $- e^{iS_2} \dfrac{\lambda_i^{(2)}}{2} \ldots \ldots$

Page 20: Line 5:   instead of   "commutable infinity"

              read   "countable infinity"

Page 32: Line 3:   read   "$\beta^{(1)} = 1$ when the whole system is at rest

                   or $\underset{\sim}{P} = 0$; for a free particle"

        Line 5:   read   "quark is at rest, or $\underset{\sim}{P}/2 + \underset{\sim}{\pi} = 0$. Thus we
                       construct"

       Eq. (5.17):   instead of $\dfrac{i\gamma^{(1)} \cdot \pi}{\pi^2}$ read $\dfrac{i\gamma^{(1)} \cdot \pi}{\sqrt{\pi^2}}$

                 instead of $\dfrac{i\gamma^{(2)} \cdot \pi}{\pi^2}$ read $\dfrac{i\gamma^{(2)} \cdot \pi}{\sqrt{\pi^2}}$

Page 24: last 2 lines: instead of $\sqrt{M_N^2 + P_z^2}$ read $\sqrt{M_N^2 + P_z^2 + k^2/4}$

                   instead of $\sqrt{M_N'^2 + P_z^2}$ read $\sqrt{M_N'^2 + P_z^2 + k^2/4}$

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Page 29: Eq. (5.5): Add to the right-hand side the term

$$\frac{(\sigma_z^{(1)} + \sigma_z^{(2)})\, p_y}{4m^2}$$

Eq. (5.6): Add to the right-hand side the term

$$- \frac{(\sigma_z^{(1)} + \sigma_z^{(2)})\, p_x}{4m^2}$$

I.   CURRENT ALGEBRA AT INFINITE MOMENTUM

Let us summarize the discussion in the first part of this work on the use of equal-time commutation relations of charge densities at infinite momentum. We consider the space of all quantum states of hadrons, to all orders in the strong interaction, and with weak, electromagnetic, and gravitational interactions treated as perturbations. We assume the validity of equal-time commutation relations among the time-components of vector currents $\mathcal{F}_{i\alpha}(x)$ and axial vector currents $\mathcal{F}_{i\alpha}{}^5(x)$, with i = 0, 1, . . . 8, such that only δ-functions appear on the right-hand sides and no derivatives of δ-functions. We sandwich these relations between hadron states of equal momentum P_z in the z-direction (with arbitrary finite P_x and P_y) and let the common value of P_z tend to infinity.

In fact, we work with transverse Fourier components of the charge densities:

$$F_i(\underset{\sim}{k}_\perp) \equiv \int d^3x \ e^{i\underset{\sim}{k}_\perp \cdot x} \ \mathcal{F}_{io} \quad , \quad F_i{}^5(\underset{\sim}{k}_\perp) \equiv \int d^3x \ e^{i\underset{\sim}{k}_\perp \cdot x} \ \mathcal{F}_{io}{}^5 \quad ,$$

(1.1)

where $\underset{\sim}{k}_\perp$ is in the x-y plane. These operators, by virtue of our assumed commutation rules, have the commutators

$$[F_i(\underset{\sim}{k}_\perp), F_j(\underset{\sim}{k}'_\perp)] = i \, f_{ijk} \, F_k(\underset{\sim}{k}_\perp + \underset{\sim}{k}'_\perp) \quad , \tag{1.2}$$

$$[F_i(\underset{\sim}{k}_\perp), F_j{}^5(\underset{\sim}{k}'_\perp)] = i \, f_{ijk} \, F_k{}^5(\underset{\sim}{k}_\perp + \underset{\sim}{k}'_\perp) \quad , \tag{1.3}$$

$$[F_i{}^5(\underset{\sim}{k}_\perp), F_j{}^5(\underset{\sim}{k}'_\perp)] = i \, f_{ijk} \, F_k(\underset{\sim}{k}_\perp + \underset{\sim}{k}'_\perp) \quad . \tag{1.4}$$

Now we label each hadron state by its momentum $(P_x, P_y, P_z = \infty)$, its helicity h (effectively equal to J_z since $P_z \to \infty$ with P_x, P_y finite), and an index N that includes all other labels, such as mass M, internal angular momentum $\mathcal{J}$, parity $\mathcal{P}$, etc. We then find that the matrix element of $F_i(\underset{\sim}{k}_\perp)$ or $F_i^5(\underset{\sim}{k}_\perp)$ has the property

$$\langle N', h', \underset{\sim}{P}'_\perp \,|F_i(\underset{\sim}{k}_\perp)|\, N, h, \underset{\sim}{P}_\perp \rangle = \langle N'h' \,|F_i(\underset{\sim}{k}_\perp)|\, Nh \rangle \quad , \qquad (1.5)$$

$$\langle N', h', \underset{\sim}{P}'_\perp \,|F_i(\underset{\sim}{k}_\perp)|\, N, h, \underset{\sim}{P}_\perp \rangle = \langle N'h' \,|F_i^5(\underset{\sim}{k}_\perp)|\, Nh \rangle \quad , \qquad (1.6)$$

that is, at $P_z = \infty$ the matrix elements are independent of the sum $\underset{\sim}{P}'_\perp + \underset{\sim}{P}_\perp$ of the final and initial transverse momenta and depend only on the difference $\underset{\sim}{P}'_\perp - \underset{\sim}{P}_\perp$, which equals $\underset{\sim}{k}_\perp$ by momentum conservation; thus we can write the matrix elements in abbreviated form as in (1.5) and (1.6).

The analogue at $P_z = \infty$ of the angular momentum $\underset{\sim}{J}$ of the system at rest is easily defined; we call it $\underset{\sim}{\mathcal{J}}$ and it is made up of $\mathcal{J}_z \equiv h$ and $\mathcal{J}_x$ and $\mathcal{J}_y$ defined by

$$\langle N'h' \,|\mathcal{J}_x|\, Nh \rangle \equiv \langle h' \,|\mathcal{J}_x|\, h \rangle\, \delta_{NN'} \quad , \quad \langle N'h' \,|\mathcal{J}_y|\, Nh \rangle \equiv \langle h' \,|\mathcal{J}_y|\, h \rangle\, \delta_{NN'} \quad , \qquad (1.7)$$

where $\langle h' \,|\mathcal{J}_x|\, h \rangle$ and $\langle h' \,|\mathcal{J}_y|\, h \rangle$ are ordinary Clebsch-Gordan coefficients. A state of intrinsic angular momentum $\mathcal{J}_N$ then has, at $P_z = \infty$,

$$\underset{\sim}{\mathcal{J}}^2 \,|N, h \rangle = \mathcal{J}_N (\mathcal{J}_N + 1)\, |N, h \rangle \quad . \qquad (1.8)$$

The operators M (mass) and $\mathscr{P}$ (intrinsic parity) have similar behavior:

$$M \,|N, h\rangle \;=\; M_N \,|N, h\rangle \quad , \quad \mathscr{P}\,|N, h\rangle \;=\; \mathscr{P}_N\,|N, h\rangle \quad . \quad (1.9)$$

We now establish the angular momentum and parity properties of the operators $F_i(\underset{\sim}{k}_\perp)$ and $F_i^{\,5}(\underset{\sim}{k}_\perp)$ according to relativistic kinematics. Since the Lorentz transformation from rest to a fixed momentum P_z in the z-direction depends on the mass M, the angular properties of the matrix elements of $F_i(\underset{\sim}{k}_\perp)$ and $F_i^{\,5}(\underset{\sim}{k}_\perp)$ involve the initial and final masses M and M' respectively. The conditions imposed by relativistic kinematics turned out as follows. First of all,

$$\langle N'h' \,|\exp\left\{i\,\mathscr{J}_y\,(\arctan \tfrac{M'-M}{k} - \arctan \tfrac{k}{M'+M})\right\}$$

$$F_i(\underset{\sim}{k}_\perp)\,\exp\left\{-i\,\mathscr{J}_y\,(\arctan \tfrac{M'-M}{k} + \arctan \tfrac{k}{M'+M})\right\}|\,Nh\rangle \qquad (1.10)$$

has $|\Delta \mathscr{J}_x| \leq 1$, and the same for $F_i^{\,5}(\underset{\sim}{k}_\perp)$, with $\underset{\sim}{k}_\perp$ taken in the x-direction. Here the matrices $\mathscr{J}_y$ act only on the initial and final helicity indices h and h', and the rotation angles depend on the initial and final mass eigenvalues, M and M' respectively. **The parity and time-reversal properties are reproduced here:**

$$F_i(k) \text{ even under } \mathscr{P}\,e^{i\pi\,\mathscr{J}_y} \text{ and } \mathscr{T}\,e^{i\pi\,\mathscr{J}_y} \quad ,$$

$$\hspace{10cm} (1.11)$$

$$F_i^{\,5}(k) \text{ odd under } \mathscr{P}\,e^{i\pi\,\mathscr{J}_y} \text{ and even under } \mathscr{T}\,e^{i\pi\,\mathscr{J}_y} \quad .$$

We repeat also the properties under k-reversal:

$F_i(k) + F_i(-k)$ has $\Delta \mathcal{G}_z$ even   , $F_i(k) - F_i(-k)$ has $\Delta \mathcal{G}_z$ odd ,

$$(1.12)$$

$F_i^5(k) + F_i^5(-k)$ has $\Delta \mathcal{G}_z$ even , $F_i^5(k) - F_i^5(-k)$ has $\Delta \mathcal{G}_z$ odd .

