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A THEOREM AND ITS APPLICATION TO FINITE TAMPERS
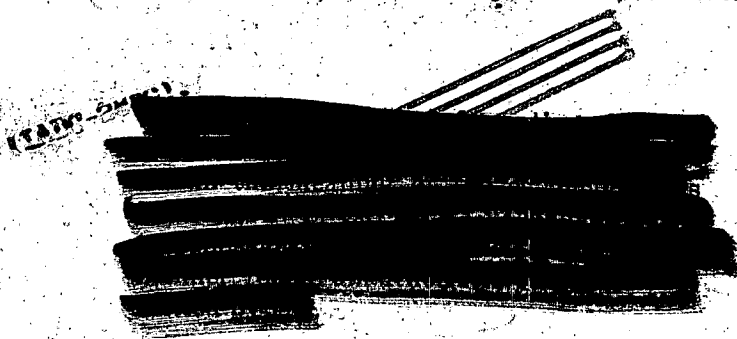
WORK DONE BY:

R. P. Feynman

REPORT WRITTEN BY:

R. P. Feynman

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ABSTRACT

A theorem is derived which is useful in the analysis of neutron problems in which all neutrons have the same velocity. It is applied to determine extrapolated end-points, the asymptotic amplitude from a point source, and the neutron density at the surface of a medium. Formulas for the effect of finite tampers are derived by its aid, and their accuracy is discussed.

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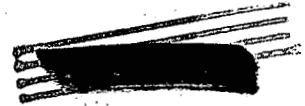
A Theorem and its Application to Finite Tampers

The problems of the characteristics of systems in which the neutrons can be considered to have only one velocity have been ably solved by Frankel and Nelson (LA 53) for the case that the core in tamper have the same mean free path, and by Serber and the members of his group (see, for example, LA 234) for the general case. It is not the purpose of this report to add appreciably to what is known about these problems. While this work was being done an interesting theorem was found to be extremely useful in obtaining approximate expressions for many of the properties of systems where the neutrons have only one velocity. In spite of the fact that, at the present time this theorem and its applications can add very little that is new to our knowledge of these systems, it was thought to be worthwhile to describe the theorem in a report. It does permit, in many cases, a simpler derivation or understanding of some of the properties. In particular it permits one to obtain a formula with the effect of a tamper of finite size. When the tamper is not absorbing this formula, (Eq.39) can be expected to be quite accurate. The corresponding formula for the absorbing tamper unfortunately cannot be expected to be as accurate and there is still room for improvement. The first part of the report will derive the theorem (Eq. 8) and apply it to various simple problems such as: The determination of the extrapolated end point; the value of the neutron density at the edge of a medium; and the determination of the asymptotic solution far from its source in an absorbing medium. In the second part of the report the theorem will be applied to calculate the effect of a finite tamper.

Suppose that at point 1 neutrons are being emitted equally in all directions. How many of these will there be per unit volume at another point 2 which have gotten to 2 without suffering any collisions on the way? Call
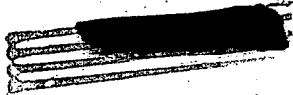
this number $(1/v)$ Q $(1,2)$.

Now suppose we have a critical system and $N(1)$ is the number of neutrons at the point (1). These suffer collisions, in number v $\sigma(1)$ $N(1)$, where $\sigma(1)$ is the total collision cross section per unit volume at (1) (equals the number of nuclei per unit volume times the nuclear cross section of each). Let the average number of neutrons liberated per collision at the point (1) be $1 + f(1)$. The quantity f may be negative. In a tamper, for example, where there is no fission it will be negative, and in that case we shall often call it $-g$. If σ_f is the fission cross section, σ_e the elastic scattering cross section and σ_r the capture cross section, then if v neutrons result from fission, the total cross section is

$$\sigma = \sigma_f + \sigma_e + \sigma_r \tag{1}$$

and the number of neutrons released is

$$(1 + f) \quad \sigma = v\sigma_f + \sigma_e \tag{2}$$

so that

$$f = \left[(v - 1) \quad \sigma_f - \sigma_r \right] / \quad \sigma \tag{3}$$

From the $(1 + f(1))$ v $\sigma(1)$ $N(1)$ neutrons liberated per unit volume at (1), $(1/v)$ $Q(1,2)$ $(1 + f(1))$ v $\sigma(1)$ $N(1)$ will be found per units volume at (2). The total neutron number at (2) is made of contributions from all the collisions occuring in volume elements such as (1) so that,

$$N(2) = \int Q(1,2) \; (1 + f(1)) \; \sigma(1) \; N(1) \; dVol_1 \tag{4}$$

This integral equation has a solution if the system is critical and it is the properties of the solution N that we shall discuss by means of an interesting

theorem which is a consequence of the fact the N satisfies (4). The theorem

hinges on the fact that Q(1,2) is a <u>symmetrical function</u> of 1 and 2. We

shall interupt our argument a moment to prove this.

Take a small volume element at (2) with area dA facing the point 1

and of depth dx in the direction of the line between 1 and 2. The number

of neutrons in the volume at 2 from a unit source at 1 is then $(1/v)$ $Q(1,2)$

dAdx. Since we are concerned with neutrons which have suffered no collision,

they must travel in straight lines from 1 to 2. The fraction of the neutrons

which are aimed to strike the area dA and hence pass through our volume element

is $dA/4\pi r_{12}^2$ where r_{12} is the distance between the points 1 and 2. These

spend a time dx/v within the volume since they traverse the volume at velocity

V. Hence, only those neutrons, in number $N \cdot dx/v$ which were liberated dur-

ing a time interval dx/v can be found in the volume element at all. Hence

$$(1/v) \ Q(1,2) \ \ dA \ dx = (dA/4\pi r_{12}^2) \ \ (dx/v) \ \ N \ P_{12} \ .$$

Where P_{12} is the probability of transversing the line between 1 and

2 without suffering a collision. This is equal to P_{21}, the probability of

transversing the line in the opposite direction. This is because there are

just as many nuclei in the way to be avoided with either direction of trans-

versal. The probability of successful avoidance of a series of hurdles (the

product of the probability of avoidance of each) is independent of the order

in which the hurdles are placed (since a product does not depend on the order

of its factors). Hence * $P_{12} = P_{21}$ and therefore,

$$Q \ (1,2) = Q(2,1) \tag{6}$$

which is what we wanted to prove.

