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PHYSICS



UNITED STATES ATOMIC ENERGY COMMISSION

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Photostat Price $ 4.80
Microfilm Price $ 2.70

Available from the
Office of Technical Services
Department of Commerce
Washington 25, D. C.

August 6, 1946

Los Alamos Scientific Laboratory
New Mexico

Technical Information Service Extension, Oak Ridge, Tenn.

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956 - [illegible] 1

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CRITICAL DIMENSIONS OF WATER-TAMPED SLABS AND SPHERES OF ACTIVE MATERIAL

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ABSTRACT

The magnitude and distribution of the fission rate per unit area produced by three energy groups of moderated neutrons reflected from a water tamper into one side of an infinite slab of active material is calculated approximately in section II. This rate is directly proportional to the current density of fast neutrons from the active material incident on the water tamper.

The critical slab thickness is obtained in section III by solving an inhomogeneous transport integral equation for the fast-neutron current density into the tamper. Extensive use is made of the formulae derived in THE MATHEMATICAL DEVELOPMENT OF THE END-POINT METHOD by Frankel and Goldberg. (cf. LA-258, LADC-76, or AECD-2056.)

In section IV slight alterations in the theory outlined in sections II and III were made so that one could approximately compute the critical radius of a water-tamper sphere of active material.

The derived formulae were applied to calculate the critical dimensions of water-tamped slabs and spheres of solid $UF_6$ leaving various (25) isotope enrichment fractions. (cf. Fig. 6.)

## I. INTRODUCTION

The primary effect of placing water on one side of a slab of active material is to return slow neutrons to the slab. Fissions produced by the slow neutrons reflected from the water give rise to a



956 - 002

distribution of fast fission neutrons which decreases rapidly as one goes into the active layer.

The multiplication of fission neutrons in the slab proceeds from this essentially asymmetric source. One may describe the process by which a time-independent fast-neutron density in the active layer is established as follows:

Of the primary fission neutrons in the slab a certain fraction reproduce fission neutrons without having left the active material. The rest (a) suffer radiative capture before leaving the slab, or (b) they escape into the water on one side of the slab, or (c) they leak off the bare side never to return. Of those that enter the water, a certain fraction are returned to the slab after moderation and produce fissions. Let us indicate the number of fast neutrons per sec per $cm^2$ incident on the water as I. The number of fission per $cm^2$ per sec that slow neutrons returning from the water produce between x and x + dx in the slab we shall call $I\, f_S(x)\, dx$

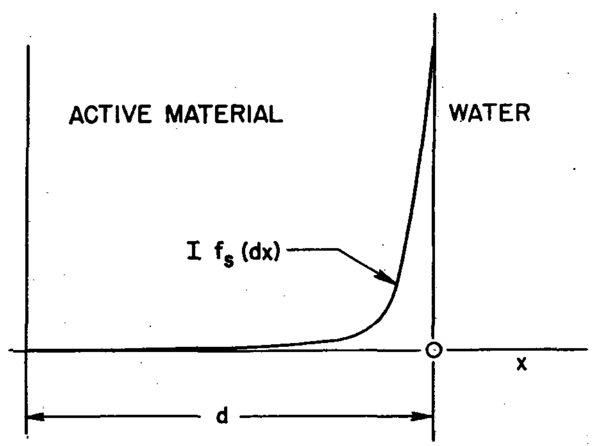


Fig. 1

The problem of calculating the critical thickness, d, is now conveniently broken up into two parts. The first is to determine the magnitude and distribution of the source fission rate per $cm^3$, $I\, f_S(x)$. The second is to compute the current of fast neutrons into the water which depends on the source strength of fission neutrons, $\nu\, I\, f_S(x)$, slab thickness, d, in units of fast neutron mean free path in the active material, and the value of f, defined in the usual manner as the excess

956-023

number of neutrons emerging per collision from an average nucleus in the active layer.

## II. DETERMINATION OF THE FISSION RATE PRODUCED BY REFLECTED NEUTRONS

In order to calculate approximately the distribution in the slab of fissions per unit fast neutron current into the water, $f_s(x)$, we defined a penetration length, $l$, of fast neutrons into the water such that $I \exp[-x/l]\, d(x/l)$ represents an isotropic source of neutrons of age $\tau_0 = l^2$ in the water. We assumed arbitrarily that neutrons entering the water had to suffer between 3 and 4 hydrogen collisions before they were sufficiently disoriented to be considered as an isotropic source. With such a source of neutrons in the water one may obtain the subsequent neutron flux distribution in the age group between $\tau_0$ and $\tau$ by solving the following diffusion-like equations:

In water $\quad \lambda(n\nu)_1''/3 - \lambda(n\nu)_1/3\tau' + I \exp[-x/l]/l = 0 \qquad$ (1a)
$x \geq 0$

In active layer $\quad \Lambda_1(N\nu)_1''/3 - \sigma_a^{(1)}(N\nu)_1 = 0 \qquad$ (1b)
$x \leq 0$

Here $\lambda$ is the transport mean free path in water, $\Lambda_1$ is the first-group average transport mean free path in the active layer (assumed to extend to $x = -\infty$), and $\sigma_a^{(1)}$ is the first-group average total absorption cross section per $cm^3$ in the active material. The latter two quantities were averaged over the first group energy range corresponding to neutrons of age $\tau_0$ to $\tau_1$.

The quantity $\lambda/3\tau'$ is the absorption cross section per $cm^3$, $\sigma_{ab}$, effective in removing neutrons from the age group between $\tau' = 0$ and $\tau' = \tau = -\tau_0$ in water. It was estimated as $\sigma_{ab} = \sigma_s(H)/N$ where $\sigma_s(H)$ is the hydrogen scattering cross section per $cm^3$ and N is the average number of collisions required to moderate a neutron from age $\tau_0$ to $\tau$ or from an energy $E_0$ to E. Assuming unit average logarithmic energy loss per collision in water, one obtains according to age theory:

$$N = ln(E_0/E) = 3(\tau - \tau_0)\, \sigma_s(H)/\lambda \qquad (2)$$

Therefore $\sigma_{ab} = \sigma_s(H)/N = \lambda/3\tau'$ where $\tau' = \tau - \tau_0$.

At the interface between active material and water (x = 0) the usual continuity of neutron density and current boundary conditions were applied, namely:

$$n_1(0) = N_1(0) \text{ and } \lambda(n\nu)_1' = \Lambda_1(N\nu)_1' \tag{3}$$

In Eqs. 1 and 3 and hereafter the neutron density and mean free path are designated respectively by lower case n and $\lambda$ in the water, and upper case N and $\Lambda$ in the active material.