Finally, there are the conditions on $|\Delta \mathcal{G}|$ for the coefficient of k^j when $M' - M = 0$ (the allowed multipole conditions) and, if we want to include them, the forbidden multipole conditions to each order in $M' - M$. These are mentioned in the first part of the notes, but we shall not use them explicitly; they often come out automatically.

II. RELATIVISTIC REPRESENTATION OF CURRENT ALGEBRA

We have postulated that, to all orders in the strong interaction, the set of all quantum states describing hadron systems with a given value A of baryon number, at $P_z = \infty$, constitute a representation of the commutation relations (1.2 - 1.4) of the local algebra of vector and axial vector charge densities. We have seen also that, defining angular momentum $\underline{\mathcal{J}}$ and parity $\mathcal{P}$ for these states, as well as a mass operator M, the angular relation (1.10) and the further conditions (1.11 - 1.12) are obeyed exactly between every pair of states.

Dashen and I now propose to construct smaller mathematical representations not only of the current commutation rules but of the whole algebraic system composed of $F_i(\underline{k}_\perp)$, $F_k^5(\underline{k}_\perp)$, $\underline{\mathcal{J}}$, $\mathcal{P}$, and M. The purpose is to describe approximately an idealized infinite set of meson or baryon states, including the well-known low-lying bound and resonant states, and perhaps including all well-defined resonances. At very high masses (many BeV) no doubt the description in terms of such resonances is highly idealized, but over the first few BeV the resonances may well be a good approximation to the real situation in which complicated continua of many particles are involved. So far it has been our experience that subsuming these continua in a set of resonances gives a reasonable description of many phenomena.

If we find such small representations of the algebraic system, with very few degrees of freedom, we may suppose that all or nearly all of the baryon bound states and resonances belong approximately to such a representation in the same sense that the lowest ten baryons with $J = 3/2^+$ belong approximately to the $\underline{10}$ representation of the U(3) algebra of the $F_i(0)$. In fact, the state $\Omega^-(1675)$ is bound and

perfectly well-defined, while $\Delta(1240)$ is a very broad resonance that actually represents a chunk of (πN) continuum; nevertheless, the decimet approximation has been highly successful. In the same way, we have hopes of describing all or most of the baryon resonances in a useful approximation as belonging to a single relativistic representation of our algebraic system, with a small number of degrees of freedom, rather than the huge representation to which all the states, including all the continua, actually belong exactly.

To represent just the commutation relations (1.2-1.4) with a small number of degrees of freedom is trivial. We could take for the mesons, for example, a representation corresponding to a mathematical quark and antiquark:

$$F_i(k_\perp) \;\rightarrow\; \frac{\lambda_i^{(1)}}{2}\, e^{ik_\perp \cdot x/2} \;+\; \frac{\lambda_i^{(2)}}{2}\, e^{-ik_\perp \cdot x/2} \quad,$$

$$F_i^{\,5}(k) \;\rightarrow\; \frac{\lambda_i^{(1)}}{2}\, \sigma_z^{(1)}\, e^{ik_\perp \cdot x/2} \;-\; \frac{\lambda_i^{(2)}}{2}\, \sigma_z^{(2)}\, e^{-ik_\perp \cdot x/2} \quad,$$

where x is an ordinary three-dimensional space variable operator (it could even be two-dimensional, since $k_\perp$ is two-dimensional), the $\lambda_i^{(1)}$ are the nine isotopic matrices of a $\underline{3}$ representation of $U(3)$, the $\lambda_i^{(2)}$ are the matrices of a $\underline{\bar{3}}$ representation, and the σ_z's are just the Pauli matrices of two spins of $1/2$ each. The minus sign in the second relation is natural since the axial vector current behaves oppositely under C to the vector current.[1,2]

To represent the system of $F_i(k_\perp)$, $F_i^{\,5}(k_\perp)$, $\mathcal{G}$, $\mathcal{P}$, and M is somewhat more complicated. We must make x three-dimensional in order to accommodate $\mathcal{G}$, which we can take to be the ordinary expression

$$\oint \rightarrow \frac{\sigma^{(1)}}{2} + \frac{\sigma^{(2)}}{2} + \underset{\sim}{x} \times \underset{\sim}{p} \qquad , \qquad (2.1)$$

where $\underset{\sim}{p}$ is the momentum conjugate to $\underset{\sim}{x}$. Parity is trivially represented:

$$\mathcal{P} \rightarrow - (-1)^L \qquad , \qquad (2.2)$$

where $L(L+1)$ is the eigenvalue of $\underset{\sim}{L}^2 = (\underset{\sim}{x} \times \underset{\sim}{p})^2$; the extra minus sign comes from the intrinsic parity of quark and antiquark. Charge conjugation[1] C takes $\lambda_i^{(1)} \leftrightarrow - \lambda_i^{(2)*}$, $\underset{\sim}{x} \leftrightarrow - \underset{\sim}{x}$, $\underset{\sim}{\sigma}^{(1)} \leftrightarrow \underset{\sim}{\sigma}^{(2)}$.

We now take a mass operator M that commutes with $\oint$ and P and with charge conjugation. M can depend on $\underset{\sim}{x}$, $\underset{\sim}{p}$, $\underset{\sim}{\sigma}^{(1)}$, $\underset{\sim}{\sigma}^{(2)}$, $\lambda_i^{(1)}$, and $\lambda_i^{(2)}$. Thus the whole spectrum of the meson states we describe, with their masses, spins, parities, and charge conjugations, is specified by the operator M, with its eigenvalues M_N and eigenfunctions ψ_{Nh},

$$M \psi_{Nh} = M_N \psi_{Nh} \qquad . \qquad (2.3)$$

Finally, we must make $F_i(\underset{\sim}{k}_\perp)$ and $F_i^{5}(\underset{\sim}{k}_\perp)$ obey the angular condition (1.10), which they will not do in general if we take the simple expressions above. We must perform a unitary transformation on these expressions in order to accommodate the angular condition. Dashen and I thought that a single unitary transformation would do the trick, but it turns out that two different ones are necessary:

$$F_i(\underset{\sim}{k}_\perp) \rightarrow e^{iS_1} \frac{\lambda_i^{(1)}}{2} e^{i\underset{\sim}{k}_\perp \cdot \underset{\sim}{x}/2} e^{-iS_1} + e^{iS_2} \frac{\lambda_i^{(2)}}{2} e^{-i\underset{\sim}{k}_\perp \cdot \underset{\sim}{x}/2} e^{-iS_2} \quad ,$$

$$(2.4)$$

$$F_i^5(\underline{k}_\perp) \rightarrow e^{iS_1} \frac{\lambda_i^{(1)}}{2} \sigma_z^{(1)} e^{i\underline{k}_\perp \cdot \underline{x}/2} e^{-iS_1}$$

$$+ e^{iS_2} \frac{\lambda_i^{(2)}}{2} \sigma_z^{(2)} e^{-i\underline{k}_\perp \cdot \underline{x}/2} e^{-iS_2} \quad , \qquad (2.5)$$

where S_1 and S_2 are so adjusted that we still have both of the first terms in (2.4) and (2.5) commuting with both the second terms, just as we do for S_1, $S_2 = 0$. We must now further adjust S_1 and S_2 so that we have the angular condition (1.10) satisfied. The parity and time-reversal properties of $F_i(\underline{k}_\perp)$ and $F_i^5(\underline{k}_\perp)$ are guaranteed if we take S_1 and S_2 (which commute with $\mathcal{J}_z$) satisfying

$$S_1, S_2 \quad \text{even under} \quad \mathcal{P} \, e^{i\pi \mathcal{J}_y} \qquad\qquad (2.6)$$

and

$$S_1, S_2 \quad \text{even under} \quad \mathcal{T} \, e^{i\pi \mathcal{J}_y} \qquad . \qquad (2.7)$$

We shall see later that there is one trivial mass operator, namely $M = 2\sqrt{m^2 + \underline{p}^2}$, corresponding to a free quark and antiquark, for which S_1 and S_2 have been found with the required properties. Let us suppose that there is a class of M operators, including some with discrete spectra, such that we can find S_1 and S_2 with the necessary behavior. We then suggest that one of these allowable M operators will give a good description of the meson system, including all or most of the well-defined excited states.