Furthermore we have seen that Q (1,2) depends only on the total

* Of course, $P_{12} = P_{21} = e^{-\int_1^2 dz/c(z)}$ where Z is the length along the line from 1 to 2

cross section as a function of position and not on the dependence of f on position. Suppose we have two systems, each with the same total cross section as a function of position, but with different functions for f say f and f'. The distribution of neutrons will also be different. For the one system N satisfies (4). For the other the distribution N', satisfies:

$$N'(2) = \int Q(1,2) \ (1 + f'(1) \ ) \ \sigma(1) \ N'(1) \ d \ Vol_1 \qquad (4')$$

The Q function is the same in both equations. If we multiply (4) by $N'(2) \ (1 + f'(2) \ ) \ \sigma(2)$ and integrate over volume 2, and likewise multiply (4') by $N(2) \ (1 + f(2) \ ) \ \sigma(2)$ and integrate, and finally subtract the two resulting equations we would find:

$$\int N'(2) \ (1 + f'(2) \ ) \ \sigma(2) \ N(2) \ dV_2 - \int N(2) \ (1 + f(2) \ ) \ \sigma(2) \ N'(2) \ dV_2 =$$

$$= \iint N'(2) \ (1 + f'(2) \ ) \ \sigma(2) \ Q(1,2) \ (1 + f(1) \ ) \ \sigma(1) \ N(1) \ dV_1 \ dV_2 -$$

$$- \iint N(2) \ (1 + f(2) \ ) \ \sigma(2) \ Q(1,2) \ (1 + f'(1) \ ) \ \sigma(1) \ N'(1) \ dV_1 \ dV_2 \qquad (7)$$

The two double integrals on the right side of (7) are equal. If the variable label 1 and 2 are interchanged in the second double integral it will be the same as the first double integral except that $Q(1,2)$ will be replaced by $Q(2,1)$. But by (6) this makes no difference, so the right side of (7) is zero. The integral on the left side can be combined and we find:

$$\int (f - f') \ \sigma \ N \ N' \ d \ Vol = 0 \qquad (8)$$

This is the theorem which we have found so useful. We shall restate it. We assume neutrons have a single velocity. Given two assemblies which differ only in the value of f (the net number of neutrons released per collision) as a function of position, but which have the same total scattering cross

section everywhere. If they are both critical it would be expected that in some sense the average f must be the same. The exact sense is given in (8). It says that there is no difference in the average of f times the total cross section, provided that this average is taken over the whole system with weight equal to the product of the neutron distributions.

If the changes in f are small, $f - f' = \delta f$ say, then N and N' are nearly equal and we obtain the well known perturbation relation

$$\int (\delta f) \; N^2 \; \sigma \; d \, Vol \; = \; 0 \tag{9}$$

Although this is a very useful relation we should like to point out that the original equation (8) for finite and large changes in f is even more useful.

In applying the theorem (8) we shall always imagine that the system is infinite in extent. If there is a finite tamper for example of radius R_T we can imagine the tamper infinite but absorbing every neutron that suffers a collision beyond the radius R_T. That is $f = -1$ (or $g = 1$) beyond R_T.

We shall give two simple examples of how the theorem can be used to derive well-known exact results. The remaining examples will involve various approximations.

First suppose we have a core with a constant f inside, surrounded by any kind of tamper or tampers with a given absorbtion function g, $(= -f)$. Suppose two different values of f say f_1 and f_2 can both make the system critical, and let N_1 and N_2 be the neutron distributions for these two values of f. Then we can apply (8) with the unprimed system being system 1, and the primed system being 2. The integrand in the tamper vanishes because even though N and N' may be quite different, the f values are equal so that $f - f' = 0$. In the core the values are different, but are constant so (8) becomes:

$$(f_1 - f_2) \int \sigma \, N_1 \, N_2 \, d \, Vol_{core} = 0$$

This is the well known orthogonality theorem which says that if two values of f can make a core critical, the neutron densities are orthogonal (ie: the integral of their product, times σ over the core vanishes). It shows that there can only be one value of f for which N is positive everywhere, for if there were two such $N_1 \cdot N_2$ would be positive and the integral in (9) could not vanish.

As a second example consider any system with f given as a function of position, and call it the unprimed system. For the prime system assume the same total cross section everywhere, but just scattering, no absorbtion, no fission. That is f' = 0 everywhere. The prime system is critical in the sense that if a uniform distribution of neutrons (N'= constant) is present at time zero then at later times distribution remains, neither rising nor falling in average number (no absorption or fission) nor changing from a uniform distribution (there is no flow if there is no gradient). Putting f'= 0 and N' = 1 (or any other constant) into (8) we obtain:

$$\int f \ \sigma \ N \ dVol \ = \ 0 \tag{10}$$

This just represents the fact that the total number of neutrons remains constant in a critical system. The number of collisions neutrons suffer is σN, and the net number generated per collision is f so that (10) expresses the fact that the net number being generated everywhere in the system adds up to zero. If the system consists of a core with constant f and σ_c and an infinite tamper with constant absorption g and σ_t this becomes

$$f \ \sigma_c \int N \ dV_{core} \ = g \cdot \sigma_t \int N \ dV_{tamper} \ = \ S \tag{11}$$

where S is the flux of neutrons through the surface of the core.

We will now apply (3) to obtain a solution for the simplest problem,
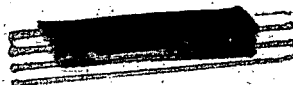
the extrapolated end-point for
a plane nonabsorbing semi-infinite
medium. Thus (see Fig. 1) for $x < 0$
there is a medium with constant σ
with no fission or absorption $(f = 0)$.
For $x > 0$ there is nothing. All
neutrons going beyond $x = 0$ are never
returned. This can be represented
equally well as far as the solution
to the left is affected by a medium, to
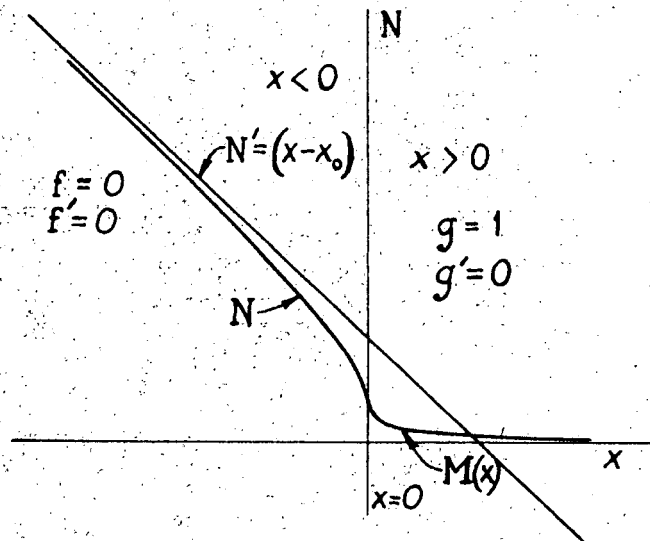the right of $x = 0$ with constant σ

Fig. 1.

(equal to that on the left) but with $g = 1$ so that every collision results in
absorption and again nothing is returned to $x < 0$. It is imagined that there is
a source deep in the interior (very negative x) and the neutron density has
constant gradient, (which we shall take $= 1$) therefore, except near the surface
$x = 0$. To what point x_o will the neutron density extrapolate? In an infinite
medium with no absorption N can be a straight line $b - ax$ or $a \cdot (x_o - x)$, and
therefore, in a finite medium except near the surface this nearly is the case. If
the surface is at $x = 0$ what is x_o? Let the unprimed system be the actual
system described above, and the shape of the solution for $x > 0$ is $M(x)$. Let
the primed system have $f' = -g' = 0$ everywhere, and therefore, $N = x - x_o$ is
a solution. Under these circumstances $f = f'$ for $x < 0$ and the equation (3)
has no integral to the left (it: for $x < 0$) [*] To the right $g = -1$, $g' = 0$ or
$g - g' = -1$, and σ is constant so we find