Applying the boundary conditions (3) the solutions satisfying Eqs. 1a and 1b are

$$(n\nu)_1 = (3I\tau')(\lambda l)^{-1}\,(1 - \sqrt{\tau'}/l)^{-1}\,A(\tau')\,[e^{-x/l} - (\sqrt{\tau'}/l)(1 + S_1 l/\lambda)\,B(\tau')\,e^{-x/l}\sqrt{\tau'}] \tag{4a}$$

and

$$(N\nu)_1 = (3I\tau')(\lambda l)^{-1}\,A(\tau')\,B(\tau')\,e^{+x/L} \tag{4b}$$

where

$$A(\tau') = (1 + \sqrt{\tau'}/l)^{-1},\ B(\tau') = (1 + S_1\sqrt{\tau'}/\lambda)^{-1} \tag{5}$$

The diffusion length in the active layer, of neutrons in the age group $\tau' = 0$ to $\tau' = \tau_1 - \tau_0$ is assumed constant; it is

$$L_1 = \sqrt{\Lambda_1/3\ \sigma_a^{(1)}} \tag{6}$$

The dimensionless quantity $S_1$ is the ratio between the active layer average mean free path and diffusion length of first-group neutrons.

$$S_1 = \Lambda_1/L_1 \tag{7}$$

The flux, $(N\nu)_1$ expressed in Eq. 4b, gives the exponential distribution of those neutrons in the active material that have been reflected by the water and have energies corresponding to the age range between $\tau_0$ and $\tau$ (i.e., 0 to $\tau'$ where $0 \le \tau' \le \tau_1 - \tau_0 \equiv \tau_1'$). By differentiating $(N\nu)_1$ with respect to $\tau'$ one obtains the flux between $\tau'$ and $\tau' + d\tau'$.

956-005

Multiplying this differential flux in the active material by the fission cross section per cm$^3$, $\sigma_+$, considered as a function of $\tau'$ and integrating from $\tau' = 0$ to $\tau' = \tau_1'$, one obtains the fission rate per cm$^2$ produced between x and x + dx by first group neutrons reflected from the water.

$$I f_s^1 \ (x)\ dx = I\ F_1 \exp(x/L_1)\ d(x/L_1),\ x \leq 0 \tag{8}$$

Here, $F_1$ is a dimensionless constant representing the total fission rate per cm$^2$ of slab, produced by first-group reflected slow neutrons, per unit incident fast-neutron current into the water.

$$F_1 = (3L_1)/(2\lambda l) \int_0^{\tau_1'} d\tau'\ \sigma_f\ AB(A + B) \tag{9}$$

A and B are the functions of $\tau'$ defined in Eq. 5. For a given active material $F_1$ may be determined by numerical integration.

A second group of neutrons, $\tau = \tau_1$ to $\tau_2$, where $\tau_2$ is the age of thermalized neutrons in water was similarly treated. The diffusion equations for this group are similar to Eqs. 1a and 1b. The source term, $Q_2$, replacing $I \exp[-x/l]/l$ in Eq. 1a is simply the number of neutrons removed per sec per cm$^3$ from group 1, namely according to Eq. 4a:

$$Q_2 \equiv [\lambda(n\nu)_1/3\tau'] = (I/l)(1 - \sqrt{\tau_1'}/l)^{-1}\ A(\tau_1')[e^{-x/l} - (\sqrt{\tau_1'}/l)(1 + S_1 l/\lambda)\cdot B(\tau_1')\ e^{-x/\sqrt{\tau_1'}}] \tag{10}$$

$\tau' = \tau_1'$

The effective absorption rate per cm$^3$ from the second-group neutrons of age between $\tau_1$ and $\tau$ ($\tau_1 \leq \tau \leq \tau_2$) is $\lambda(n\nu)_2/3\tau''$, where $\tau'' = \tau - \tau_1$. Equation 1b is altered only insofar as to replace $\Lambda_1$ and $\sigma_a^{(1)}$ by $\Lambda_2$ and $\sigma_a^{(2)}$, the average mean free path and absorption cross section per cm$^3$ of second group neutrons in the active material.

The solutions for the second-group neutron flux are similar to Eqs. 4a and 4b.

$$(n\nu)_2 = [(3\ I\ \tau'')/(\lambda l)]\ [A_2 e^{-x/l} + B_2 e^{-x/\sqrt{\tau_1'}} + C_2 e^{-x/\sqrt{\tau''}}] \tag{11a}$$

$$(N\nu)_2 = [(3\ I\ \tau'')/(\lambda l)]\ D_2 e^{x/L_2} \tag{11b}$$

956-006

where $A_2 = (1 - \tau_1'/l^2)^{-1} (1 - \tau''/l^2)^{-1}$

$$B_2 = A(\tau_1') B(\tau_1')(1 - \tau''/\tau_1')^{-1} - (1 - \tau_1'/l^2)^{-1}(1 - \tau''/\tau_1')^{-1}$$

$$C_2 = (1 + S_2\sqrt{\tau''}/\lambda)^{-1} [(\sqrt{\tau''}/l)(1 + S_2 l/\lambda) A_2 + \sqrt{\tau''/\tau_1'} (1 + S_2\sqrt{\tau_1'}/\lambda) B_2] \quad (12)$$

and

$$D_2 = (1 + S_2\sqrt{\tau''}/\lambda)^{-1} [(1 - \sqrt{\tau''}/l) A_2 + (1 - \sqrt{\tau''/\tau_1'}) B_2]$$

The expression for the distribution of fissions produced by second-group neutrons obtained by differentiating Eq. 11b with respect to $\tau''$, multiplying by $\sigma_f$ considered as a function of $\tau''$, and integrating from $\tau'' = 0$ to $\tau'' = \tau_2'' = \tau_2 - \tau_1$ is

$$I f_s^{(2)}(x)\, dx = I F_2 \exp(x/L_2)\, d(x/L_2), \; x = 0 \quad (13)$$

where

$$F_2 = [(3L_2)/(2\lambda l)(1 - \tau_1'/l^2)] \{\int_0^{\tau_2''} d\tau''\, \sigma_f A'B'(A' + B') - (\sqrt{\tau_1'}/l)(1 + S_1 l/\lambda) B(\tau_1') \int_0^{\tau_2''} d\tau''\, \sigma_f C'B'(C' + B')\} \quad (14)$$

The functions $A'$, $B'$, and $C'$ are similar to A and B defined in Eq. 5. They are:

$$A' = (1 + \sqrt{\tau''}/l)^{-1}, \quad B' = (1 + S_2\sqrt{\tau''}/\lambda)^{-1}, \quad C' = (1 + \sqrt{\tau''/\tau_1'})^{-1} \quad (15)$$

Here $S_2$ is the ratio between the second-group average mean free path ($\Lambda_2$) and diffusion length ($L_2$) in the active material.