The eigenvalues M_N give the masses of bound and resonant states; the corresponding wave functions $\psi_{Nh}(\underset{\sim}{x})$ give the angular momentum and parity of each state, as well as the isotopic spin and strangeness, and the value of C when $Y = 0$. We can calculate all the Regge trajectories on which our mesons lie by solving the eigenvalue equation (2.3) for non-integral values of J.

All the matrix elements at $P_z = \infty$ of the operators $F_i(\underset{\sim}{k}_\perp)$ and $F_i^5(\underset{\sim}{k}_\perp)$ are found by sandwiching the operators (2.4) and (2.5) between $\psi_{N'h'}^*$ and ψ_{Nh}. Thus nearly all the electromagnetic and weak form factors between meson states can be calculated. All pion couplings to two mesons can be calculated in the PCAC approximation from the matrix elements of $F_i^5(0)$ and similar techniques can be used to calculate certain other meson couplings approximately.

Finally, we can look at the resulting form factors and see if all the vector current form factors exhibit poles at the same set of negative values, $-(\mu_n^V)^2$ of $\underset{\sim}{k}_\perp^2$. If they do, and if the axial vector current form factors all exhibit poles at the same set of negative values, $-(\mu_n^A)^2$, of $\underset{\sim}{k}_\perp^2$, then we can impose the bootstrap condition by requiring that these values agree with the calculated values $-(M_N^V)^2$ and $-(M_N^A)^2$ of the negative squares of the masses of the vector and normal axial vector mesons with the appropriate isotopic spins, strangeness, and values of C. In this way, if the formalism permits, we can find the conditions imposed on the mass operator M by the bootstrap requirement, and perhaps nearly determine its form.

If the form factors can be made to exhibit the right poles, then of course the formalism gives definite predictions of the coupling

constants of all vector and normal axial vector mesons to all pairs of mesons, even without assuming the validity of methods like PCAC. The bootstrap also requires, of course, that the meson-meson-meson couplings determined in this way be totally symmetric in the three mesons and we must see if the formalism can satisfy that, too.

It is easy to see how to apply our remarks to the baryon system. We take a three quark representation, with two coordinate operators described by $\underset{\sim}{x}_1$, $\underset{\sim}{x}_2$, and $\underset{\sim}{x}_3$ and the constraint $\underset{\sim}{x}_1 + \underset{\sim}{x}_2 + \underset{\sim}{x}_3 = 0$. The two independent coordinate operators belong to the two-dimensional representation of the permutation group on three objects, the one given by the Young diagram ⊟ . The angular momentum is represented in the obvious way

$$\underset{\sim}{\mathcal{J}} \rightarrow \underset{\sim}{\sigma}_1/2 + \underset{\sim}{\sigma}_2/2 + \underset{\sim}{\sigma}_3/2 + \underset{\sim}{L} \tag{2.8}$$

and the parity by the total orbital parity. The mass operator M is invariant under $\mathcal{P}$, $\underset{\sim}{\mathcal{J}}$, and permutations of the three quarks and depends on $\underset{\sim}{x}$'s, $\underset{\sim}{\sigma}$'s, and λ's. The current algebra is represented as follows:

$$F_i(\underset{\sim}{k}_\perp) \rightarrow e^{iS_1}\frac{\lambda_i^{(1)}}{2}e^{i\underset{\sim}{k}_\perp \cdot \underset{\sim}{x}^{(1)}}e^{-iS_1} + e^{iS_2}\frac{\lambda_i^{(2)}}{2}e^{i\underset{\sim}{k}_\perp \cdot \underset{\sim}{x}^{(2)}}e^{-iS_2}$$

$$+ e^{iS_3}\frac{\lambda_i^{(3)}}{2}e^{i\underset{\sim}{k}_\perp \cdot \underset{\sim}{x}^{(3)}}e^{-iS_3} \quad , \tag{2.9}$$

$$F_i^5(\underset{\sim}{k}_\perp) \rightarrow e^{iS_1}\frac{\lambda_i^{(1)}}{2}\sigma_z^{(1)}e^{i\underset{\sim}{k}_\perp \cdot \underset{\sim}{x}^{(1)}}e^{-iS_1}$$

$$+ e^{iS_2}\frac{\lambda_i^{(2)}}{2}\sigma_z^{(2)}e^{i\underset{\sim}{k}_\perp \cdot \underset{\sim}{x}^{(2)}}e^{-iS_2} + e^{iS_3}\frac{\lambda_i^{(3)}}{2}\sigma_z^{(3)}e^{i\underset{\sim}{k}_\perp \cdot \underset{\sim}{x}^{(3)}}e^{-iS_3} \quad ,$$

$$\tag{2.10}$$

where S_1, S_2, and S_3 are adjusted so that the first terms commute with the second terms, and so forth, and so that the angular condition (1.10) is obeyed. Again, the parity and time-reversal properties of the currents are guaranteed if S_1, S_2, and S_3 obey the conditions (2.6) and (2.7).

Again, we have no idea how wide is the class of mass operators M for which all this can be done, but we hope that it includes one that gives a good description of the baryons. Once again, the spectrum of states, the Regge trajectories on which they lie, the form factors of vector and axial vector currents, and the couplings of vector and pseudo-scalar mesons can all be computed. We can try to impose once more the condition that the operators $F_i(\underset{\sim}{k}_\perp)$ and $F_i^5(\underset{\sim}{k}_\perp)$ have poles in $\underset{\sim}{k}_\perp^2$ precisely at the negative squared masses of the vector and axial vector mesons respectively.

III. RELATION TO APPROXIMATE SYMMETRY

Let us assume that we can find suitable mass operators for the meson and baryon systems to carry out the program outlined in Section II and obtain reasonable agreement with experiment. What do we have, and how is it related to attempts that have been made so far to describe the mesons and baryons?

First of all, we have described representations in which the higher states are given by orbital and radial excitations, staying within the mathematical $\bar{q}q$ or qqq configurations, not by excitations of "quark spin" and unitary spin, which would correspond to configurations like $\bar{q}\bar{q}q$ or $qqq\bar{q}$ and higher. Thus we are ignoring all baryon states with SU(3) representations other than $\underset{\sim}{1}$, $\underset{\sim}{8}$, and $\underset{\sim}{10}$, and all meson states with SU(3) representations other than $\underset{\sim}{1}$ and $\underset{\sim}{8}$. So far, there is no very convincing evidence that resonances with such exotic properties exist, but of course there are <u>continua</u> with such values of the quantum numbers, and there may very well be some bumps or even genuine resonances. Such resonances could be described by other representations, but our formalism is approximate in that no connection of the usual resonances with these "exotic channels" is permitted.

Second, our resonances all have zero width to begin with and their decays are described in perturbation theory, using the coupling constants to π, ρ, etc., determined by the formalism. This is a considerable idealization, especially for the very high states, which are probably not well-defined at all.

Third, all current sum rules, including the Adler-Weisberger relation, the Bjorken-Cabibbo-Radicati relation, etc., are obeyed exactly in our formalism.

Fourth, if we wish to introduce antisymmetric tensor currents, $\mathcal{T}_{i\mu\nu}$, we may do so. At $P_z = \infty$, the components with non-vanishing matrix elements are $\mathcal{T}_{ixo}$ and $\mathcal{T}_{ixz}$ (which become equal) and $\mathcal{T}_{iyo}$ and $\mathcal{T}_{iyz}$ (which become equal). We can define

$$T_i^{\,x}(k) \equiv \int \mathcal{T}_{iyz} \, e^{i\underset{\sim}{k}_\perp \cdot \underset{\sim}{x}} \, d^3x \quad , \tag{3.1}$$

$$T_i^{\,y}(k) \equiv - \int \mathcal{T}_{ixz} \, e^{i\underset{\sim}{k}_\perp \cdot \underset{\sim}{x}} \, d^3x \quad , \tag{3.2}$$

and postulate that these together with $F_i^{\,5}(\underset{\sim}{k}_\perp)$ and $F_i(\underset{\sim}{k}_\perp)$, obey the commutation relations of a local "$[U(6)]_W$" at $P_z = \infty$. The operators $T_i^{\,x}(0)$, $T_i^{\,y}(0)$, $F_i^{\,5}(0)$, and $F_i(0)$ then form the algebra of a regular "$[U(6)]_W$" at $P_z = \infty$. There is a natural "home" in the theory for the representation of these additional operators. For instance, in the baryon representation we can put

$$
\begin{aligned}
T_i^{\,x}(\underset{\sim}{k}_\perp) \rightarrow\; & e^{iS_1} \frac{\lambda_i^{(1)}}{2} \sigma_x^{(1)} e^{i\underset{\sim}{k}_\perp \cdot \underset{\sim}{x}_1/2} e^{-iS_1} \\
& + e^{iS_2} \frac{\lambda_i^{(2)}}{2} \sigma_x^{(2)} e^{i\underset{\sim}{k}_\perp \cdot \underset{\sim}{x}_2/2} e^{-iS_2} \\
& + e^{iS_3} \frac{\lambda_i^{(3)}}{2} \sigma_x^{(3)} e^{i\underset{\sim}{k}_\perp \cdot \underset{\sim}{x}_3/2} e^{-iS_3} \quad , \tag{3.3}
\end{aligned}
$$

etc. We can then see if the suitable commutation and angular relations hold; presumably they do. If we wish to estimate the "magnetic" couplings of the vector mesons by means of an approximate "PCTC" principle, we may do so.