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[*] One need not be concerned that (8) was derived assuming no sources, while we
are now imagining a source far to the left of $x = 0$. For in this problem the source
could be a slab of fissioning material with positive f. Since N and N' are almost
exactly equal deep inside, it will require the same f in the slab to make the primed
and unprimed systems critical. Hence, $f - f' = 0$ in the slab and therefore the inte-
gral in (8) coming from the source makes no contribution.
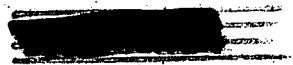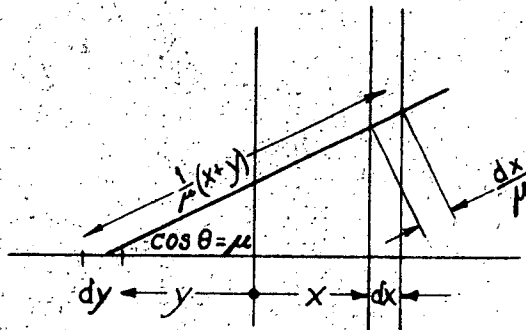
$$(-\sigma)\int_0^\infty (x_0 - x)\ M(x)\ dx = 0$$

or

$$x_0 = \int_0^\infty x\ M(x)\ dx \Big/ \int_0^\infty M(x)\ dx \tag{12}$$

That is, the extrapolated end point equals the average depth of penetration of neutrons into the completely absorbing tamper with the same mean free path.

Since this expresses X as an average over M(X), even an approximate form for M(X) may give a sufficiently accurate value for X. M(X) is simply related to the solution N(X) for negative X. Because if we know N(X) we can easily find out how many neutrons go into the tamper and where they are absorbed. For simplicity we shall call $y = -x$ for negative x, and shall speak simply of N(Y). Every second $v\sigma\ N(Y)\ dy$ collisions occur in the range dy and of these $2\pi \sin\theta\ d\theta/4\pi$ or $1/2\ d\mu$ ($\mu=\cos\theta$) come out at angle θ in the range $d\theta$. The probability that they get to x without collision is $e^{-\sigma l}$, where $l = (x+y)/\mu$ is the slant distance between x and y. The chance that they are found in the range dx is equal to the time spent in slant range, or $dx/v\mu$. All angles from 0 to $90°$ (μ from o to 1) are possible, and all ranges dY contribute. Hence, the total number in the range dx (which is M(x) dx) is given by

$$M(x) = (1/2)\int_0^1 \int_0^\infty \sigma\ N(Y)\ dY\ e^{-\sigma(x+Y)/\mu}\ (1/\mu)\ d\mu \tag{13}$$

We can get a first approximation to M(x), say $M_0(x)$ by using for N(y) the asymptotic expression $x_0 + y$. If we do this we find

$$M_0(x) = (1/2)\int_0^1 \int_0^\infty \sigma(x_0+y)\ dy\ e^{-\sigma(x+y)/\mu}\ (1/\mu)\ d\mu$$

$$= (1/2)\int_0^1 (x_0+\mu/\sigma)\ e^{-\sigma x/\mu}\ d\mu \tag{14}$$

From which one finds

$$\int_0^\infty M_o(x)\,dx = (1/2)\int_0^\infty\int_0^1 (x_o + \mu/\sigma)\,e^{-\sigma x/\mu}\,d\mu = (1/2)\int_0^1 (x_o + \mu/\sigma)(\mu/\sigma)\,d\mu =$$

$$(1/4)\ x_o/\sigma + 1/6\sigma^2$$

and

$$\int_0^\infty xM_o(x)\,dx = (1/2)\int_0^\infty\int_0^1 (x_o + \mu/\sigma)\,x e^{-\sigma x/\mu}\,d\mu = (1/2)\int_0^1 (x_o + \mu/\sigma)(\mu^2/\sigma^2)\,d\mu =$$

$$1\ x_o\ /\ 6\ \sigma^2 + 1/\ 8\ \sigma^3$$

If we use $M_o(x)$ instead of $M(x)$ in (12) we can obtain an approximate expression for x_o:

$$x_o = \left[(1/6)\ x_o/\sigma^2 + 1/8\sigma^3\right]\ /\ \left[(1/4)\ x_o/\sigma + 1/6\sigma^2\right]$$

which has the solution:

$$x_o = \sqrt{2}/\sigma = .7071/\sigma \tag{15}$$

This differs by only a fraction of a percent from the true value $x_o = .7104/\sigma$ which is derived in LA-53. (Welton has computed x_o by means of (8) but has used for $M(x)$ the next higher approximation obtained by taking into account a first perturbation on $N(y)$ in (13). He finds $x_o\sigma = .7094$ in this manner).

Encouraged by this result we can go on to find the extrapolated end point in a multiplying ($f > 0$) or absorbing ($f < 0$) medium. To make things definite, suppose we have a multiplying system so that the asymptotic solution takes the form $N(x) = \sin k\sigma(x_o - x)$, where k and f are related by the well-known relation $(\tan^{-1}k)\ /\ k = 1\ /\ (1 + f)$ \hfill (16)

Again we use for the prime system in (8) one which has the region to the right ($x > 0$) identical to that on the left. We find in this case then that

$$\int_0^\infty \sin kc \, (x_o - x) \quad M(x) \, dx = 0 \tag{17}$$

We use for $M(x)$ the expression (13) with $N(y) = \sin k \, (x_o + y)$ for the first approximation. Again the integrate on y can be performed, the result for $M(x)$ put into (16), the integrals on x calculated, and finally the μ integration can be done. This results in an equation for x_o which can be put into the form:

$$\cos (2kox_o) = \frac{2 k^2}{(1 + k^2) \, \ln(1 + k^2)} - 1 \tag{18}$$

For absorbing media ($f < o$, call $g = -f$) the asymtotic solution in $\sin h \, h \, (x_o - x)$ where

$$\frac{\tanh^{-1} h}{h} = \frac{1}{1 - g} \tag{19}$$

and x_o is determined by

$$\cosh (2ho \, x_o) = - \frac{2 h^2}{(1 - h^2) \, \ln(1 - h^2)} - 1 \tag{20}$$

These formulas for x_o give results which agree extremely well with the correct results obtained by much more difficult methods.[*] They are compared in the Table I below. The remarkable constancy of $x_o \, o(1 + f)$ is reproduced by our approximate formula.

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[*] From LA - 53.