Neutrons which have been thermalized in the water (i.e., reached an age $\tau_2$) are considered as the source of a thermal neutron group designated by the subscript, 3. Their energy is limited to an approximately Maxwellian distribution. The strength of the source of thermal neutrons, $Q_3$, is simply the number of neutrons per $cm^3$ removed per sec from the second group given by Eq. 11a.

$$Q_3 \equiv [\lambda(n\nu)_2/3\tau'']_{\tau''=\tau_2''} = (I/l)[A_2(\tau'')\, e^{-x/l} + B_2(\tau_2'')\, e^{-x/\sqrt{\tau_1'}} + C_2(\tau_2'')\, e^{-x/\sqrt{\tau_2''}}] \quad (16)$$

955-017

In the water the diffusion of thermal neutrons is governed by the simple diffusion equation,

$$(n\nu)_3'' - (n\nu)_3/L^2 + Q_3/L^2\, \sigma_a^{(H)} = 0 \tag{17}$$

Here $\sigma_a^{(H)}$ is the hydrogen thermal absorption cross section per $cm^3$ in water and L is the thermal diffusion length in water; $\sigma_a^{(H)} = 0.0208\ cm^{-1}$ and $L = 2.88$ cm.

In the active material the thermal neutron mean free path and diffusion length are designated as $\Lambda_3$ and $L_3$, respectively. Thermal neutrons which enter the active layer obey the diffusion equation,

$$(N\nu)_3'' - (N\nu)_3/L_3^2 = 0 \tag{18}$$

The boundary conditions at $x = 0$ that the solutions of Eqs. 17 and 18 satisfy are:

$$L^2\, \sigma_a^{(H)}(n\nu_3)' = (\Lambda_3/3)(N\nu)_3', \text{ and } (n\nu)_3 = (N\nu)_3 \tag{19}$$

Multiplying the thermal-neutron flux in the active material $(N\nu)_3$ by the thermal-neutron fission cross section per $cm^3$, $\sigma_f^{(3)}$, one obtains the third-group fission distribution,

$$I\, f_s^{(3)}(x)\, dx = I\, F_3 \exp(x/L_3)\, d(x/L_3), \quad x \le 0 \tag{20}$$

where $F_3$ is the long expression,

$$\begin{aligned} F_3 = {} & (L_3/l)(\sigma_f^{(3)}/\sigma_a^{(H)})(1 - \tau_1'/l^2)^{-1}\,(1 + S_3/3\, \sigma_a^{(H)}\, L)^{-1} \\ & \times \{(1 - \tau_2''/l^2)(1 + L/l)^{-1} - (\sqrt{\tau_1'}/l)(1 + S_1 l/\lambda)(1 + L/\sqrt{\tau_1'})^{-1} \\ & \times (1 - \tau_2''/\tau_1')^{-1}\, B(\tau_1') \\ & + (\sqrt{\tau_1'}/l)(1 + S_1 l/\lambda)(1 - \sqrt{\tau_2''/\tau_1'} + B'^{-1}(\tau_2''))(1 - \tau_2''/\tau_1')^{-1} \\ & + (1 + L/\sqrt{\tau_2''})^{-1}\, B(\tau) \\ & - (\sqrt{\tau_2''}/l)(1 + S_2 l/\lambda)(1 + L/\sqrt{\tau_2''})^{-1}(1 - \tau_2''/l)^{-1}\, B'(\tau_2'')\} \end{aligned} \tag{21}$$

In the next section we will consider the fast neutron multiplication that proceeds from the fission neutron source, $\phi(x)$, given by $\nu$ times the sum of the three fission rate distributions, Eqs. 8, 13, and 20.

$$\phi(x)\,dx = I\,\nu \sum_{i=1}^{3} f_S^{(i)}(x) = I\,\nu \sum_{i=1}^{3} F_i \alpha_i \exp(-\alpha_i x)\,dx \qquad (22)$$

Here we have used as unit of length the inverse transport cross section per $cm^3$, $\sigma^{-1}$, of the fast neutrons in the active material. The three constants $\alpha_i$ are thus $\alpha_i = 1/\sigma L_i$; i = 1, 2, and 3. For convenience the sign of x has been reversed to conform with Fig. 2. The constants $F_i$ are given by Eqs. 9, 14, and 21.

## III. DETERMINATION OF THE FAST-NEUTRON CURRENT INTO THE WATER

The treatment of the fast-neutron multiplication in the slat of active material outlined here makes extensive use of the work done by Frankel, Goldberg, and Nelson in solving the inhomogeneous transport integral equation as reported in (LA-258). Throughout the treatment we make use of the fact that $\phi(x)$ [Equation (22)] becomes negligibly small at a distance x a few mean free paths into the active material as shown in the figure below.

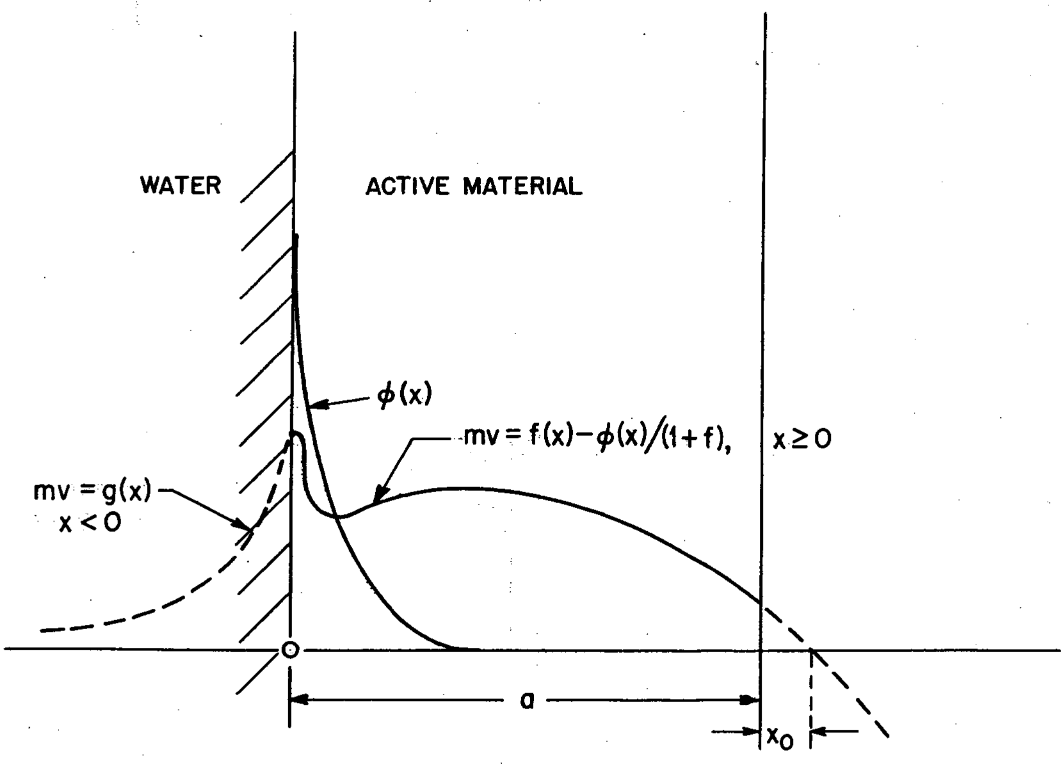


Fig. 2

956-009

The two quantities necessary to describe the flux of fast neutrons in the active material of thickness "a" = $\sigma d$ are the transport cross section per $cm^3$, $\sigma$, and the net number of neutrons emerging per collision $1 + f = (\nu\sigma_f + \sigma_s)/\sigma$. Here $\sigma$, $\sigma_f$ and $\sigma_s$ are the one-group fast neutron total transport, fission, and scattering cross sections $cm^3$, respectively.

