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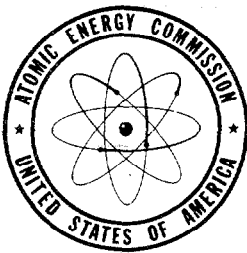UNITED STATES ATOMIC ENERGY COMMISSION

TAYLOR INSTABILITY OF INCOMPRESSIBLE LIQUIDS

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TAYLOR INSTABILITY OF INCOMPRESSIBLE LIQUIDS

PART 1.   TAYLOR INSTABILITY OF AN INCOMPRESSIBLE LIQUID

by Enrico Fermi

This is an attempt to discuss in a very simplified form the problem of the growth of an initial ripple on the surface of an incompressible liquid in presence of an acceleration, g, directed from the outside into the liquid.

The model is that of a heavy liquid occupying at t = 0 the half space above the plane z = 0. It is well known that this is a state of unstable equilibrium. Any tiny ripple on the surface at the initial time grows in amplitude, first exponentially and later, when its amplitude has become comparable to the wave length, by a more complicated law.

The case will be considered that there is initially a small amplitude sinusoidal ripple of wave length λ. In a first phase this amplitude will increase exponentially like

$$\exp\left(\sqrt{\frac{2\pi g}{\lambda}}\, t\right) \tag{1}$$

This exponential law, however, will break down when the amplitude has become comparable to $\lambdabar = \lambda/2\pi$. We propose to discuss what happens in the subsequent phase.

This will be done by grossly schematizing the shape of the wave as indicated in Fig. 1.

Instead of a wave profile like the curve, a profile like ABCDEFGHIJ will be assumed.

It is clear from the symmetry of the problem that the points at the maximum and the minimum of the wave move in vertical directions. In Fig. 2 a half wave, from a maximum to the successive minimum is represented with the notations adopted. OO' is the initial level of the liquid. On account of the incompressibility the amount of liquid below the plane OO', namely CO'DE
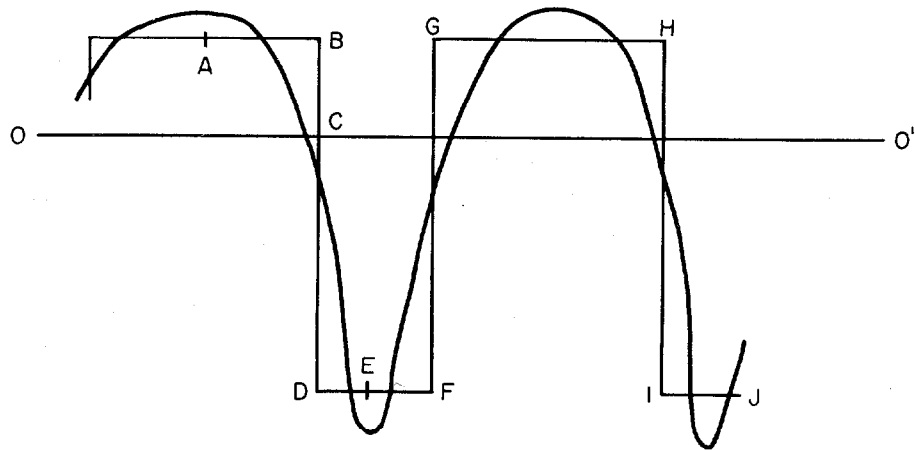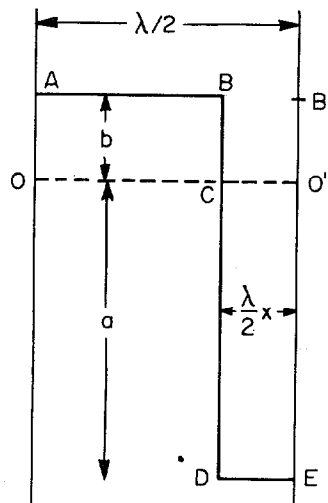
Fig. 1

Fig. 2

must be equal to the amount of liquid ABCO missing from above. This condition leads immediately to the relationship

$$b = \frac{ax}{1-x} \tag{2}$$

Our schematic wave profile is then characterized by the two parameters a, x. The problem is to determine how they vary with time.