In any case, whether we use $U(3) \times U(3)$ or $[U(6)]_W$ at $P_z = \infty$, we can see the relation of our theory to the notion of approximate symmetry. Suppose that we can write $M = M^{(0)} + M^1$, where $M^{(0)}$ is completely independent of spins and isotopic spins, while M^1 can be treated in some reasonable approximation as a perturbation. Then in the limit of ignoring M^1, M is invariant under the group $U(6) \times U(6)$ with generators

$$\sum_{\text{quarks}} \frac{\lambda_i}{2} \pm \sum_{\text{antiquarks}} \frac{\lambda_i}{2} \quad \text{and} \quad \sum_{\text{quarks}} \frac{\lambda_i \, \underline{\sigma}}{2} \pm \sum_{\text{antiquarks}} \frac{\lambda_i \, \underline{\sigma}}{2} \quad ,$$

$$(3.4)$$

and under orbital angular momentum $\underline{L}$. The mesons belong to the representation $(\underline{6}, \underline{\bar{6}})$ with different values of L and with parity $- (-1)^L$, while the baryons belong to $(\underline{56}, \underline{1})$, $(\underline{70}, \underline{1})$, and $(\underline{20}, \underline{1})$ with different values of L and parity.

With simple forces, the lowest set of mesons would be a $(\underline{6}, \underline{\bar{6}})$ with $L = 0^+$, corresponding to the known nonets of pseudoscalar and vector mesons. The next set would be a $(\underline{6}, \underline{\bar{6}})$ with $L = 1^-$, giving nonets of tensor mesons (observed), both kinds of axial vector mesons, and scalar mesons. Examples of the last three kinds may have been detected experimentally. We can refer to the mesons we have listed so far as $^1S_0$ and $^3S_1$, then $^3P_2$, $^3P_1$, $^3P_0$, and $^1P_1$ nonets. The

first Regge recurrence of the $^1S_0$ mesons will correspond to $^1D_2$, that

of the $^3S_1$ mesons to $^3D_3$, etc. (The Regge trajectories passing through

$^3D_2$ and $^3D_1$ have no physical particles at lower values of J, where the

trajectories correspond to "nonsense".) If the forces in the mass operator

M are mostly ordinary forces, we will have "exchange degeneracy", i.e.,

the Regge trajectories of even and odd signature will nearly coincide, as

they seem to do experimentally. Likewise, spin-orbit forces will split

the trajectories for different orientations of $\underset{\sim}{L}$ and $\underset{\sim}{S}$ apart, but in the

limit of $U(6) \times U(6)$ symmetry they are degenerate. We can think of S, P,

D, . . . crudely as lying on a single "trajectory". The next "trajectory"

corresponds to the first radial excitation, and gives us again a full

panoply of S, P, D, . . . states. This is repeated for an infinite number

of radial excitations or "trajectories", the later ones being, no doubt,

not very physical, but the early ones corresponding, we hope, to observable

trajectories and observable resonances.

Now the baryon ground state, in the limit of $U(6) \times U(6)$ symmetry,

is certainly a $(\underset{\sim}{56}, \underset{\sim}{1})$ with $L = 0^+$. Taking the simplest point of view,

we suppose that the wave function of our three mathematical quarks is

always totally symmetric (rather than antisymmetric, as it would be for

real fermionic quarks). The ground state wave function is then an overall

s-state, as it would be for forces that are mostly ordinary. Above it,

we expect to find a $(\underset{\sim}{70}, \underset{\sim}{1})$ with $L = 1^-$ and, indeed, all the negative

parity baryon states lying reasonably low can be fitted into the $^2P_{1/2}$

and $^2P_{3/2}$ singlets, octets, and decimets and the $^4P_{1/2}$, $^4P_{3/2}$, and

$^4P_{5/2}$ octets that are predicted by this assignment. Other configurations

that may lie low, with ordinary forces, are another $\underset{\sim}{56}$, $L = 0^+$,

a $\underset{\sim}{70}$, $L = 0^+$, a $\underset{\sim}{56}$, $L = 2^+$, a $\underset{\sim}{70}$, $L = 2^+$, and a $\underset{\sim}{20}$, $L = 1^+$. The $\underset{\sim}{56}$, $L = 2^+$ no doubt contains the first Regge recurrence of the ground state.

Now the symmetry under $U(6) \times U(6)$ of the mass operator and the meson and baryon spectra, when we neglect the perturbation term M^1, is by no means reflected in an obvious way in the matrix elements of $F_i(\underset{\sim}{k}_\perp)$ and $F_i^{\,5}(\underset{\sim}{k}_\perp)$, i.e., the electromagnetic and weak form factors, or in the matrix elements of $T_i^{\,x}(\underset{\sim}{k}_\perp)$ and $T_i^{\,y}(\underset{\sim}{k}_\perp)$ either. Indeed, let us consider the algebra of $[U(6)]_W$ at ∞, composed of $F_i(0)$, $F_i^{\,5}(0)$, $T_i^{\,x}(0)$, and $T_i^{\,y}(0)$. In the limit of neglecting M^1, there are still the unitary transformations e^{iS} that make these operators different from the corresponding generators of the $[U(6)]_W$ subgroup of $U(6) \times U(6)$. For example, for the baryon, we have

$$F_i^{\,5}(0) = e^{iS_1} \frac{\lambda_i^{(1)}}{2} \sigma_z^{(1)} e^{-iS_1} + e^{iS_2} \frac{\lambda_i^{(2)}}{2} \sigma_z^{(2)} e^{-iS_2}$$

$$+ e^{iS_3} \frac{\lambda_i^{(3)}}{2} \sigma_z^{(3)} e^{-iS_3}$$

$$\neq \frac{\lambda_i^{(1)}}{2} \sigma_z^{(1)} + \frac{\lambda_i^{(2)}}{2} \sigma_z^{(2)} + \frac{\lambda_i^{(3)}}{2} \sigma_z^{(3)} \quad . \tag{3.5}$$

The operators S vanish only when we let <u>all the masses of all baryon states be equal and furthermore let the common mass value tend to ∞</u>. Only in such a drastic limit are the generators of the $[U(6)]_W$ subgroup of $U(6) \times U(6)$ (under which $M^{(0)}$ is invariant) equal to the generators of $[U(6)]_W$ at $P_z = \infty$. Only in such a drastic limit are

the representations of $[U(6)]_W$ at $P_z = \infty$ pure. Representation mixing of $[U(6)]_W$ at $P_z = \infty$, giving, for the nucleon, $- G_A/G_V \neq 5/3$ and anomalous magnetic moment $\neq 0$, is compulsory as soon as we have reasonable kinematics and the S operators are non-zero. Experimentally, of course, the various contributions to the Adler-Weisberger relation show how there is extensive representation mixing at $P_z = \infty$, with $F_1^5(0)$ connecting $\underline{56}$, $L = 0^+$ to $\underline{70}$, $L = 1^-$ and to other states.

It is probably useful, in fact, to consider first a mass operator $M^{(0)}$ symmetrical under $U(6) \times U(6)$, giving a degenerate spectrum but still a great deal of representation mixing with respect to $[U(6)]_W$ at $P_z = \infty$, and then to add the perturbation $M^{(1)}$ that causes <u>spin-orbit splitting, octet-singlet splitting for the spin singlet case</u>, and <u>violation of SU(3)</u>. The operator $M^{(0)}$ may be very simple, especially in the meson case. If it turns out to be an operator allowed by the formalism, we might try $M^{(0)} = 2\sqrt{a^2 + p^2 + c^2 x^2}$, so that M^2, in the $U(6) \times U(6)$ limit, is just given by the Hamiltonian of a three-dimensional harmonic oscillator. The result is to have "trajectories" for which J is perfectly linear in M^2, all with the same slope, a situation that is not so far from the experimental one.