The actual flux at x, generated by last collisions at x′ is given by

$$m(x)\,\nu = (1 + f)\int_0^a dx'\,[E_1|x - x'|/2][m(x')\,\nu + \phi(x')/(1 + f)] \tag{23}$$

where $E_1(x)$ is the exponential integral, $\int_1^\infty dy\, e^{-xy}/y$. Let us define a function n(x) for both positive and negative values of x as follows:

$$n(x) = \begin{cases} m(x)\,\nu + \phi(x)/(1 + f) & x \geq 0 \\ m(x)\,\nu & x < 0 \end{cases} \tag{24}$$

From Eq. 23 one then obtains an inhomogeneous integral equation for n,

$$n(x) = (1 + f)\int_0^a dx'\,[E_1|x - x'|/2]\,n(x') + \phi(x)/(1 + f) \tag{25}$$

where $\phi(x)$ is zero for $x < 0$ and is given by Eq. 22 for $x \geq 0$.

The water acts as an absorber of fast neutrons, none being returned to the active material without having been moderated. One may interpret the quantity n(x) dx in the region $x < 0$ given by Eq. 25 as the absorption rate per unit area between x and x + dx of a medium having zero elastic scattering cross section for fast neutrons and an absorption mean free path just equal to the transport mean free path of fast neutrons in the active material to the right of $x = 0$. Thus the current density crossing the surface at $x = 0$ to the left is simply

$$I = \int_{-\infty}^{0} dx\, n(x) \tag{26}$$

Now one may distort Eq. 25 by letting the upper limit of the integral approach $\infty$ and at the same time impose the restriction that $m(x)\,\nu$ and thus n(x) (since $\phi$ becomes negligibly small) vanishes at $x = a + x_0$. Here $x_0$ is the usual extrapolated end point and a is the critical slab thickness.

$$x_0 = 0.71/(1 + f) \quad \text{and} \quad a = \sigma d \tag{27}$$

956-010

If one defines two functions f(x) and g(x) as follows:

$$n(x) = f(x) + g(x) \begin{cases} f(x) = 0 & \text{for} \quad x < 0 \\ g(x) = 0 & \text{for} \quad x \geq 0 \end{cases} \tag{28}$$

Eq. 25 may be rewritten as

$$f(x) + g(x) = (1 + f) \int_{-\infty}^{+\infty} dx' \, [E_1|x - x'|/2] \, f(x') + \phi(x)/(1 + f) \tag{29}$$

with the condition that $f(a + x_0) = 0$.

We shall outline the procedure used in solving the above inhomogeneous Weiner-Hopf-type integral equation for the current density given by Eq. 26. Let us define the following Laplace transforms:

$$\begin{aligned} F(k) &= \int_0^{+\infty} dx \, f(x) \, e^{-kx} = \int_0^{\infty} dx \, n(x) \, e^{-kx} \\ G(k) &= \int_{-\infty}^{+\infty} dx \, g(x) \, e^{-kx} = \int_{-\infty}^{0} dx \, n(x) \, e^{-kx} \\ F_1(k) &= \int_{-\infty}^{+\infty} dx \, \phi(x) \, e^{-kx} = \int_0^{\infty} dx \, \phi(x) \, e^{-kx} \\ P(k) &= \int_{-\infty}^{+\infty} dx \, [E_1|x|/2] \, e^{-kx} \end{aligned} \tag{30}$$

The last two transforms are simply

$$F_1(k) = I\nu \sum_{i=1}^{3} F_i \int_0^{\infty} dx \, \alpha_i e^{-(\alpha_i + k) dx} = I\nu \sum_{i=1}^{3} F_i \alpha_i/(\alpha_i + k) \tag{31}$$

and

$$P(k) = (2k)^{-1} \ln \left[(1 + k)/(1 - k)\right] \tag{32}$$

By taking the Laplace transform of Eq. 29, using the definitions (30), and inserting the expressions (31) and (32), one obtains

$$\begin{aligned} K(k) + G(k) &= (1 + f) \; F(k) \; P(k) + F_1(k)/(1 + f) \\ &= (1 + f) \; F(k)(2k)^{-1} \ln \left[(1 + k)/(1 - k)\right] \frac{I\nu}{1 + f} \sum_{i=1}^{3} \\ &= F_i\alpha_i/(\alpha_i + k) \end{aligned} \tag{33}$$

Clearly the current to the left across the interface at x = 0 as defined by Eq. 26 is just G(0), which, according to the above equation, is

$$I = G(0) = f\ F(0) + I\nu F_S/(1 + f) \tag{34}$$

where

$$F_S = \sum_{i=1}^{3} F_i$$

We are left with the problem of finding F(0) if the current is to be calculated by Eq. 34.

Consider now the homogeneous integral equation (i.e., set $\phi = 0$ in Eq. 29). The homogeneous solutions in the regions $x \geq 0$ and $x < 0$ are designated with a zero subscript. They are respectively, $f_0(x)$ and $g_0(x)$ and their Laplace transforms were shown to be given by the following expressions derived exactly by Frankel and Nelson in (LA-258).

$$F_0(k) = [B/(k^2 + k_0^2)] \exp[(k/n) \int_0^1 d_S T_c/(1 + ks)] \tag{35}$$

$$G_0(k) = [Bf/k_0^2] \exp[(k/n) \int_0^1 d_S T_c/(1 - ks)] \tag{36}$$

where

$$T_c = \tan^{-1}\left(\frac{n/2}{\tanh^{-1} s - 1/Cs}\right) \quad c = 1 + f$$

The constant $k_0$ is given by the relation,

$$k_0/\tan^{-1} k_0 = 1 + f \tag{37}$$

In Appendix II of (LA-258) it is shown that the particular solution of Eq. 29, where $\phi(x)/(1 + f)$ is a single exponential term, $e^{-\alpha x}$, has the Laplace transform,

$$F(k) = - F_0(k)/[(k + \alpha)\ G_0(-\alpha)] \tag{38}$$

For our problem the inhomogeneous term is a sum of exponentials, $\phi(x)$ being given by Eq. 22. The general solution thus has the Laplace transform

958 012

$$F(k) = A\ F_0(k) - \frac{I\,\nu}{1+f} \sum_{i=1}^{3} \frac{F_i\ \alpha_i\ F_0(k)}{(k+\alpha_i)\ G_0(-\alpha_i)} \tag{39}$$

Here A is a constant determining the relative amount of homogeneous and particular solution. A is to be chosen in such a manner as to satisfy the condition, $f(a + x_0) = 0$.