In principle the problem so simplified could be solved by expressing the kinetic energy T and the potential energy U of the liquid contained between the two boundaries OA, O'E as functions of a, x, $\dot{a}$, $\dot{x}$.

$$T = T(a, x, \dot{a}, \dot{x})$$
$$U = U(a,x)$$

One can then write the Lagrange equations

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{x}} - \frac{\partial T}{\partial x} = -\frac{\partial U}{\partial x}; \quad \frac{d}{dt} \frac{\partial T}{\partial \dot{a}} - \frac{\partial T}{\partial a} = -\frac{\partial U}{\partial a} \tag{3}$$

which describe the law of variation of the two wave parameters a, x.

The potential energy U can be written down immediately. It is due to having moved the liquid originally contained in ABOC (weight per unit length perpendicular to the plane of the drawing = $\rho g \frac{\lambda}{2} b$ (1-x), height of the center of gravity = b/2) to the lower position CDEO' with the center of gravity at a height -a/2.

In what follows the following units will be used: Unit of length, $\frac{\lambda}{2}$; unit of acceleration, g; unit of density, ρ.

One finds, then, the potential energy

$$U = -\frac{1}{2} \frac{a^2 x}{1-x} \tag{4}$$

The calculation of the kinetic energy is more difficult. In principle it could be carried out for a prescribed motion of the profile of the liquid by solving a Dirichlet problem. Instead of doing this, a much cruder method was followed in keeping with the crude approximation chosen for the profile of the wave.

When the amplitude of the wave is very large it is evident that the kinetic energy is due primarily to the vertical component of the liquid velocity inside the domain BDEB'. The corresponding kinetic energy can be computed easily on the assumption that the vertical component of the velocity is constant on each horizontal section of BDEB'. One finds that this part of the kinetic energy is given by

$$T_1 = \frac{a^3 \dot{x}^2}{6x(1-x)} \tag{5}$$

For small and moderate amplitudes of the wave, additional terms in the kinetic energy become important. One of them is the kinetic energy due to the horizontal component of the motion of the liquid BDEB'. This term of the kinetic energy is given approximately by

$$T_2 = \frac{a \dot{x} \dot{x}^2}{6(1-x)} \tag{6}$$

Finally, the kinetic energy due to the motion of the liquid above the line AB' should be estimated. An approximate expression for this term of the kinetic energy yields

$$T_3 = \frac{\pi}{4} \frac{a^2 \dot{x}^2}{(1-x)^2} + \frac{\pi}{2} \frac{a \dot{x} \dot{a} \dot{x}}{1-x} + \frac{\pi}{4} x^2 \dot{a}^2 \tag{7}$$

The kinetic energy is the sum of the three terms (5), (6), (7)

$$T = T_1 + T_2 + T_3 \tag{8}$$

As pointed out, the leading term at high amplitude is the first. For low amplitude all the three terms need to be considered.

Using the expressions (4) and (8) for potential and kinetic energy, one can write the Lagrange equations (3). That enables one to express the second time derivatives x and a in terms of x, a, $\dot{x}$, $\dot{a}$. One finds

$$\ddot{x} = \frac{ED-FB}{AD-BC}, \quad \ddot{a} = \frac{AF-EC}{AD-BC} \tag{9}$$

where

where

$$A = \frac{a^2}{3y} + \frac{y}{3} + \frac{\pi}{2} a, \quad D = a + \frac{\pi}{2} y$$

$$(10)$$

$$B = \frac{a}{2} + \frac{\pi}{2} y, \quad C = \frac{a^2}{2y} + \frac{\pi}{2} a$$

$$E = \frac{a}{2} - \frac{\dot{a}^2}{2} - \frac{1}{6} \frac{(4x-1)a^2\dot{x}^2}{y^2} - \frac{a\dot{a}\dot{x}}{y} - \frac{\dot{x}^2}{6} - \frac{y\dot{x}\dot{a}}{3a} - \frac{\pi}{2} \frac{xy\dot{a}^2}{a} - \frac{\pi}{2} \frac{a\dot{x}\dot{x}^2}{y} - \pi\dot{x}\dot{a} \quad (11)$$