If such a trivial $M^{(0)}$, with some simple perturbation term $M^{(1)}$, explains the meson spectrum and is consistent with the formalism, we must still ask whether it obeys the bootstrap conditions. Our program is hardly one with an unlimited amount of freedom!

Now let us ask how our program compares with the work of those who have tried to fabricate mesons and baryons as realistic bound states of real, heavy, fermionic quarks and antiquarks. First of all, we know

that, in the sense of dispersion theory, the ρ, for example, is mostly made up of 2π, $K\bar{K}$, $\pi\omega$, $N\bar{N}$, $\Xi\bar{\Xi}$, etc., etc., and so, even if real heavy quarks exist, the configuration of real quark and real antiquark con- stitutes only a minute fraction of the ρ state. How, then, could such a realistic configuration describe in a good approximation the behavior of ρ? Thus, without prejudice to the question of whether real, heavy quarks exist, we can say that trying to make baryons and mesons mostly out of them means facing some really unpleasant difficulties that we, with our mathematical quarks, are free of. Another feature of the mathematical quarks is that our model might be consistent with the boot- strap.

We are free also to treat potentials that rise to infinity, like the harmonic oscillator case discussed above. We are free to treat the 3q configuration in the baryon as symmetrical, without having to worry about real particles obeying unusual statistics.

Finally, our formalism, assuming it works, is consistent with relativity, whereas the heavy quark bound state calculations are done, typically, in a non-relativistic manner. In the next section we shall even present a covariant formalism describing our meson and baryon representations.

In practice, the work done by the realistic quark investigators on the structure of the mass operator M can often be taken over by us, especially for the mesons, but the calculation of form factors and coupling constants requires the relativistic operators S and we believe that most conclusions about form factors based on a naïve quark model must be wrong.

IV. COVARIANT FORMALISM

It is not difficult to present in a covariant manner the relativistic representations of local current algebra that we have discussed for baryon and meson states. Let us consider the meson case first.

We want a relativistic wave function to describe an idealized infinite set of mesons, consisting of a commutable infinity of 0^- states, a countable infinity of 1^- states, etc. All these states at rest, with masses M_N, correspond to the degrees of freedom represented by two spins (and two unitary spins, one quark and one antiquark) and a three-dimensional relative coordinate. Relativistically, we must describe at the same time the same states moving with any momentum $\underset{\sim}{P}$ and having energy $\sqrt{\underset{\sim}{P}^2 + M_N^2}$.

To obtain a covariant wave function, we generalize each spin 1/2 variable to a four-valued Dirac index (call them α and β) and we generalize the internal coordinate $\underset{\sim}{x}$ to a four-vector x. Let the four-dimensional coordinate describing the position of the meson be called X. Then we have a meson wave function $\Psi_{\alpha\beta;rs}(X, x)$, where r and s are three-valued indices acted on by the matrices $\lambda_i^{(1)}$ and $\lambda_i^{(2)}$ respectively. We let the Dirac matrices $\gamma_\mu^{(1)}$ and $\gamma_\nu^{(2)}$ act on the indices α and β respectively.

Now, to compare this formalism with our earlier work, let us expand Ψ in a sum over states labeled by N and h and over states of motion labeled by a momentum $\underset{\sim}{P}$:

$$\Psi = \sum_{\underset{\sim}{P}} \sum_{N,h} \phi_{NhP}(x)\, e^{iP^{(N)} \cdot X} \cdot C_{NhP} \qquad ,$$

where $P_0^{(N)} = \sqrt{M_N^2 + \underset{\sim}{P}^2}$. Let us now isolate the terms with $\underset{\sim}{P} = 0$, describing all the various mesons at rest. We want to have in this case the internal wave functions $\psi_{Nh}(\underset{\sim}{x})$ that we discussed earlier. Since ψ_{Nh} is to be a function of a three-dimensional coordinate $\underset{\sim}{x}$, we want to have the ϕ's describing rest-states obey $\frac{\partial}{\partial x_0} \phi = 0$. Covariantly, that gives us the equation

$$P_\mu \frac{\partial}{\partial x_\mu} \Psi = 0 \qquad \text{or} \qquad \frac{\partial}{\partial X_\mu} \frac{\partial}{\partial x_\mu} \Psi = 0 \quad . \qquad (4.1)$$

Since ψ_{Nh} is a function of two one-component spin variables rather than two four-component spin variables, we want the ϕ's describing rest states to obey $\beta^{(1)} \phi = \phi$ and $\beta^{(2)} \phi = \phi$. Covariantly, that gives us the equations[1)]

$$(\gamma_\mu^{(1)} \frac{\partial}{\partial X_\mu} + M) \Psi = 0 \qquad \text{or} \qquad (i\, \gamma^{(1)} \cdot P + M) \Psi = 0 \quad , \qquad (4.2)$$

$$(\gamma_\mu^{(2)} \frac{\partial}{\partial X_\mu} + M) \Psi = 0 \qquad \text{or} \qquad (i\, \gamma^{(2)} \cdot P + M) \Psi = 0 \quad , \qquad (4.3)$$

where M is an operator on the internal coordinate x and on the relativistic spin indices α and β and the SU(3) indices r and s. For the case of rest, M is a function of $\underset{\sim}{x}$ and two two-component spins and two sets of λ matrices. The manifestly covariant form of M must be such that (4.2) and (4.3) are compatible with (4.1). For example, if the rest form of M (as discussed in the previous section) is $2\sqrt{a^2 + \underset{\sim}{p}^2 + c^2 \underset{\sim}{x}^2}$, we may write M covariantly as

$$2\sqrt{a^2 + \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x_\mu} + c^2 \left(x_\mu - \frac{x \cdot P\, P_\mu}{P^2}\right)\left(x_\mu - \frac{x \cdot P\, P_\mu}{P^2}\right)} \qquad ,$$

where we recall that the internal four-momentum $\frac{1}{i}\frac{\partial}{\partial x_\mu}$ is orthogonal to P_μ by (4.1). A term in M_{rest} of the form $\underset{\sim}{\sigma} \cdot \underset{\sim}{L}$ can be represented covariantly as a term proportional to

$$\sigma_{\alpha\beta}\left(x_\alpha - \frac{x \cdot P \; P_\alpha}{p^2}\right)\frac{\partial}{\partial x_\beta} \qquad .$$

And so forth. To some extent, the mass operator is defined implicitly, since P^2 occurs in the formula for M, but that is not serious and the physical content is clear when we look at the case of rest and obtain our old formalism.

The equations (4.1), (4.2), and (4.3) are a combination of familiar tricks, the writing of spin triplet and singlet as the direct product of two spins of 1/2 and the writing of an infinite number of states of all angular momenta in terms of a wave function with an internal coordinate. The latter trick has been discussed by Yukawa and his associates for more than 15 years. There is no real physics in the equation, except the specification of the masses and spins and other quantum numbers by the operator M. Any assembly of meson states with these masses and quantum numbers is correctly described by Eqs. (4.1) - (4.3). Nor is there any real physics in what we do next. We write a general expression for the matrix element of a vector current between two wave functions. This expression allows any form factors between any of the states, compatible with relativity and the usual conservation laws. It does not place any dynamical restrictions on the vector current matrix element. For convenience, we work in momentum space, using the variable π conjugate to x.

The internal wave functions $\phi_{NhP}(\pi)$ are simply generalizations to the case of motion of the rest wave functions $\psi_{Nh}(\underset{\sim}{p})$ written in Fourier transform. We normalize the ϕ's arbitrarily by the condition

$$\int \bar{\bar{\phi}}_{NhP} \; \phi_{NhP} \; \delta(\pi \cdot P) \; d^4\pi \; = \; 2 \, M_N \qquad , \qquad (4.4)$$

where the double bar means hermitian conjugation and multiplication by $\beta^{(1)} \; \beta^{(2)}$. Then we write the matrix element of the appropriate Fourier component of the vector current as

$$\langle n'h'P' \, | \mathcal{F}_{i\alpha} | \, nhP \rangle \; \propto \; \frac{1}{\sqrt{4 \, P_o \, P_o'}} \; \int d^4\pi \; \sqrt{\delta(\pi \cdot P)} \; \int d^4\pi' \; \sqrt{\delta(\pi' \cdot P')}$$

$$\bar{\bar{\phi}}_{N'h'P'} \; G_{i\alpha} \; \phi_{NhP} \qquad , \qquad (4.5)$$

where $G_{i\alpha}$ is a sum over $\lambda_i^{(1)}/2$ and $\lambda_i^{(2)}/2$ times all possible independent vector operators, such as $\gamma_\alpha^{(1)}$, $\gamma_\alpha^{(2)}$, π_α, $\sigma_{\alpha\beta}^{(1)} k_\beta$, $\sigma_{\alpha\beta}^{(2)} k_\beta$, etc., times Lorentz-invariant functions $F(\pi \cdot \pi', \; \pi' \cdot k, \; \pi \cdot k, \; k^2, \; \pi^2, \; \pi'^2)$, where we define $k \equiv P' - P$ and bear in mind the conditions $\pi \cdot P = 0$, $\pi' \cdot P' = 0$. We may write formally

$$G_{i\alpha} \; = \; \sum_\Theta \; \Theta_{i\alpha} \; F_\Theta(\pi \cdot \pi', \; \pi' \cdot k, \; \pi \cdot k, \; k^2, \; \pi^2, \; \pi'^2) \qquad . \qquad (4.6)$$

We now have a general wave function describing mesons with the right masses and quantum numbers and a general covariant form for all vector current matrix elements between pairs of such meson states. We can impose current conservation where appropriate and we can introduce a similar formula for the axial vector currents. Nothing controversial

has yet been said and there is so far no question of consistency.