By taking the inverse Laplace transform of Eq. 39 one obtains the general solution

$$f(x) = A\ f_0(x) - \frac{I\,\nu}{1+F} \sum_{i=1}^{3} \frac{F_i\,\alpha_i}{G_0(-\alpha_i)}\ f_i(x) \tag{40}$$

Here we shall make use of the asympototic expressions for $f_0(x)$, $f_i(x)$ and their corresponding Laplace transforms.

$$F_0(k) = \int_0^\infty dx\ e^{-kx}\ f_0(x) \approx \int_0^\infty dx\ e^{-kx}\ \sin K_0\ (x + x_0) \tag{41}$$

$$F_0(k)/(k + \alpha_i) = \int_0^\infty dx\ e^{-kx}\ f_i(x) \tag{42}$$

Multiplying Eq. 42 by $k + \alpha_i$ and integrating the first term of the right member by parts, one obtains:

$$\begin{aligned} F_0(k) &= \int_0^\infty dx\ ke^{-kx}\ f_i(x) + \int_0^\infty dx\ \alpha_i\ e^{-kx}\ f_i(x) \\ &= f_i(0) + \int_0^\infty dx\ e^{-kx}\ (d/dx + \alpha_i)\ f_i(x) \end{aligned} \tag{43}$$

By equating Eqs. 43 and 41 one obtains the following differential equation for $f_i(x)$:

$$(d/dx + \alpha_i)\ f_i(x) = \sin k_0(x + x_0), \quad f_i(o) = 0 \tag{44}$$

The solution of Eq. 44 is simply

$$\begin{aligned} f_i(x) = (k_0^2 + \alpha_i^2)^{-1} \{&\alpha_i \sin k_0(x + x_0) - k_0 \cos k_0(x + x_0) \\ &+ (k_0 \cos k_0 x_0 - \alpha_i \sin k_0 x_0)\ e^{-\alpha_i x}\} \end{aligned} \tag{45}$$

Inserting Eq. 45 and $f_0(x) = \sin k_0(x + x_0)$ into Eq. 40 one evaluates the constant A by setting $f\ (a + x_0) = 0$.

756 013

$$A = [I\nu/(1+f)] \sum_{i=1}^{3} [F_i/G_0(-\alpha_i)]\,[(\alpha_i^2 - k_0\alpha_i \cot k_0(a+2x_0))](k_0^2+\alpha_i^2)^{-1}$$
$$+ \cos(k_0x_0+\delta_i)\sin\delta_i \operatorname{cosec} k_0(a+2x_0)\, e^{-\alpha_i(a+x_0)} \tag{46}$$

where $\delta_i = \tan^{-1}(\alpha_i/k_0)$.

One may neglect the terms containing $e^{-\alpha(a+x_0)}$ as factors because, for all cases treated, $\alpha_i$ has values sufficiently large to make these terms negligibly small compared to

$$(\alpha_i^2 - k_0\alpha_i \cot k_0(a+2x_0)(k_0^2+\alpha_i^2)^{-1}$$

Inserting A into Eq. 39, neglecting the last terms of Eq. 46, and setting $k = 0$, one obtains:

$$F(o) = -[(I\nu)/(1+f)] \sum_{i=1}^{3} F_i[F_0(o)/G_0(-\alpha_i)][k_0^2 + k_0\alpha_i \cot k_0(a+2x_0)]$$
$$= (k_0^2+\alpha_i^2)^{-1} \tag{47}$$

From Eqs. 35 and 36 one obtains the ratio:

$$F_0(o)/G_0(\alpha_i) = e^{\alpha_i A_i}/f \tag{48}$$

Here $A_i = (1/n)\int_0^1 ds\, T_c/(1+\alpha_i S)$ is the expression tabulated in Table II of (LA-258).* Upon inserting F(o), given by Eqs. 47 and 48 into Eq. 34, one obtains the current density of fast neutrons into the water tamper,

$$I = [(I\nu)/(1+f)]\{F_s - \sum_{i=1}^{3}[F_i e^{\alpha_i A_i}/(k_0^2+\alpha_i^2)][K_0^2 + k_0\alpha \cot k_0(a+2x_0)]\} \tag{49}$$

The relation (49) enables one to compute the critical slab thickness "d" in cm. as a function of $\nu$, $\sigma$, f, $\alpha_1$, $\alpha_2$, $\alpha_3$, $F_1$, $F_2$, and $F_3$.

$$\cot k_0(\alpha d + 2x_0) = \frac{F_2 - (1+f)/\nu - k_0^2 \sum_{i=1}^{3} F_i e^{\alpha_i A_i}/(k_0^2+\alpha_i^2)}{k_0 \sum_{i=1}^{3} F_i \alpha_i e^{\alpha_i A_i}/(k_0^2+\alpha_i^2)} \tag{50}$$

where $k_0/\tan^{-1} k_0 = 1 + f > 1$, and $x_0 = 0.71/(1+f)$.

If the active material is just critical in an infinite amount, no water being present, (f = 0) a finite critical thickness is possible for

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* The notation used in (LA-258) identifies 1 + f as c, and $\alpha_i$ as k. S. Frankel has had Table II extended to the much larger values of $k = \alpha_i$ required for this problem. We are greatly indebted to Frankel for suggesting the method outlined in this section.

a half water-tamped slab. The number of fission neutrons, $\nu F_s$, produced per cm$^2$ per sec be reflected slow neutrons must be greater than the unit flux of neutrons entering the water. In this case the critical thickness as given by Eq. 50 reduces to

$$d = \sigma^{-1}\left\{\sum_{i=1}^{3} [F_i e^{\alpha_i A_i}/(F_s - 1/\nu)] - 2x_0\right\} \tag{51}$$

for $f = k_0 = 0$

In case the water is a sufficiently good tamper to make $\nu F_s > 1$ it is possible, even though f is less than zero in the active material, for the tamped slab to become critical with a finite thickness. As usual we designate a negative f as $-g$. Equation 50 is then altered to read:

$$\coth k_0(\sigma d + 2x_0) = \frac{F_s - (1-g)/\nu + k_0^2 \sum_{i=1}^{3} F_i e^{\alpha_i A_i}/(\alpha_i^2 - k_0^2)}{k_0 \sum_{i=1}^{3} F_i \alpha_i e^{\alpha_i A_i}/(\alpha_i^2 - k_0^2)} \tag{52}$$

where $k_0/\tanh^{-1}$, $k_0 = 1 - g < 1$, and $x_0 = 0.71/(1-g)$

Finally, for this case ($\nu F_s > 1$) a semi-infinite slab is just critical when g reaches the value,

$$g = 1 - \nu F_s + \nu k_0 \sum_{i=1}^{3} F_i e^{\alpha_i A_i}/(\alpha_i + k_0) \qquad \text{limit } d \to \infty \tag{53}$$