TABLE I   Extropolated End Points

	Multiplying Systems				Absorbing Systems		
k	f	from (18) $(1+f)x_0\sigma$	true value[*] $(1+f)x_0\sigma$	h	g	from(18) $(1-g)x_0\sigma$	true value[*] $(1-g)x_0\sigma$
3.0	1.402	.7169	.7174	0.0	.0000	.7071	.7104
2.0	.8064	.7121	.7141	0.2	.0135	.7071	.7104
1.0	.2732	.7080	.7109	0.4	.0558	.7072	.7105
.8	.1856	.7075	.7106	0.6	.1344	.7075	.7108
.6	.1102	.7073	.7104	0.8	.2718	.7091	.7120
.4	.0512	.7071	.7104	0.9	.3887	.7123	.7147
.2	.0132	.7071	.7104				
0.0	.0000	.7071	.7104				

Many more properties of the solutions can be gotten from (8), but we shall obtain an even wider class of results from a generalization of (8):

$$\int_{ins} (f - f')\ \sigma\, N\, N'\ dV + \int_{ins} (SN' - S'N)\ dV =$$

$$= \int_{2=outs} \int_{1=ins} \Big[(1+f(1))\, N(1)\cdot(1+f'(2))\, N'(2) - (1+f(2))\, N(2)\cdot(1+f'(1))\, N'(1) \Big]$$

$$\sigma(1)\, \sigma(2)\, Q(1,2)\ dV_1\, dV_2 \tag{21}$$

Here we imagine first that there may be external sources of neutrons, S, in the unprimed problem, and possibly a different source S' in the primed problem. In addition we have generalized (8) by only integrating over a finite volume (called INS for "Inside"). As a consequence we expect some sort of terms involving the neutron density near the surface. These are given by the integral on the right-hand side. It is a double integral, of one solution on the inside and the other on the outside (minus vice versa) with the kernel $Q(1,2)$ between them. As 1 and 2 get far apart the kernel falls off rapidly, and since 1 and 2 are on opposite sides of the surface, the integral only involves knowledge of the solutions near the surface. We have assumed in (21) that the sources S or S' are very far from

[*]From LA - 53
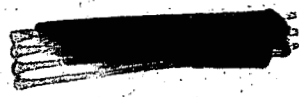
the surface and therefore, do not appear in the surface integrals. If this is not the case, the term $(1 + f)\, \sigma N$ should be replaced by $(1 + f)\, \sigma N + S$ and $(1 + f')\, \sigma N'$ by $(1 + f')\, \sigma N' + S'$ in the integral on the right-hand side of (21). The source terms on the left-hand side are, in fact, obtained from the same substitution. This is because, with external sources S, the fundamental equation (4) is just altered by a replacement of $(1 + f)\, \sigma N$ by $(1 + f)\, \sigma N + S$ in the integral on the right side. Otherwise, (that is if $S = S' = 0$) (21) can be derived directly from (7) imagining the integrals on dV_2 to be only over the inside. The integrals on dV_1 on the right of (7) are over inside and also over the outside. The inside integrals cancel since Q is symmetrical and what remains is just the right side of (21).

As a first very simple example, we can calculate the extrapolated end point for a plane problem again. We shall use (21) with the boundary between inside (left) and outside (right) to be actually the tamper surface. The prime case (see Fig. 4) has, as before, the same value of f on both sides, and on asymtotic solution say, $\sin k (x_o - x)$. The unprimed has $(1 + f) = 0$ to the right. There may be a source S deep to the left (for the absorbing case), the same in both cases. Since N does not change appreciably here, the second integral on the left of (2) vanishes. The first integral does likewise since $f = f'$. The second term in the integral on the right of (21) vanishes also because $(1 + f) = 0$ outside, so (21) is simply,

$$(1 + f)^2 \sigma^2 \int_{x_1 = -\infty}^{0} \int_{x_2 = 0}^{+\infty} N(x_1)\, Q(x_1, x_2)\, \sin k(x_o - x_2)\, dx_1\, dx_2 \qquad (22)$$

If we use the fact that

$$Q(x_1, x_2) = 1/2 \int_0^1 e^{-|x_2 - x_1|/\mu}\, d\mu/\mu \qquad (23)$$

and approximate $N(x_1)$ by its asymtotic form, we obtain as a formula for x_o

$$(1/2)\int_0^1 \int_{x_1 = -\infty}^0 \int_{x_2 = 0}^\infty \sin k(x_o - x_1)\ e^{-x_2 - x_1 / \mu} \sin k(x_o - x_2)\, dx_1 dx_2\, d\mu/\mu \qquad (24)$$

This is the same integral as was obtained in a less direct way by using (3) and gives, on integration, the relations (16) and (20) for x_o.
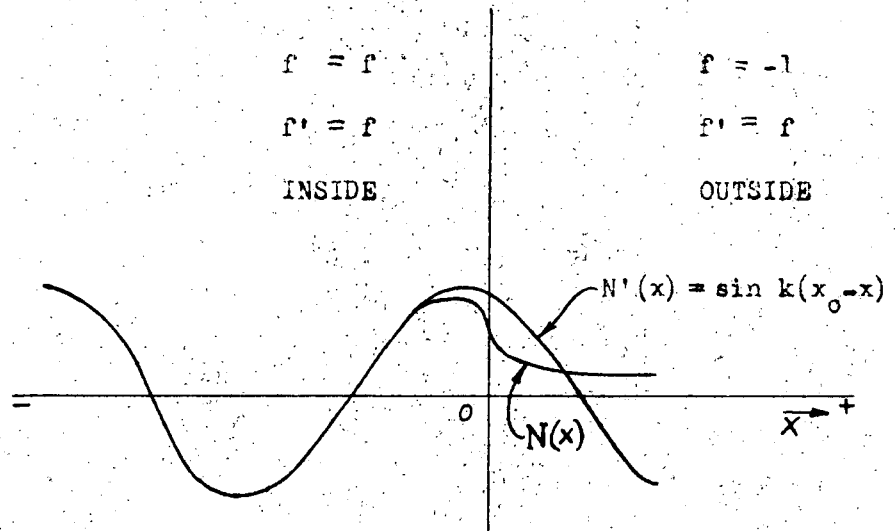