$$F = a - \frac{\dot{a}^2}{2} + \frac{(1-2x)a^2\dot{x}^2}{2y^2} - \frac{a\dot{x}\dot{a}}{y} + \frac{\dot{x}^2}{6} - \pi\frac{y\dot{x}\dot{a}}{x}$$

and

$$y = x(1-x) \qquad (12)$$

These equations have been integrated numerically by Miriam Caldwell. Initial conditions corresponding to a wave of very low amplitude were chosen as follows: $a = .01$, $\dot{a} = .0177$, $x = .5$, $\dot{x} = 0$. The results of the numerical integration are given in Table I.

t	a	b	x
0	.0100	.0100	.500
.5	.0243	.0228	.484
1.0	.0628	.0468	.427
1.5	.192	.083	.303
2.0	.584	.115	.165
2.5	1.218	.144	.106
3.0	2.195	.170	.072

Table I

The four columns of the table give, respectively: the time in units $\sqrt{\frac{\lambda}{2g}}$; the two amplitudes of the wave, a and b, below and above the original surface of the liquid expressed in units $\lambda/2$; and the quantity x that measures the asymmetry of the wave ($x = .5$ corresponding to a symmetrical wave). $x < .5$ corresponds to a wave in which the half wave below the original liquid sur- face is narrower than the half wave above. From an inspection of the table one will recognize that up to about $t = 1$, the two amplitudes, a and b, have rather close values and they grow approximately exponentially with a period

not far from the one computed from the correct hydrodynamical theory of small amplitude waves

$$T = \sqrt{\frac{\lambda}{2\pi g}} = \frac{1}{\sqrt{\pi}} = .56 \quad \text{(in our units)} \tag{13}$$

Already, at t = 1, an appreciable asymmetry of the wave has developed. This becomes more and more noticeable for later times. At t = 3, for example, b is less than 1/10th of a.

The asymptotic behavior of $\underline{a}$, $\underline{b}$, and $\underline{x}$ for large values of the time is obtained from a discussion of the equations (9). One finds that $\underline{a}$ increases proportionally to the square of the time, b increases proportionally to the square root of the time, and x is inversely proportional to the 3/2 power of the time. More precisely, one finds the following limiting expressions

$$a \rightarrow \frac{4}{7}(t - 1.04)^2 \tag{14}$$

$$b \rightarrow .12(t - 1.04)^{1/2} \tag{15}$$

$$x \rightarrow .21(t - 1.04)^{-3/2} \tag{16}$$

In other words, the lower tip of the wave falls with uniformly accelerated motion and with acceleration equal to 8/7 g. The upper half wave grows much more slowly and its velocity decreases with time.

It is interesting to compare the results of this crude approximation with the experimental results obtained by D. J. Lewis[1], as well as with the results of G. I. Taylor[2] and of Taylor and Davis[3]. The present theory seems to represent correctly one feature of experimental results, namely the fact that the half wave of the heavy liquid into the vacuum becomes rapidly narrower, whereas the half wave pushing into the heavy liquid becomes more and more blunt. On the other hand, the present theory fails to account for the experimental results according to which the front of the wave pushing into the heavy liquid moves with constant velocity. According to the present theory the displacement is expected instead to be proportional to the square root of the time.