## IV. APPROXIMATE TREATMENT OF A WATER-TAMPED SPHERE

One may estimate the critical radius of a water-tamped sphere of active material by altering the theory developed in sections II and III slightly. We shall assume that the source of fission neutrons produced by the three groups of slow neutrons reflected from the water is distributed as

$$\phi(x) = I\nu \sum_{i=1}^{3} B_i \alpha_i e^{-\alpha_i x} \tag{54}$$

where x is the distance in units of $\sigma^{-1}$ along a radius of the sphere from the sphere surface. Here again when x has reached the value "a" (sphere radius) we obtain a negligible value for $\phi$. In order to

956-015

put the inhomogeneous integral equation in the form of the equivalent slab problem $\phi(x)$ must be interpreted as radial coordinate, $(a - x)$, times source per unit volume per unit time. Thus if the total number of fissions per unit current into the water is defined as $F'$ for each group, one may determine the constants $B_i$ from

$$F_i' = 4\pi B_i \alpha_i \int_0^a dx\,(a - x)\, e^{-\alpha_i x} \tag{55}$$

Here we allow the upper limit to approach $\infty$ and obtain approximately:

$$B_i = F_i'/[(4\pi a)(1 - 1/\alpha_i a)] \tag{56}$$

The values of $F_i'$ are obtained approximately from the $F_i$ values for the slab (Eqs. 9, 14, and 21) by taking into account the sphericity of the water-active material boundary according to a suggestion by E. Teller.

$$F_i' = F_i/(1 + l\,\sigma/a) \tag{57}$$

Thus Eq. 56 becomes

$$B_i = F_i/[(4\pi a)(1 - 1/\alpha_i a)(1 + l\,\sigma/a)] \tag{58}$$

One may solve for the current of fast neutrons leaving the sphere in a manner exactly analogous to the treatment given in section III. The integral equation to be solved is identical to Eq. 29 except that not $f(x)$ and $g(x)$ are respectively $n(x) \cdot (a - x)$, $x \geq 0$ and $n(x) \cdot (a - x)$, $x < 0$. Thus $G(o)$ is no longer the current but instead of Eq. 26 one has

$$I = 4\pi \int_{-\infty}^{0} dx(a - x)^2\, n(x) = 4\pi \int_{-\infty}^{0} dx(a - x)\, g(x) \approx 4\pi a(1 + 1/a)\, G(o) \tag{59}$$

The equations analogous to (34) and (47) are obviously

$$G(o) \doteq \frac{I}{4\pi a(1 + 1/a)} = f\, F(o) + \frac{I\nu}{1 + f} \sum_{i=1}^{3} B_i \tag{60}$$

and

$$f\, F(o) = \frac{I\nu}{1 + f} \sum_{i=1}^{3} B\, e^{\alpha_i A_i} [k_0^2 + k_0 A_i \cot(a + x_0)]\, (k_0^2 + \alpha_i^2)^{-1} \tag{61}$$

956-016

Here we have made use of the fact that f (a) = 0 instead of f $(a + x_0) = 0$ as was the case for the slab. Inserting the quantities $B_i$ given by Eq. 58, one obtains from Eqs. 60 and 61 the following relation from which the critical radius may be obtained:

$$\cot k_0(a + x_0) = \frac{\sum_{i=1}^{3} [F_i/(1 - 1/\alpha_i a)][1 - k_0^2\, e^{\alpha_i A_i}/(k_0^2 + \alpha_i^2)] - \dfrac{(1 + f)(a + l\sigma)}{\nu\,(a + 1)}}{k_0 \sum_{i=1}^{3} [F_i/(1 - 1/\alpha_i a)][\alpha_i e^{\alpha_i A_i}/(k_0^2 + \alpha_i^2)} \qquad (62)$$

where a = R σ, R = critical radius in cm, and $k_0/\tan^{-1} k_0 = 1 + f > 0$.

The above critical equation cannot be expected to yield a satisfactory critical radius for active material of high enrichment (i.e., small a) because then $e^{-\alpha_i a}$ is not negligible and inclusion of the factors $(1 - 1/\alpha_i a)$ is inconsistent with the neglection of the exponential terms.

## V. APPLICATION TO WATER-TAMPED SLAB AND SPHERE OF ENRICHED SOLID $UF_6$

The three values of $F_i$ were calculated according to Eqs. 9, 14, and 21 for six enrichments of a $UF_6$ (density = 4.68 gm/cm$^3$) half water tamped slab; (25)/(25 + 28) = 6, 8, 15, 25, 50, and 100%. In each case the penetration length, $l$, corresponds to a degradation of neutron energy from $E_I$ to $E_I/39 = E_0$, where $E_I$ is a guess at the average energy of the neutrons that are incident on the water tamper and $E_0$ is their energy in the water at age $\tau_0 = l^2$ when they are considered as an isotropic source for further age diffusion. Below is tabulated the percentage enrichment, (25)/(25 + 28), incident energy $E_I$, penetration length $l$, the three fission rates per unit incident current, and their sum.

TABLE I

| (25)/(25 + 28) | $E_I$ Mev | $l$ cm | $F_1$ | $F_2$ | $F_3$ | $F_S$ |
|---|---|---|---|---|---|---|
| 6% | 0.92 | 3.0 | 0.024 | 0.071 | 0.304 | 0.399 |
| 8% | 1.05 | 3.25 | 0.027 | 0.081 | 0.286 | 0.394 |
| 15% | 1.27 | 3.60 | 0.038 | 0.107 | 0.265 | 0.410 |
| 25% | 1.38 | 3.76 | 0.048 | 0.126 | 0.253 | 0.427 |
| 50% | 1.47 | 3.88 | 0.064 | 0.148 | 0.242 | 0.454 |
| 100% | 1.51 | 3.94 | 0.080 | 0.167 | 0.233 | 0.480 |

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The decrease of the thermal-neutron fission rate, $F_3$, with increasing enrichment is produced by the increase of the penetration length. Spreading thermal neutrons further into the tamper because of their higher incident energy allows the absorption by water to compete successfully against an increasing fission cross section in the active material.

In all cases the first and second groups correspond respectively to the range of ages $\tau_1' = 2\ \text{cm}^2$ and $\tau_2'' = 3.3\ \text{cm}^2$. It was convenient to split the two groups in this manner because, at an energy corresponding to the age $\tau_1 = l^2 + \tau_1'$ from incidence, a sharp increase in the $UF_6$ absorption cross section with age set in. The correlation between neutron energy and age in water was made by referring to the report by Nordheim, Nordheim, and Soodak. (CP-1251).

The three diffusion lengths $L_1$, $L_2$, and $L_3$ as a function of percentage enrichment, (25)/(25 + 28), in solid $UF_6$ are shown in Fig. 3.

The calculations of the critical slab thickness by Eq. 50 requires knowledge of $\nu$, $\sigma$, and f. Throughout we used $\nu = 2.47$ and $\sigma$ and f are shown in Fig. 4 as functions of enrichment percentage. They depend on $E_f$, the average fast neutron energy in the $UF_6$, which is shown in the same figure.

The rapid decline of $\sigma$ between 6 and 15% enrichment shows the effect of the low energy resonance in the Fluorine cross section.