f = f   f = -1

f' = f   f' = f

INSIDE   OUTSIDE

$N'(x) = \sin k(x_o - x)$

$N(x)$

Fig. 4

INSIDE

f = -g
f' = -g

OUTSIDE

f = -g
f' = -g

$N = A e^{-\sigma h(L+x)}$

$B e^{-\sigma h L}$

$N' = B e^{+\sigma h x}$

L

x

Fig. 5

$$f = -1 \qquad \begin{cases} f = -1 \\ f' = f + \Delta f \end{cases} \qquad \begin{cases} f = f \\ f' = f - \Delta f \end{cases} \qquad \begin{array}{l} N = \cos kx \\ N' = \cos(k - \Delta k)x \end{array}$$
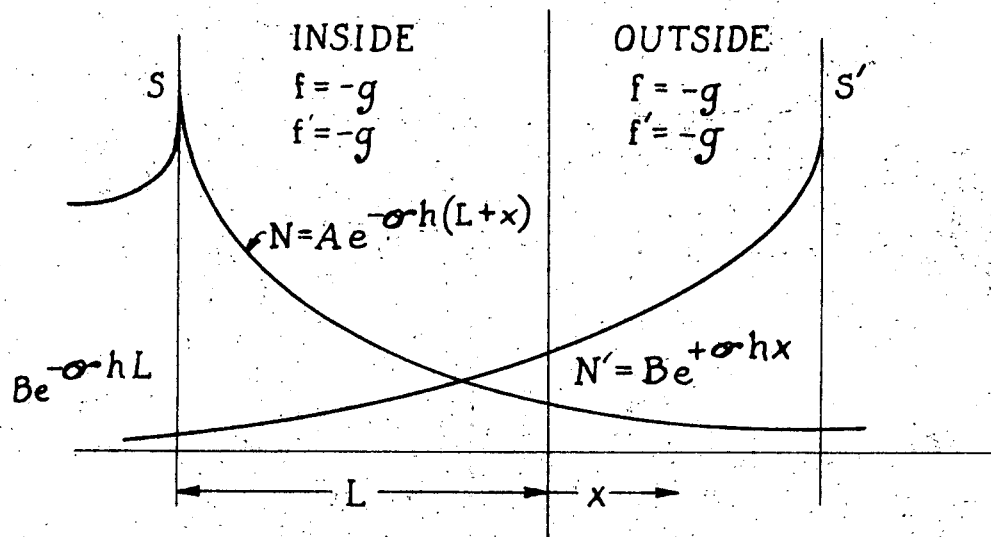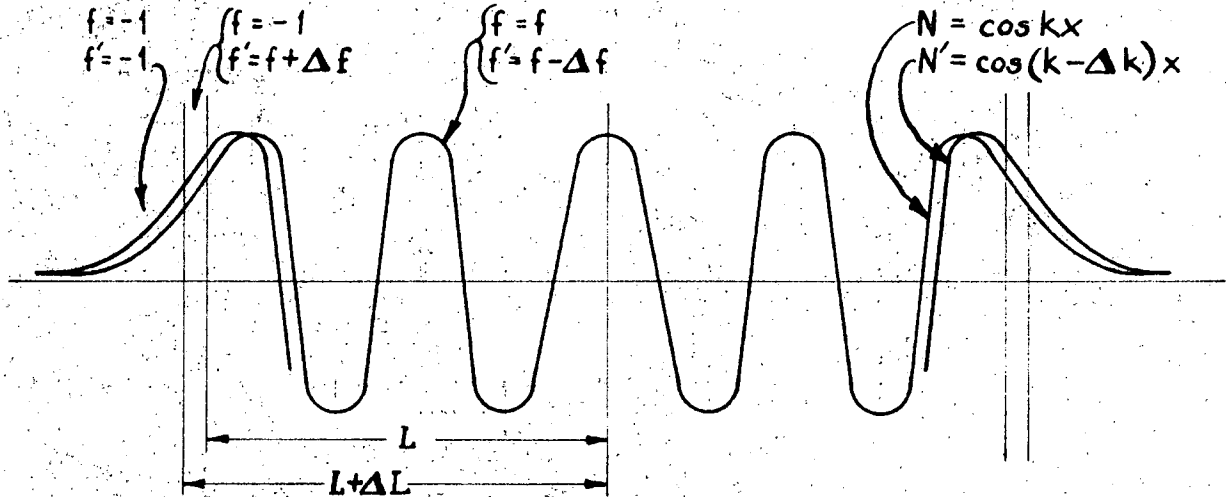
L

L + ΔL

Fig. 6

We also use this Eq. 21 to obtain a different kind of information about the solutions for one velocity problems. Suppose we have a homogeneous medium with a constant absorption g and constant cross section σ . If we have a plane source emitting S neutrons per second then we know that at greater distances from this source the neutron intensity falls off exponentially with distance x away from the source, as $Ae^{-h\sigma x}$ (see Eq. 19). The problem is to determine the magnitude A of the asymptotic solution for a source of strength S. We shall use Eq. 21 in the following manner: (See Fig. 5) We shall imagine that the plane which separate the "inside" from the "outside" region is at a distance L away from the source which is in the inside region in the unprimed case. For the primed solution we shall assume that there is some source at a great distance from the plane in the outside region and that, therefore, the solution in the inside region from this source behaves exponentially with an increasing exponential. In both the primed and unprimed cases

the value of f is the same and is the same in both the inside and outside
regions. There is no source in the inside region in the primed case. A
glance at Fig. 5, which indicates the coordinates used will show that Eq. 21
gives the formula:

$$SBe^{-hL} = \int_{x_2=0}^{\infty} \int_{x_1=-\infty}^{0} (1-g)^2 \left[Ae^{-\sigma h(L+x_1)}Be^{+\sigma hx_2} - Ae^{-\sigma h(L+x_2)} \right.$$

$$\left. Be^{+\sigma hx_1} \right] Q(x_1,x_2) \, dx_1 \, dx_2 \tag{25}$$

We can substitute the expression (23) for $Q(x_1, x_2)$ in the integrals on
the right side of (25) and then perform the integrals on x_1, x_2 and then on
μ. When this is done we obtain: (the value of B cancels out of both sides)

$$S = A \cdot \frac{(1-g)}{h} \left(\frac{h^2 - g}{1-h^2} \right) \tag{26}$$

which gives the connection between the source and the coefficient of the
exponential in the neutron density at greater distances. The formula is exact.
This can also be applied to a spherical problem in which we have the source of
strength S located at a point. At a very large distance r from this source
the neutron density is:

$$N(R) = \sigma S \cdot \frac{2h^2(1-h^2)}{(1-g)(h^2-g)} \cdot \frac{e^{-h\sigma R}}{4\pi R} \tag{27}$$

It is interesting to find that it is comparatively simple to find
the neutron density exactly at the boundary of an untamped region. For this
purpose we can apply our theorem (8) in the form (9) which is valid when there
is a small difference between the prime and unprimed systems. Suppose we have
a slab of multiplying medium of very large thickness, 2L (see Fig. 6). The
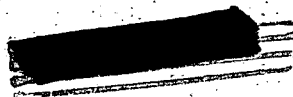
solution in this medium for the neutron density will be proportional to cos kx, say A cos kx. The relation between k and f is given by (16). In accordance with our extrapolated end point formula we know that M must go to 0 when x is equal to $L + x_o$, therefore:

$$k \ (L + x_o) = (\pi / 2) + 2\pi m \tag{28}$$

where m is some integer. Now, what will happen if we increase by an amount ΔL the half width of the slab? If it is still to be critical the f value need not be as great. We shall call it $f - \Delta f$. The new value of the wave number k which we shall call $k - \Delta k$ must satisfy:

$$(k - \Delta k) \ (L + \Delta L + x_o) = (\pi/2) + 2\pi_m \tag{29}$$

This implies that $\Delta k/k$ is equal to $\Delta L/L$. In the region between $x = L + \Delta L$ and $X = L$ the value of $1 + f$, which is 0 in the unprimed system, changes to $1 + f + \Delta f$ in the primed system, or by $1 + f$ to the first order. The neutron density here is approximately N_s, the value at the surface for the unprimed problem. In the region from $x = -L$ to $x = L$, the change in f is just Δf and the square of the neutron density (or more accurately, NN'), has an average value of $1/2A^2$ to the first order so that an application of Eq. 8 gives immediately:

$$(1 + f) \ N_s^2 \ \Delta L = (1/2) \ A^2 \cdot 2L \cdot \Delta f \quad \text{or} \quad N_s^2 = \left[A^2 k / (1 + f) \right] \ df/dk \tag{30}$$

We can compare N_s to Next, the value of the neutron density that would be at the surface if the approximate solution A cos kx were extrapolated all the way to the surface. That is Next is equal to $A \sin kx_o$. We find, finally, as an (exact) expression for the value of the neutron density at the
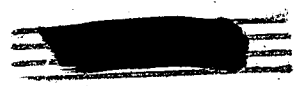
edge of a bounded multiplying medium, the expression: (where we have gotten
df/dk from 16)

$$N_s^2 / N_{ext}^2 = \left\{ \left[(1 + f) / (1 + k^2) \right] - 1 \right\} / \sin^2 k\, x_0 \tag{31}$$

If the medium is an absorbing medium a similar expression can be found; however,
the method of derivation given here must, of course, be altered. What one can
do is keep a source in the material, at $x = 0$, and use our expression (27) to
obtain the size of the neutron density at the edge of the medium. A method
exactly analagous to the method we have just used for the multiplying medium
can then be applied to the multiplying medium with the source at the center.
This gives the result:

$$N_s^2 / N_{ext}^2 = \left[1 - (1-g) / (1-h^2) \right] / \sinh^2 h x_0 \sigma \tag{32}$$

In the limit when the absorption or the multiplication goes to 0, N_x / N_{ext}
approaches $1 / (\sqrt{3} x_0 \sigma)$.