Now we want to introduce the algebra of charge densities at $P_z = \infty$. That can no doubt be done in a covariant manner by generalizing the covariant way we have introduced the same algebra between spin zero states. However, I don't know the generalization yet. For the moment, let us put in the algebra of charge densities at $P_z = \infty$ by brute force. We express the current matrix element (4.5) in terms of the rest internal wave functions $\psi_{Nh}(\underset{\sim}{p})$. We have

$$\phi_{NhP} = \frac{(\underset{\sim}{\alpha}^{(1)} \cdot \underset{\sim}{P} + M_N + \sqrt{M_N^2 + \underset{\sim}{P}^2})(\underset{\sim}{\alpha}^{(2)} \cdot \underset{\sim}{P} + M_N + \sqrt{M_N^2 + \underset{\sim}{P}^2})}{\sqrt{2} \ (M_N + \sqrt{M_N^2 + \underset{\sim}{P}^2})} \ \psi_{Nh}(\Lambda^{-1} \pi) \quad ,$$

$$(4.7)$$

where ψ is the ordinary function of a three-momentum and two-component spin, but written as a function of a four-dimensional variable with vanishing fourth component, and as a function of four-component spins but with $\beta^{(1)} = 1$ and $\beta^{(2)} = 1$ acting on it; the Lorentz transformation Λ^{-1} takes P into $(0, 0, 0, M_N)$ and converts π into $(p_x, p_y, p_z, 0)$. The normalization (4.4) is now equivalent to a standard normalization

$$\int \psi_{N'h'}^{*}(\underset{\sim}{p}) \ \psi_{Nh}(\underset{\sim}{p}) \ d^3p \ = \ \delta_{NN'} \ \delta_{hh'} \quad . \quad (4.8)$$

Now let us take $P_z \to \infty$. We let $\underset{\sim}{P}_\perp = (- k/2, 0)$ and $\underset{\sim}{P}'_\perp = (k/2, 0)$. The Lorentz transformation Λ takes $(0, 0, 0, M_N) \to (- k/2, 0, P_z, \sqrt{M_N^2 + P_z^2})$ and $(p_x, p_y, p_z, 0) \to \pi$, while Λ' takes $(0, 0, 0, M_N') \to (+ k/2, 0, P_z, \sqrt{M_N'^2 + P_z^2})$ and

$(p'_x, p'_y, p'_z, 0) \to \pi'$. We consider only the time-component of each vector operator $\Theta_{i\alpha}$ and obtain effectively the operator

$$\frac{1}{\sqrt{4\,P_o\,P'_o}} \; \frac{(\underline{\alpha}^{(1)} \cdot \underline{P}' + M_N' + \sqrt{M_N'^2 + \underline{P}'^2})(\underline{\alpha}^{(2)} \cdot \underline{P}' + M_N' + \sqrt{M_N'^2 + \underline{P}'^2})}{\sqrt{2}\,(M_N' + \sqrt{M_N'^2 + \underline{P}'^2})}$$

$$\beta^{(1)}\,\beta^{(2)}\,\Theta_{1o}\; \frac{(\underline{\alpha}^{(1)} \cdot \underline{P} + M_N + \sqrt{M_N^2 + \underline{P}^2})(\underline{\alpha}^{(2)} \cdot \underline{P} + M_N + \sqrt{M_N^2 + \underline{P}^2})}{\sqrt{2}\,(M_N + \sqrt{M_N^2 + \underline{P}^2})}$$

between $\beta^{(1)} = +1$ and $\beta^{(1)} = +1$ and between $\beta^{(2)} = +1$ and $\beta^{(2)} = +1$. Let us call the result of that operation, between two-component spinors, $\hat{\Theta}_i$. The expression (4.5) for the current now leads us to the result

$$\langle N'h' | F_i(\underline{k}_\perp) | Nh \rangle = \int d^3p' \int d^3p \; \psi^*_{N'h'}(\underline{p}')$$

$$\times \; \sum_\Theta \hat{\Theta}_i \; F_\Theta(k^2, \underline{p}'^2, \underline{p}^2, k\,p'_x - \frac{M'^2 - M^2 - k^2}{2M'}\,p'_z, \; k\,p_x - \frac{M'^2 - M^2 + k^2}{2M}\,p_z,$$

$$p'_x\,p_x + p'_y\,p_y + \frac{M'^2 + M^2 - k^2}{2MM'}\,p'_z\,p_z + \frac{k}{M'}\,p'_z\,p_x - \frac{k}{M}\,p_z\,p'_x)$$

$$\times \; \psi_{Nh}(\underline{p}) \qquad\qquad , \qquad\qquad (4.9)$$

where the quantities that appear as arguments of F are merely k^2, π'^2, π^2, $k \cdot \pi'$, $k \cdot \pi$, and $\pi' \cdot \pi$, expressed by means of Λ and Λ' in terms of $\underline{p}$ and $\underline{p}'$, in the limit $P_z \to \infty$. Taking a linear combination of the last argument and the product of the two preceding arguments, we can write

$$\langle N'h' | F_i(\underset{\sim}{k}_\perp) | Nh \rangle = \int d^3p' \int d^3p\; \psi^*_{N'h'}(\underset{\sim}{p}') \sum_\theta \hat{\theta}_i\; f_\theta(k^2, (R'_y \underset{\sim}{p}')^2,$$

$$(R_y \underset{\sim}{p})^2, \underset{\sim}{k}_\perp \cdot R'_y \underset{\sim}{p}', \underset{\sim}{k}_\perp \cdot R_y \underset{\sim}{p}, R'_y \underset{\sim}{p} \cdot R_y \underset{\sim}{p})\; \psi_{Nh}(\underset{\sim}{p}) \qquad , \quad (4.10)$$

where R_y is a rotation around the y-axis by the angle

$$\phi = \text{arc tan } \frac{M'-M}{k} - \text{arc tan } \frac{k}{M'+M} \qquad\qquad (4.11)$$

while R'_y is a rotation around the y-axis by the angle

$$\phi' = \text{arc tan } \frac{M'-M}{k} - \text{arc tan } \frac{k}{M'+M} \qquad . \qquad (4.12)$$

We need then only examine the trivial angular properties of the various $\hat{\theta}_i$ to arrive at the general result that

$$\langle N'h' | e^{i \mathcal{G}_y \phi'}\; F_i(\underset{\sim}{k}_\perp)\; e^{-i \mathcal{G}_y \phi} | Nh \rangle \qquad \text{has} \qquad \Delta \mathcal{G}_x = 0, \pm 1$$

$$(\underset{\sim}{k}_\perp \text{ in x-direction}) \qquad , \qquad (4.13)$$

which is just the same as (1.10).

The covariant formalism gives us automatically the angular condition. The only problem is to impose on the formalism the equal time charge density commutation relations at $P_z = \infty$, by requiring that in (4.9) the F_θ be such that the algebra is obeyed. That is the main physical content of our work (the rest being largely kinematics) and

the condition is non-trivial to impose in this language. In the earlier, non-covariant formalism the algebra was easy to satisfy but the angular condition was difficult to impose.

In the next section we present the case of mesons made of a free Dirac quark and a free Dirac antiquark, which has been solved exactly in both formalisms.

V. A SIMPLE CASE SOLVED EXACTLY

In this section, we describe, by means of our formalism, both covariant and non-covariant, a trivial model of the mesons in which they are built of a free Dirac quark and antiquark, each of mass m.