In estimating $E_f$ one must bear in mind the fact that it is certainly less than the average energy of neutrons incident on the water tamper, $E_I$, because a sizable fraction of the total current I going into the water consists of fission neutrons that come directly from the slow neutron fission source, $\phi(x)$, without suffering any collisions in the $UF_6$. This part of the current of neutrons which suffers no moderation in the $UF_6$ is

$$I_0 = \int_0^a dx\ \phi(x)\ E_2(x)/2 \tag{63}$$

where

$$E_2(x) = \int_1^\infty dy\ e^{-xy}/y^2$$

By extending the upper limit of integration to $\infty$ ($\phi(x)\ E_2(x)$ is negligible for $x > a$), one obtains, upon inserting Eq. 22 for $\phi(x)$ into Eq. 63

$$I_0 = I(\nu/2) \sum_{i=1}^{3} F_i[1 - \alpha_i^{-1}\ ln\ (1 + \alpha_i)] \tag{64}$$

According to Eq. 64 the fraction of the current into the water, $I_0/I$, which is made up of unmoderated fission neutrons, increases from 30% to 41% in going from an uranium enrichment of 15% to 100%. We estimated the average fast neutron energy in the $UF_6$, $E_f$, by assuming $E_I = 1.3\ E_f$.

The critical thickness in centimeters of a slab of $UF_6$ tamped by water on one side as given by Eq. 50 is the curve (b) plotted versus enrichment percentage in Fig. 5.

Removal of the water is equivalent to allowing the source strengths $F_i$ to approach zero in Eq. 50. Thus one obtains for the untamped slab a critical thickness in centimeters, d, given by:

$$\cot k_0(\sigma d + 2x_0) \rightarrow -\infty \quad \text{or} \quad d = (\pi/k_0 - 2x_0)/\sigma \tag{65}$$

Curve (a) in Fig. 5 shows the above untamped $UF_6$ critical slab thickness for comparison.

The lowest curve, (c) in Fig. 5 is the critical radius in centimeters, R, calculated according to Eq. 62 for a water-tamped sphere of $UF_6$.

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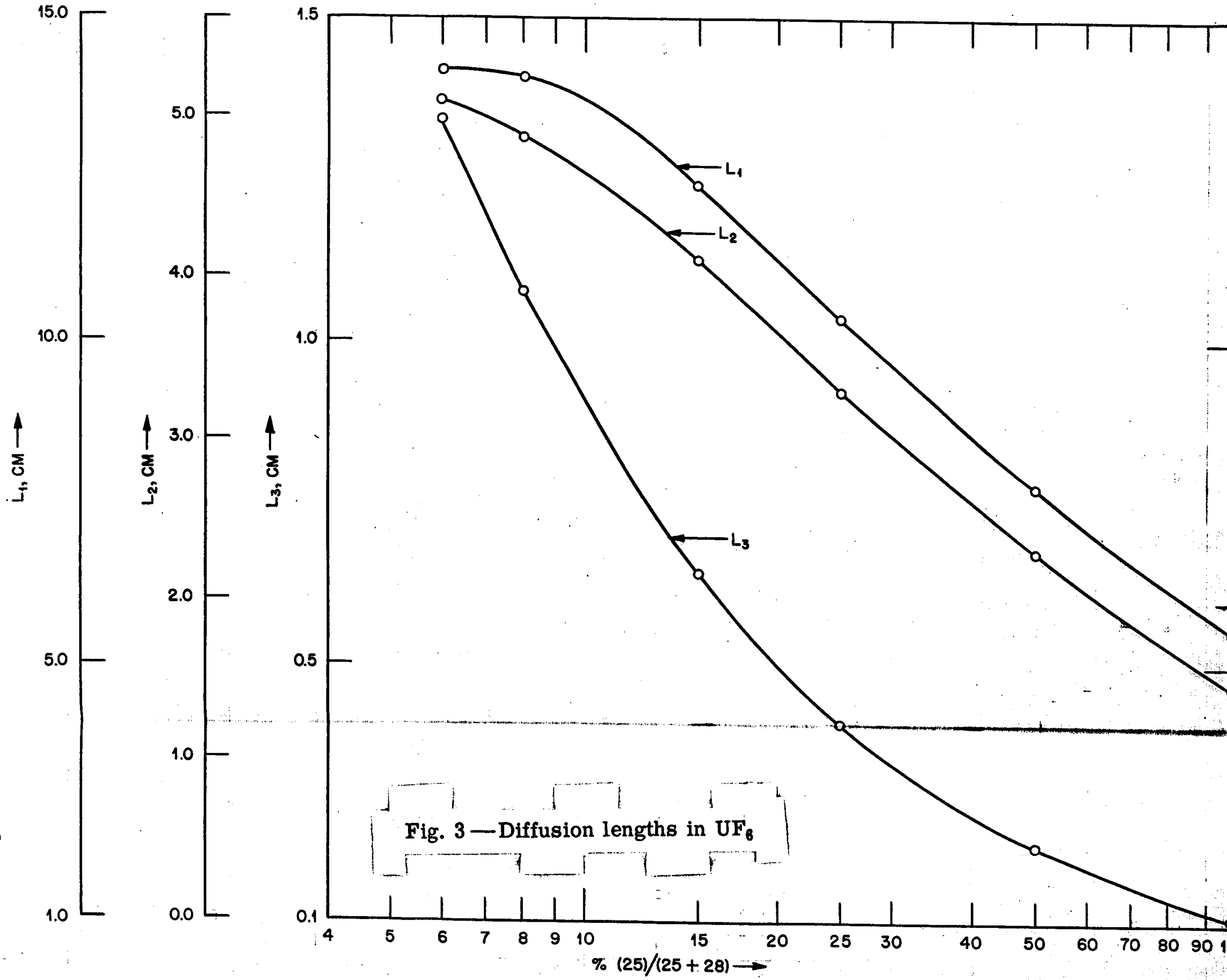