Finite Tamper

We can now apply the theorem to a more complicated problem, namely,
to determine the effect of finiteness of a tamper. We shall apply it in the
case of nonabsorbing ($g = 0$) tamper of radius b. Suppose then we have a spherical
core of radius a and a tamper of radius b. For the system to be critical a
certain f value, say f_b , is required in the core. Were the tamper infinite
the requisite value would be f_∞ . We
shall let the primed system in (8) be
this system with infinite tamper (see   *Fig. 7*
Fig. 7). The unprimed system will be
the system with finite tamper, or rather

Fig. 7
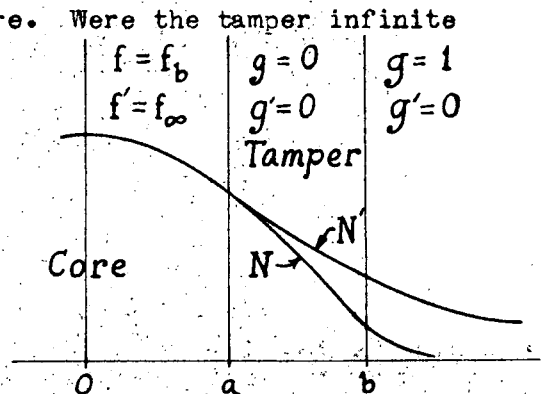
with tamper with complete absorption (g = 1) beyond b. Applying (3) we find,

(with σ_c and σ_t total cross section in core and tamper).

$$\sigma_c (f_b - f_\infty) \int N_b N_\infty \, dV_{core} = \sigma_t \int_b^\infty N_\infty N_b \cdot 4\pi r^2 dr \tag{33}$$

We shall now need approximate expressions for N_∞ and N_b. For very large r, N_∞ varies asymtotically as $1/r$, but there is a transition effect near the core. This transition effect does not penetrate more than a fraction of a mean free path into the tamper. If b-a exceeds a mean free path, then, to an excellent approximation N_∞ varies as $1/r$ for $r > b$ and we have

$$N_\infty (r) = (3 S \sigma_t / 4\pi) \cdot (1/r) \tag{34}$$

where

$$S = \sigma_c f_\infty \int N_\infty \, dV_{core} \tag{35}$$

is the net number of neutrons generated in the core. The coefficient in (34) can be found from the integral theory, or more simply by noting that very far out (large r) the neutron density varies very slowly in a mean free path $1/\sigma_t$. Hence, diffusion theory holds, which says the flux is $(1/3\sigma_t) \, 4\pi r^2 \, \partial N/\partial r$ and since this flux must just equal S, (34) follows.

The quantity N_b can be found approximately in the following way. As is well known, for one velocity spherical problems with constant mean free path everywhere there is a close relation to a corresponding slab problem. The solution for N for the sphere problem times r, is a solution for the slab. Now if our problem is looked at as a slab problem, we would imagine a source on one side of a finite slab of width b-a, with absorption on the other side, outside b. If b-a exceeds a mean free path or so, the solution outside b will be the same as though the source were much deeper and b-a were nearly infinite. This problem we have already considered above, and have called the solution M(X), so

that

$$N_b(r) = (A/r) \, M(r - b) \qquad \text{for } r > b \qquad (36)$$

where A is a constant. It may be objected to this procedure that the conversion from sphere to slab is invalid when the core and tamper mean free path are unequal. But surely the shape of the solution outside the layer of tamper of thickness of a mean free path or so is nearly independent of the exact properties of the core. The size, of course, depends on the net number of neutrons generated in the core. This must equal the total number absorbed outside the tamper, so that we have

$$\sigma_t \int_b^\infty N_b(r) \cdot 4\pi r^2 dr = \sigma_c f_b \int N_b \, dV_{core} = 4\pi \, \sigma_t A \int_0^\infty (y + b) \, M(y) \, dy \qquad (37)$$

where we used expression (36) and put $y = r-b$ in obtaining the last integral.

If we substitute (34) and (36) into (33) we find,

$$\sigma_c (f_b - f_\infty) \int N_b \, N_\infty \, dV_{core} = \sigma_t \int_b^\infty (3S\sigma_t/4\pi) \, (1/r) \, (A/r) \, M(r-b) \cdot 4\pi r^2 dr$$

$$= 3S\sigma_t^2 \, A \int_0^\infty M(y) \, dy.$$

Using (35) to eliminate S and the second equality in (37) to eliminate A, we find

$$\sigma_c (f_b - f_\infty) \int N_b \, N_\infty \, dV_{core} = \frac{3\sigma_c^2 \sigma_t f_b f_\infty}{4\pi} \cdot \frac{\int_0^\infty M(y) \, dy}{\int_0^\infty (y + b) M(y) \, dy} \cdot \int N_b \, dV_{core} \cdot \int N_\infty \, dV_{core}.$$

This can be still further simplified, by noting that the mean of Y for the function M(y) is x_0 (see (12)), so that the ratio of integrals involving M is just $1/(x_0 + b)$. Furthermore $\int dV_{core} = 4\pi A^3/3$ so a final rearrangement gives:

$$\frac{1}{f_\infty} - \frac{1}{f_b} = \frac{\sigma_c \sigma_t a^3}{b + x_0} \left\{ \frac{\int N_b \, dV_{core} \cdot \int N_\infty \, dV_{core}}{\int N_b \, N_\infty \, dV_{core} \cdot \int dV_{core}} \right\} \qquad (38)$$
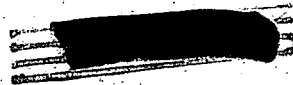
The expression in the bracket is very nearly 1 for practical cases. It has been calculated (see table) assuming the neutron density is a parabola $1-c(r/a)^2$ in the core, for various values of c for N_b and N_∞ (which we will call c_b and c_o). The ratio of N at the core surface to the value at the center is 1-c. This is rarely less than 1/2 so that c rarely exceeds 0.5. For an untamped core of material of f = 0.7 (like 49) C is about 3/4, however. For a core tamped with a non-absorbing tamper of equal mean free path C is .36. For smaller f, c is larger in the untamped case, and the better the tamper the larger is C. Numbers can be obtained from the table for any particular case, but for

TABLE II

c_b	$c_\infty \rightarrow$ 0	1/4	1/2	3/4	1
0	1.0000	1.0000	1.0000	1.0000	1.0000
1/4	1.0000	.9941	.9858	.9732	.9520
1/2	1.0000	.9858	.9662	.9371	.8909
3/4	1.0000	.9732	.9371	.8369	.8105
1.0	1.0000	.9520	.8909	.8105	.7000

nearly all purposes the expression can be replaced by 1, which makes (38) very much simpler.