Let us **first** use the notation of Section II. The mass operator is

$$M = 2 \sqrt{m^2 + \underset{\sim}{p}^2} \equiv 2 W \qquad . \qquad (5.1)$$

We notice that in the angular condition (1.10) both arc tan $\frac{M'-M}{k}$ and arc tan $\frac{k}{M'+M}$ are expansible in power series in m^{-1}, starting in order m^{-1}. Thus in (2.4) we can take S_1 and S_2 as being $\mathcal{O}(1/m) + \mathcal{O}(1/m^2) + \ldots$. Taking into account the conditions (2.7) and the manner in which angular momentum and parity properties of the $F_1(k)$ and $F_1{}^5(k)$ are affected to higher and higher order in $1/m$ by kinematics at $P_z = \infty$, we make the ansatz

$$S_1 = \frac{v_z}{m} + \frac{t_{zz}}{m^2} + \ldots \qquad (5.2)$$

and find we can satisfy the angular condition with

$$v_z = \frac{(\sigma_x^{(1)} - \sigma_x^{(2)}) P_y - (\sigma_y^{(1)} - \sigma_y^{(2)}) P_x}{2} - \frac{P_z \, \underset{\sim}{x} \cdot \underset{\sim}{p} + \underset{\sim}{p} \cdot \underset{\sim}{x} \, P_z}{2} \qquad , \qquad (5.3)$$

$$t_{zz} = \frac{(\sigma_y^{(1)} P_x - \sigma_x^{(1)} P_y) P_z}{2} + \frac{z \, P_z \, \underset{\sim}{p}^2 + \underset{\sim}{p}^2 \, P_z \, z}{4} \qquad , \qquad (5.4)$$

giving

$$e^{iS_1} x e^{-iS_1} = x - \frac{p_z x}{m} - \frac{(\sigma_y^{(1)} - \sigma_y^{(2)})}{2m} + \frac{(3\sigma_y^{(1)} - \sigma_y^{(2)}) p_z}{4m^2}$$

$$+ \frac{(z p_z + p_z z) p_x}{2m^2} + \ldots \quad , \qquad (5.5)$$

$$e^{iS_1} y e^{-iS_1} = y - \frac{p_z y}{m} + \frac{\sigma_x^{(1)} - \sigma_x^{(2)}}{2m} - \frac{(3\sigma_x^{(1)} - \sigma_x^{(2)}) p_z}{4m^2}$$

$$+ \frac{(z p_z + p_z z) p_y}{2m^2} + \ldots \quad , \qquad (5.6)$$

$$e^{iS_1} \sigma_z^{(1)} e^{-iS_1} = \sigma_z^{(1)} + \frac{\sigma_y^{(1)} p_y + \sigma_x^{(1)} p_x}{m}$$

$$- \frac{\sigma_z^{(1)} p_y^2 + \sigma_z^{(1)} p_x^2 + \sigma_x^{(1)} p_x p_z + \sigma_y^{(1)} p_y p_z}{2m^2} + \ldots , \qquad (5.7)$$

with $-e^{iS_2} x e^{-iS_2}$, $-e^{-iS_2} y e^{-iS_2}$, $e^{iS_2} \sigma_z^{(2)} e^{-iS_2}$ obtainable from these by the parity operation and the exchange of $\sigma^{(1)}$ and $\sigma^{(2)}$.

We can calculate, to order $1/m$, the operator that, sandwiched between a meson of spin $\mathcal{G}$ and itself, gives the y-component of its "anomalous" magnetic moment (for our purposes, the difference between its total magnetic moment and $2\mathcal{G}$ Bohr magnetons). The operator is the M1 part of $i \frac{\partial F_i(k_\perp)}{\partial k_x}$ at $k_\perp = 0$ and is given in our representation by

M1 part of $\left(-\frac{\lambda_i^{(1)}}{2} e^{iS_1} \frac{x}{2} e^{-iS_1} + \frac{\lambda_i^{(2)}}{2} e^{iS_2} \frac{x}{2} e^{-iS_2} \right)$

$$\rightarrow \frac{(\lambda_i^{(1)} - \lambda_i^{(2)})(\sigma_y^{(1)} - \sigma_y^{(2)})}{8m} + \ldots \qquad (5.8)$$

between s-states.

To obtain the y-component of the total magnetic moment operator, we add our "normal" magnetic moment, which is

$$\frac{\lambda_i^{(1)} + \lambda_i^{(2)}}{2} \quad \frac{g_y}{2m}$$

to order $1/m$, and obtain

y-component of magnetic moment operator $= \dfrac{\lambda_i^{(1)}}{2} \dfrac{\sigma_y^{(1)}}{2m} + \dfrac{\lambda_i^{(2)}}{2} \dfrac{\sigma_y^{(2)}}{2m}$

$$+ \ldots \qquad (5.9)$$

between s-states,

which is just what we expect for two free quarks.

Next, we can calculate the axial-vector coupling constant "renormalization" (as compared with the value in the U(6)-symmetric static limit). We notice that between an s-state and itself the axial vector part of the operator

$$\frac{\lambda_i^{(1)}}{2} e^{iS_1} \sigma_z^{(1)} e^{-iS_1} - \frac{\lambda_i^{(2)}}{2} e^{iS_2} \sigma_z^{(2)} e^{-iS_2} \qquad (5.10)$$

is just

$$\left\{ \frac{\lambda_i^{(1)}}{2} \sigma_z^{(1)} - \frac{\lambda_i^{(2)}}{2} \sigma_z^{(2)} \right\} \left(1 - \frac{1}{3} \frac{\langle p^2 \rangle}{m^2} + \ldots \right) \quad , \qquad (5.11)$$

and we see the analog of the mechanism by which we believe the nucleonic value of $-G_A/G_V$ is reduced from $5/3$ to the experimental number, around 1.2.

The series expansions (5.5) - (5.7) may be replaced by the exact values of the transformed operators, as determined below by the covariant formalism. We find

$$e^{iS_1} x e^{-iS_1} = \left(x + \frac{\frac{1}{2}\{z, p_z\} p_x}{m^2 + p_x^2 + p_y^2}\right)\left(1 - \frac{p_z}{W}\right)$$

$$- \sigma_y^{(1)} \frac{W + m + p_z}{2(W+p_z)(W+m)}\left(1 - \frac{p_z}{W}\right) + \sigma_z^{(1)} \frac{p_y}{2(W+p_z)(W+m)}\left(1 - \frac{p_z}{W}\right)$$

$$+ \frac{\sigma_y^{(2)}}{2W} - \frac{\sigma_y^{(2)} p_z}{2W(W+m)} + \frac{\sigma_z^{(2)} p_y}{2W(W+m)} \quad , \tag{5.12}$$

$$e^{iS_1} \sigma_z^{(1)} e^{-iS_1} = \frac{\sigma_z^{(1)} m}{W + p_z} + \frac{\sigma^{(1)} \cdot p}{W + p_z}\left(1 + \frac{p_z}{W + m}\right) \quad , \tag{5.13}$$

and so forth. These expressions are not particularly perspicuous. However, they arise naturally in the covariant formalism, which we proceed to use.

We note that our covariant wave equations (4.1) - (4.3) for this problem are

$$(i\,\gamma^{(1)} \cdot \frac{P}{2} + \sqrt{m^2 + \pi^2})\,\Psi = 0 \quad , \tag{5.14}$$

$$(i\,\gamma^{(2)} \cdot \frac{P}{2} + \sqrt{m^2 + \pi^2})\,\Psi = 0 \quad , \tag{5.15}$$

$$\pi \cdot P\,\Psi = 0 \quad . \tag{5.16}$$

We can convert these equations to Dirac equations for two free quarks by performing a Lorentz transformation on the γ's. For example, we now have $\beta^{(1)} = 1$ when the whole system or $P = 0$ is at rest; for a free particle Dirac equation for the first quark, we want $\beta^{(1)} = 1$ when the first quark is at rest, or $P/2 + \pi = 0$. Thus we construct

$$\chi = T \Psi = \left(\sqrt{\frac{W+m}{2m}} - \frac{i\gamma^{(1)} \cdot \pi}{\pi^2} \sqrt{\frac{W-m}{2m}} \right) \left(\sqrt{\frac{W+m}{2m}} + \frac{i\gamma^{(2)} \cdot \pi}{\pi^2} \sqrt{\frac{W-m}{2m}} \right) \Psi \quad , \quad (5.17)$$

with $W = \sqrt{m^2 + \pi^2}$, and observe that it satisfies

$$(i \gamma^{(1)} \cdot (\tfrac{P}{2} + \pi) + m) \chi = 0 \quad , \quad (5.18)$$

$$(i \gamma^{(2)} \cdot (\tfrac{P}{2} - \pi) + m) \chi = 0 \quad , \quad (5.19)$$

free particle Dirac equations for the two quarks. It is clear, then, that the current operator $G_{1\alpha}$ in (4.5) for this case is

$$G_{1\alpha} = i \, T' \left\{ \frac{\lambda_i^{(1)}}{2} \, \delta(\pi' - \pi - \tfrac{k}{2}) \, \gamma_\alpha^{(1)} + \frac{\lambda_i^{(2)}}{2} \, \delta(\pi' - \pi + \tfrac{k}{2}) \, \gamma_\alpha^{(2)} \right\} T \quad , \quad (5.20)$$

while the axial vector analog is the same operator with $\gamma_\alpha \gamma_5$ replacing γ_α. We can now carry out the indicated operations and calculate any quantity in the theory, including the transformed "rest" operators as in (5.12) and (5.13).