Fig. 3 — Diffusion lengths in $UF_6$

LA-609



956-020

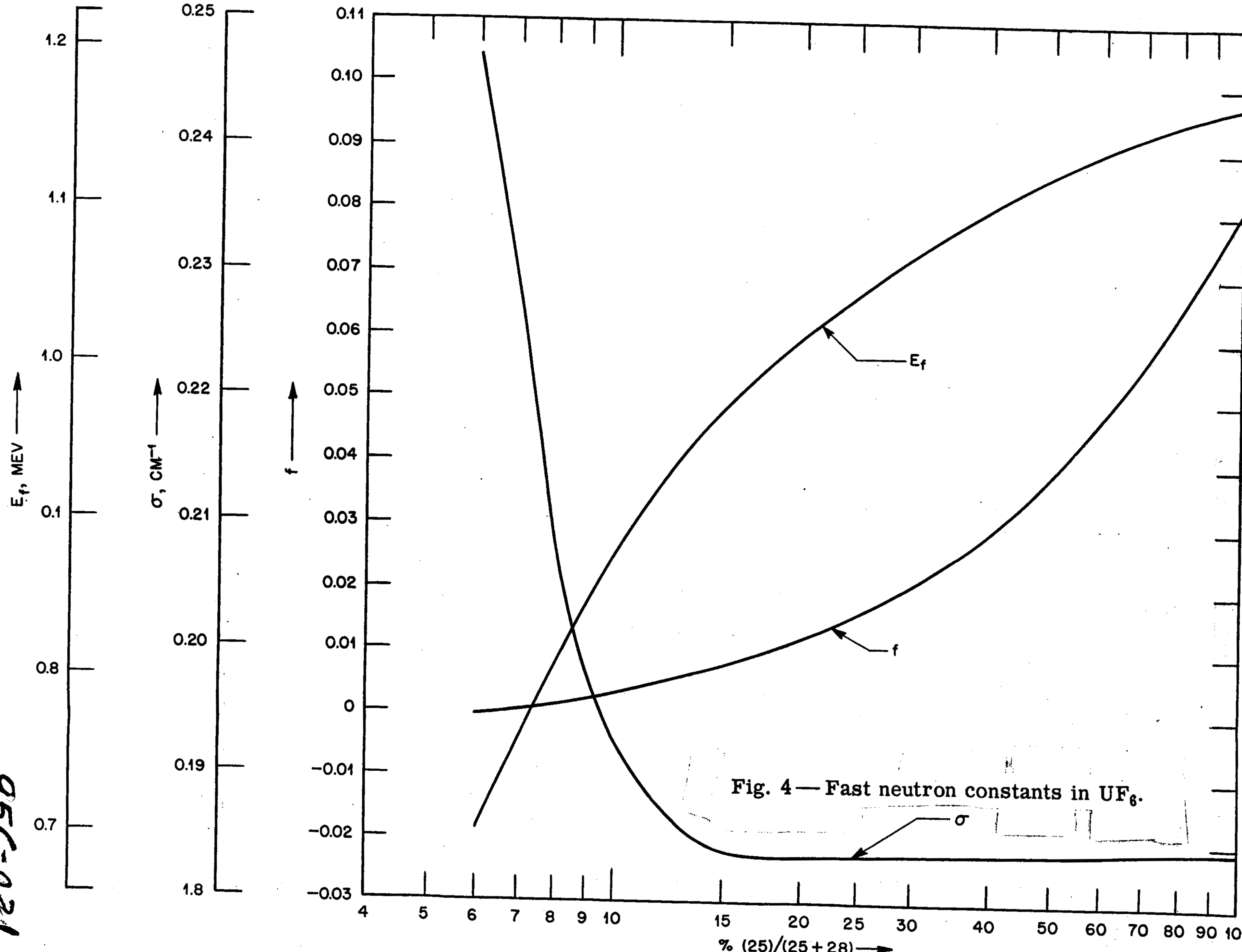


Fig. 4 — Fast neutron constants in $UF_6$.

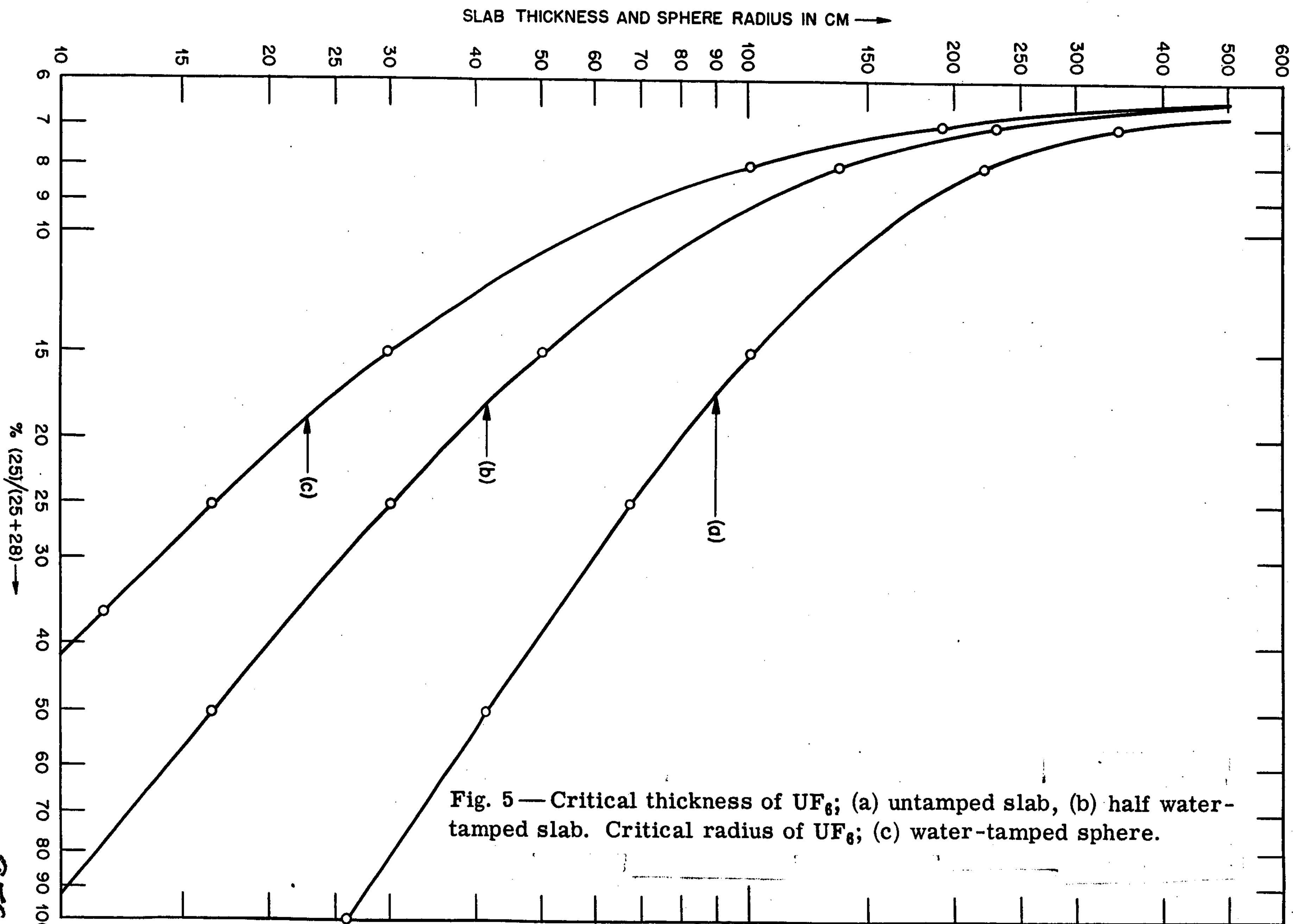


Fig. 5 — Critical thickness of $UF_6$; (a) untamped slab, (b) half water-tamped slab. Critical radius of $UF_6$; (c) water-tamped sphere.

956-022