Therefore, our formula for the effect of a finite non-absorbing tamper is: (noting (15))

$$1/f_\infty - 1/f_b = \sigma_c \sigma_t a^3 / \left[b + .71/\sigma_t \right] \tag{39}$$

From the derivation it is seen that (28) is not exact. It should be very nearly correct if b-a exceeds $1/\sigma_t$. We have tested it on the extreme case where b-a = 0; our tamper is so "finite" it is no tamper at all. Then, of course, our formula (39) cannot be expected to be correct. However, the results

(Table III) show that we are not off to an extreme degree. They apply to the case $\sigma_t = \sigma_c$ and the necessary data comes from the results of Frankel and Nelson (LA 53A) for tamped and untamped spheres. (They have also done a few cases with finite tampers and these agree with (39) as accurately as their graphs can be read).

<div align="center">TABLE III</div>

Core Radius a	Critical f Infinite Tamper f_∞	Critical f Correct $f_{b=a}$ LA 53A	No Tamper from Eq.(39) $f_{b=0}$
.9	.857	1.299	1.208
1.1	.517	.874	.834
1.4	.345	.642	.625
1.8	.221	.460	.454
2.3	.142	.350	.333

A formula (Eq. 40 below) has been proposed to give the amount of absorption (g) in an infinite tamper which is equivalent to having a finite tamper of radius b. The attempt is made to make the slope-to-value-ratio of rN in the tamper at the surface of the core equal in the two cases. Assymptotic solutions in the tamper are assumed to hold up to the core surface. Thus, with absorption g, the asymptotic solution for rN varies as $e^{-h\sigma r}$ (see (19) for the relation of g and h.) so that its slope to value ratio is $h\sigma$. For the finite tamper the "asymptotic" N is a straight line going to zero at the extrapolated point $r = b + x_o$ so that rN is proportional to $b + x_o - r$. Its slope to value ratio at $r = a$ is $1/(b + x_o - a)$. Setting these two quantities equal we find

$$\sigma_t h = 1 / (b + x_o - a) \tag{40}$$

This then gives the amount of absorption (or rather h) an infinite
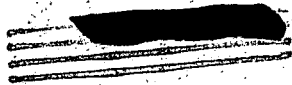
tamper should have to be equally effective in reflecting neutrons to a tamper with no-absorption but with a finite outer radius b. The formula is correct in the limit of diffusion theory. Unfortunately, however, it is not very accurate for many practical cases and its use is to be discouraged. Formula (39) should be used instead.

We can derive the relation (40) in a less intuitive way, which, however, will help to show up its errors. For this purpose, we shall find the change in f consequent on changing g from o to g in an infinite tamper (we will say f changes from f_o to f_g). If the prime system is for g = 0, and the unprimed for finite g (so that f_o and N_o here is what f_∞ and N_∞ represents in (21)) our theorem (8) tells us,

$$\sigma_t \, (f_g - f_o) \int N_o \, N_g \, dV_{core} = \sigma_t \, g \int_a^\infty N_o \, N_g \, 4\pi \, r^2 \, dr \qquad (41)$$

We shall have to use estimates for N_o and N_g in the tamper in order to make (41) useful. We shall make the approximation that N_o, N_g can be replaced in (41) by their asymptotic forms, equation (34) for N_o and

$$N_g = (B/r) \quad e^{-h\sigma(r - a)} \qquad (42)$$

for N_g. This will turn out not to be a particularly good approximation. The constant B in (42) is determined from the conservation of neutrons, (see 11)

$$S_g = \sigma_c \, f_g \int N_g dV_{core} = \sigma_t \, g \int_a^\infty N_g \, 4\pi r^2 dr = 4\pi \, \sigma_t g B(a + 1/h\sigma_t) \qquad (43)$$

where S_g is the net number of neutrons generated in the core. These forms (34) and (42) can now be substituted for N_o and N_g in (41), the integral performed (43) used to determine B, and the result rearranged (just analogously to the derivation of 39) to read:
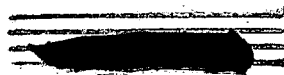
$$\frac{1}{f_o} - \frac{1}{f_g} = \frac{\sigma_c \, \sigma_t \, a^3}{a + \frac{1}{h\sigma_t}} \left\{ \frac{\int N_b \, dV_{core} \cdot \int N_o \, dV_{core}}{\int N_b N_o \, dV_{core} \cdot \int dV_{core}} \right\} \tag{44}$$

The expression in the bracket is always almost exactly 1, and can be forgotten.

If the tamper with absorption is to have the same effect as the finite tamper, then they must make equal changes in $1/f$. This will be the case if

$$a + 1/h\sigma_t = b + x_o \tag{45}$$

which is equivalent to (40).

The reason (45) or (40) is not accurate is not because (39) is not accurate, but rather that (44) is not. Table IV gives the results of some calculations made with it, and these results are compared with the correct results of LA-173. The result is very poor when the tamper mean free path is much smaller than that of the core.

TABLE IV

| $\sigma_c a$	σ_c/σ_t	g	$f_{g=o}$	f_g true	f_g calc.(44)
4.0	2.0	.1	.0780	.112	.115
1.5	2.0	.1	.410	.485	.471
4.0	2.0	.02	.0780	.096	.098
1.5	2.0	.02	.410	.449	.442
2.0	1.2	.1	.2048	.3025	.3006
2.0	1.2	.02	.2048	.2555	.2549
1.0	0.2	.02	.224	.563	1.186
1.0	0.2	.10	.224	.392	.581
1.0	0.2	.40	.224	.766	2.703

The reason is that in deriving (44) we used for N_0 and N_g simply their asymptotic values in an integral which extends right up to the core surface, near which these asymptotic forms are known to be false. This error was not made in deriving (39) since only integrals beyond $r = b$ were involved. This is sufficiently far from the surface $r = a$, to permit use of the asymptotic form.

We shall now go on to derive a formula like (39) for tampers which do absorb neutrons. This formula will, however, not be anywhere nearly as reliable as formula (39) for, as we shall see, it will require the integral over tamper solutions near the surface of the core where they are not accurately known. We shall assume a core of radius a and the tamper of radius b. The value of f needed to make the core critical will be called f_b. The absorption of the tamper will be g. We shall compare this problem to a problem in which the radius of the tamper is infinite and for which the value of f, which makes the core critical, if f_∞. As we have already seen, the finite tamper is equivalent to an infinite tamper if one arranges that the neutron density goes to 0 to some point, a distance x_0 outside the outer radius of the tamper. The quantity, x_0, is given by equation (20) or, more simply, by the approximate expression, $.71/1-g$. This neutron density can be made to go to 0 at the point $r = b + x_0$ in an infinite tamper by having a negative source very far out on the tamper and at distance c from the origin. The solution in the tamper, except very near $r = a$ or $r = b$ is:

$$(A/r)\ e^{-h\sigma r}\ (1 - e^{-2h\sigma\ (b + x_0 - r)})\ \tag{46}$$

The second term in this expression arises from the source at c and the strength of the source, per square centimeter, must be:
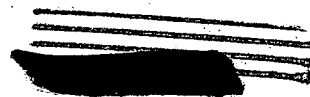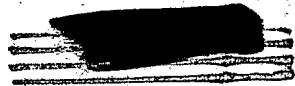
$$-S_c = \frac{(1-g)}{h} \quad \frac{(h^2 - g)}{(1 - h^2)} \quad \frac{A\,e^{h\sigma c}}{c} \quad e^{-2h\sigma(b + x_0)} \tag{47}$$

in accordance with Eq. (26). Now, we shall apply our theorem (21) to compare this problem, which we will call the unprimed problem, to the problem of an infinite tamper with no source in the tamper for which the solution is $B\,e^{-\sigma hR}/R$. In applying Eq. (21) we will assume that the inside region extends to infinity and that, therefore, the integrals on the right hand side of Eq. (21) do not make any contribution. Using a notation similar to that which we applied in the case that the tamper was not absorbing, an application of Eq. (21) gives the result:

$$(f_b - f_\infty) \int N_b\, N_\infty\, dV_{core} = -\frac{B\,e^{-\sigma h c}}{c} \quad S_c \cdot 4\pi c^2 \tag{48}$$