VI. OUTLOOK

I have presented a rather ambitious program; it is not at all clear how much of it can be carried through successfully. The essential point is that we write down a large number of relations that we believe to be exactly or nearly exactly true (relativistic kinematics, dispersion relations, current algebra, etc.) of the huge space of all hadron quantum states, and we try to satisfy as many of them as possible in a tiny space of functions of one or two three-dimensional variables and a few spins and isotopic spins. The unanswered question is: how many of these relations can we satisfy in such a little space without, so to speak, bursting it?

Actually, we can try to add more relations and calculate more quantities. We can introduce not only the strong coupling constants of vector and normal axial vector mesons, but all strong meson-meson-meson and baryon-baryon-meson coupling constants and attempt to satisfy all the superconvergence relations, not only the ones that come out of V and A current algebra. We can go even further and introduce the coupling parameter $\beta_{abn}(t)$ of each Regge trajectory n to particles a and b as a generalization of the coupling constant g_{abn} and try to satisfy relations among all the β's and thus to calculate all the β's. We can introduce the divergences of the non-conserved currents $\mathscr{F}_{4\alpha}$, $\mathscr{F}_{5\alpha}$, $\mathscr{F}_{6\alpha}$, $\mathscr{F}_{7\alpha}$, and $\mathscr{F}_{i\alpha}{}^5$, and postulate the obvious commutation relations between these and the charge densities. These give "good-bad" commutation relations at $P_z = \infty$, which we can multiply by P_z to give a finite limit and attempt to satisfy. Finally we can try, especially with the covariant formalism,

to calculate the decay amplitudes of pseudoscalar and vector mesons into lepton pairs, i.e., the matrix element of $\mathcal{J}_{i\alpha}{}^5$ between vacuum and pseudoscalar meson or of $\mathcal{J}_{i\alpha}$ between vacuum and vector meson.

Since we do not know how many "truths" can be packed into our small representation, it is not clear in what order to proceed, which relations to satisfy first. For example, instead of trying to solve the meson problem with a given M operator first and then imposing the bootstrap condition later (e.g., that the vector form factors have poles at the negative mass squared of the vector mesons), we could try to write first

$$F_i(\underset{\sim}{k}_\perp) = \sum_n \frac{1}{\underset{\sim}{k}_\perp^2 + \mu_n^2} (a_n + b_n k_{\perp x} + c_n k_{\perp y}) \quad , \qquad (6.1)$$

where a_n, b_n, and c_n are suitable operator functions of $k_{\perp x}/k_{\perp y}$, and then impose the current algebra afterward. In this way, the sum rules of current algebra appear as conditions on the vector meson coupling constants; the relevant superconvergence relations are automatically included.

Another track that future research can follow is to develop the covariant formalism further, especially by writing the charge density commutation relations at $P_z = \infty$ in a covariant manner, as they have already been written when sandwiched between states of spin zero.

Meanwhile, let me make a few concluding remarks about the most straightforward approach, namely to solve the non-covariant form of the problem for some simple, non-trivial M operator, calculating the unitary transformations e^{iS} necessary in order to satisfy the angular condition. Consider, for example, the meson problem with an approximate guess for the mass operator like

$$M^2 = 4 (m^2 + \underset{\sim}{p}^2 + U(|\underset{\sim}{x}|)) \qquad . \qquad (6.2)$$

Now we may ask what conditions are imposed on U by the requirement that we be able to find transformations e^{iS} with the right properties so that relativity and the current algebra are satisfied. We know that the case $U = 0$ is all right, but that is trivial. What potentials $U \neq 0$ are allowed? There has been much discussion of essentially this question in connection with Foldy's suggestion (Phys. Rev. 122, 275 (1961)) of solving the relativistic two-body problem "directly", i.e., in a form with all extra variables eliminated and enough Foldy-Wouthuysen transformations performed to reduce it to a sort of Schrödinger equation with two-component spins. Our suggestion is similar to Foldy's in form; we impose in addition the substance of the equal-time commutation relations.

In connection with Foldy's work, it has been alleged that his form adds nothing to the richness of field theory and that the only potentials he is allowed to have are those coming from the F-W reduction of the relativistic exchange of various quanta. Thus the positronium problem, for example, suitably transformed by F-W methods, is allowed, and is presumably allowed for us as well. (This would give spin-dependent and velocity-dependent terms in the exact U, of course.) However, since we do not require that our quarks be real and separable, it is possible that we are permitted a much wider range of "potentials", perhaps even including things like the harmonic oscillator.

Zachariasen and I have begun a brute force attempt to find out what potentials are allowed for our problem by taking (6.2) and imposing the angular and commutation conditions on the unitary transformations e^{iS}

in power series in $1/m$. From the first few orders in $1/m$, we should get a good idea of what restrictions are imposed on U. In the meson problem, we put

$$S_1 = \frac{v_z^{(1)}}{m} + \frac{t_{zz}^{(1)}}{m^2} + \ldots \ldots \tag{6.3}$$

and found

$$v_z^{(1)} = \frac{(\sigma_x^{(1)} - \sigma_x^{(2)}) \, p_y - (\sigma_y^{(1)} - \sigma_y^{(2)}) \, p_x}{2} - \frac{p_z \, \underset{\sim}{x} \cdot \underset{\sim}{p} + \underset{\sim}{p} \cdot \underset{\sim}{x} \, p_z}{2} , \tag{6.4}$$

$$t_{zz}^{(1)} = \frac{(\sigma_y^{(1)} \, p_x - \sigma_x^{(1)} \, p_y) \, p_z}{2} + \frac{z \, p_z \, \underset{\sim}{p}^2 + \underset{\sim}{p}^2 \, p_z \, z}{4}$$

$$- \frac{z^2 \, r^{-1} \, dU/dr \, \underset{\sim}{x} \cdot \underset{\sim}{p} + \underset{\sim}{p} \cdot \underset{\sim}{x} \, dU/dr \, r^{-1} \, z^2}{8} . \tag{6.5}$$

The magnetic moment of s-states to order $1/m$ and the axial vector coupling constants for s-states to order $1/m^2$ come out exactly as in Section V for free particles.

For a similar baryon problem, we obtain corresponding results. Noting that in this case, say for the nucleon, $M_N = 3m + \mathcal{O}(1/m)$, we obtain for the total magnetic moment to order $1/m$ for an overall s-state just the sum of the Dirac moments of three free quarks of mass $M_N/3$, giving for the total magnetic moment of the proton three Bohr magnetons and for the neutron -2 Bohr magnetons. This excellent agreement is not so significant as it looks, however, since we can see from the axial vector coupling constant that the expansion in p/m is not

rapidly convergent. For $-G_A/G_V$ we obtain for an overall s-state, much as in Section V,

$$- \frac{G_A}{G_V} = \frac{5}{3} \left(1 - \frac{1}{3} \langle \underset{\sim}{p}^2 \rangle / m^2 + \mathcal{O}(\underset{\sim}{p}^4/m^4) \right) \tag{6.6}$$

and we see that to get 1.2 or 1.25 we need $\langle \underset{\sim}{p}^2 \rangle / m^2$ to be nearly unity and the expansion is not so good quantitatively, however useful it may be qualitatively. (Here $\langle \underset{\sim}{p}^2 \rangle$ is the expected value of the square of the momentum of one of the quarks in the nucleon.)

Anyway, up to this order we have not found any restriction on U; we must see what happens in third and fourth order in $1/m$. If there are still no restrictions, then it is very likely that any potential will do.

As a final suggestion for further work, may I say that perhaps we have introduced too many variables into our meson and especially into our baryon representation. It is conceivable that we could hold fixed the radial variable in the meson problem, thus abolishing the radial quantum number. In the baryon problem, we might be able to effect an even more drastic reduction of the number of variables, getting a much sparser level spectrum. Consistency and comparison with experiment must guide us here, as always.

FOOTNOTES

1. Actually, I have treated both mathematical particles as quarks, rather than quark and antiquark, out of laziness; I should have performed a charge conjugation transformation on the spin matrices for the antiquark.

2. The operator that I call $F_1^5(\underset{\sim}{k}_\perp)$ here and represent by the z-component of a spin is really $-F_1^5(\underset{\sim}{k}_\perp)$ in the language of previous work. In the notation of the Dirac matrices, $-\gamma_5 = \alpha_z \sigma_z$ and so $-\gamma_5$ and σ_z are equal when $\alpha_z = 1$.

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