If we substitute the expression for S_c, which we obtained above, (47) we will find:

$$\sigma_c (f_b - f_\infty) \int N_b\, N_\infty\, dV_{core} = 4\pi \frac{(1-g)\,(h^2 - g)}{h(1 - h^2)} e^{-2\sigma h(b + x_0)} A \cdot B. \tag{49}$$

In order, therefore, to find $f_b - f_\infty$, we shall have to have an expression for A and B. We shall attempt to find B by using the fact that all of the neutrons which are generated in the core are absorbed somewhere in the tamper (see Eq. (11)). However, although we know the solution in the tamper at large distances from the core we do not know the solution near the core. This will make an error in our analysis which should certainly be studied further; but until this is done we shall use the asymptotic solution right up to the edge of the core. Therefore, if we substitute this asymptotic solution in Eq. (11) we will obtain an expression for B. The equation gives:

$$S_\infty = f_\infty \sigma_c \int N_\infty\, dV_{core} = g\sigma_t \int_a^\infty 4\pi r^2\, \frac{B\,e^{-h\sigma r}}{r}\, dr = \frac{B \cdot 4\pi g \sigma_t}{h^2 \sigma_t^2}\, (1 + ah\sigma_t)\,e^{-ah\sigma_t} \tag{50}$$

We shall compute A by a similar procedure. However, in this case we have a source in the tamper and so shall have to subtract the flux from our negative source from the total number of neutrons absorbed in the tamper. This results in:

$$S_b = \sigma_c f_b \int N_b dV_{core} = A \cdot \frac{4\pi g \sigma_t}{h^2 \sigma_t^2} \left[(1 + ah\sigma_t) \ e^{-ah\sigma_t} - (1-ah\sigma_t) e^{-\sigma_t h(2b+2x_0 -a)} \right] \tag{51}$$

Substituting the expressions for B and A, obtained from (50) and (51) into (49) and rearranging terms somewhat we will obtain finally an equation for the finite tamper:

$$\frac{1}{f_\infty} - \frac{1}{f_b} = \left(\frac{h^2(1-g)(h^2-g)}{6g^2(1-h)} \right) \frac{2\sigma_t^2 \ \sigma_c \ h \ a^3}{(1+ah\sigma_t)^2 \left[e^{2h\sigma_t(b + x_0 -a)} - \frac{1-ah\sigma_t}{1+ah\sigma_t} \right]} \cdot$$

$$\cdot \left\{ \frac{\int N_b \ dV_{core} \cdot \int N_\infty \ dV_{core}}{\int N_b \ N_\infty \ dV_{core} \cdot \int dV_{core}} \right\} \tag{52}$$

As usual the integrals of the neutron density over the core in the bracket combine to form a ratio approximately 1 for practical purposes. Furthermore, the complicated expression, depending upon g and h, in the parenthesis of expression (52), is equal to 1 within 10% as long as g is less than .3. Since this is usually the case, we can usually use a more simple expression:

$$\frac{1}{f_\infty} - \frac{1}{f_b} = \frac{2\sigma_t^2 \ \sigma_c \ ha^3}{(1+ ah\sigma_t)^2 \left[e^{2h\sigma_t(b + x_0 - a)} - \frac{(1-ah\sigma_t)}{(1+ah\sigma_t)} \right]} \tag{53}$$

In order to improve this formula it would be necessary to make some estimate of the effect of the deviations of the true neutron density from the asymptotic expression near the core surface. In the case of a core in which
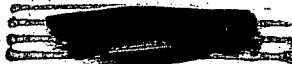
the mean free path is the same as in the tamper, it is possible to obtain

considerable information about this deviation from a direct analysis of the

integral equation (see LA 53). Another method in this case, to obtain the

relation between A and S_b, is a method analagous to the one by which we obtain

this relation, when the source was all located at a point (see the deriva-

tion of equation (26). When the core and tamper mean free paths are not equal,

the deviations between the asymptotic and true solutions near the core are more

difficult to find, and depend very strongly on this difference in mean free

path. Some information can be obtained if an attempt is made to find how

much change in the value of f in the core of an infinity tamped gadget is

required to compensate a given change in the absorption g of the tamper.

This requires, essentially, the integration of n^2 over the tamper. Since the

changes in f for given changes in g are know for infinitely tamped gadgets,

this gives a method of comparing the true value of the integral of n^2 with

the value of this integral that would be obtained in using the asymptotic

expression for the neutron density, In this way some idea can be obtained as

to the importance of the deviations. In addition, the value of the neutron

density at the surface of the core also be found (by variation of f with core

radius, as in derivation of (31)) so that still further information can be

obtained as to the deviations. The combination of these facts to give an

improved formula has not been carried out.

One can try also to find the extra absorption in an infinite tamper

which would give the same absorption as a finite one. This method is clear

from the derivation of (40). The result is that the equivalent absorption

constant h' for infinite tamper is given by

$$h' = h \,/\, \tanh \ \sigma_t h \ (b + x_o - a) \tag{54}$$
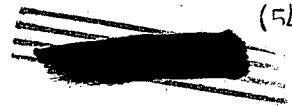
This formula is much simpler than (53). At present, it is not known in general which is more accurate. For tampers with very small absorption (53) does become the more accurate, as the descrepency terms make little contribution.

The extrapolated end point method has been extended by Frankel and Goldberg LA-258 to apply to finite tampers having the same mean free path as the core. The following table (due to Welton) gives a comparison between their results and the results obtained by equation (53). The radius of the core is a = 1.4 mean free paths, and the tamper radius b in mean free path is 3, 5 or ∞.

g	b	AECD-2056 (LA-258) f_b	This Report f_b	AECD-2056 (LA-258) $\frac{1}{f_\infty} - \frac{1}{f_b}$	This Report $\frac{1}{f_\infty} - \frac{1}{f_b}$
0.0	∞	0.3369			
0.0	5	0.4033	0.4054	0.4887	0.5015
0.0	3	0.4505	0.4552	0.7485	0.7714
0.1	∞	0.4773			
0.1	5	0.4795	0.4796	0.0096	0.0100
0.1	3	0.4945	0.4969	0.0729	0.0826
0.2	∞	0.5191			
0.2	5	0.5194	0.5196	0.0011	0.0019
0.2	3	0.5255	0.0235	0.5274	0.0303