1. PRS 202A 81, 1950
2. PRS 201A 192, 1950
3. PRS 200A 375, 1950

PART 2. TAYLOR INSTABILITY AT THE BOUNDARY OF TWO INCOMPRESSIBLE LIQUIDS

by Enrico Fermi and John von Neumann

In a previous memorandum, "Taylor Instability of an Incompressible Liquid," one of us has discussed the Taylor instability at the surface between an incompressible liquid and in a vacuum by using a very simplified model which consists in assuming that at all times the interface may be represented by a surface of a shape
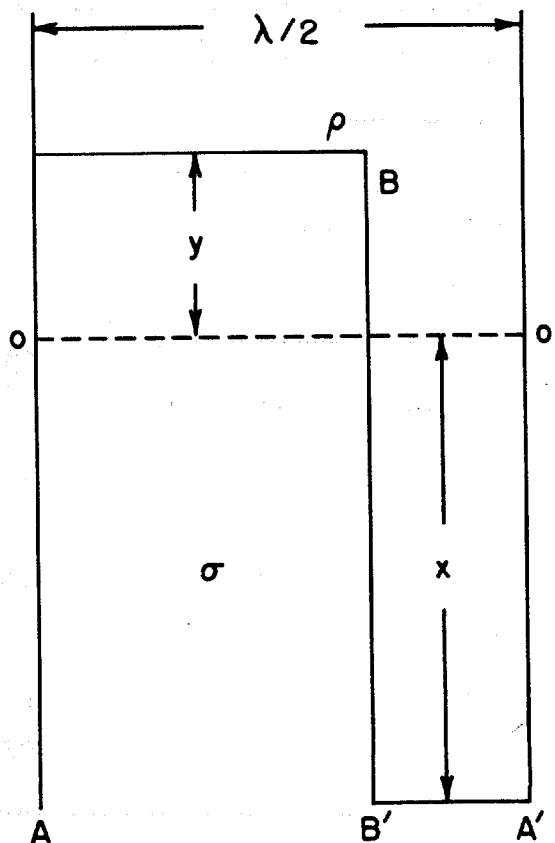

Figure 1

The vertical lines OA and O'A' are traces of planes of symmetry and their distance is half a wave length.

In the case previously discussed this model succeeded in representing correctly at least some features of Taylor instability. In particular, it was found that the heavy fluid penetrates into the vacuum with a spike which becomes thinner as the phenomenon progresses. Actually the front of this spike moves with uniformly accelerated motion with an acceleration that evidently should be equal to the gravitational acceleration g and which, due to the crudeness of the model, turns out to be $8/7$ g. The upward motion of the vacuum bubble into the fluid is represented less correctly. According to the results of Taylor, this bubble should move upward with a constant limiting velocity. The model fails to reproduce correctly this feature and the displacement of the top of the bubble is asymptotically proportional to $\sqrt{t}$.

As a contribution to the discussion of the Taylor instability between two fluids of different densities, ρ and σ $(\rho > \sigma)$, we have tried to explore a similar model for this more complicated case. The notations are slightly different from those used in the previous memorandum and are clearly shown in Figure 1.

In order to write the Lagrangean equations for the system, it is necessary, to obtain an expression for the kinetic energy of the system as a function of the two parameters, x and y, that characterize its position and of their time derivatives, $\dot{x}$ and y. This has been done using essentially the same procedure followed in the previous memorandum. In the present case we were interested particularly in a description of the late phases of the phenomenon and for this reason only one of the three terms of the kinetic energy previously used was

maintained. This term represents the kinetic energy of the vertical motions in the two channels of length $x + y$ through which the heavy fluid of density ρ moves downwards and the light fluid of density σ moves upwards. The expression of this kinetic energy is given by two terms similar to expression (5) of the previous memorandum, rewritten with new notations. The expression of the kinetic energy is

$$T = \frac{\lambda}{6}\left(\rho + \sigma\,\frac{y}{x}\right) y\,\dot{x}^2 + \frac{\lambda}{6}\left(\sigma + \rho\,\frac{x}{y}\right) x\,\dot{y}^2 + \frac{\lambda}{6}\left(\rho x + \sigma y\right)\dot{x}\,\dot{y}. \tag{1}$$

The potential energy U is given by

$$U = -\frac{\lambda(\rho - \sigma)}{2}\,g\,x\,y. \tag{2}$$

The Lagrangean equations corresponding to (1) and (2) can be written immediately. One of them is

$$\left(2\rho y + 2\sigma\frac{y^2}{x}\right)\ddot{x} + \left(\rho x + \sigma y\right)\ddot{y} - \sigma\,\frac{y^2}{x^2}\,\dot{x}^2 + \left(2\rho + 4\,\frac{\sigma y}{x}\right)\dot{y}\,\dot{x} - 2\rho\frac{x}{y}\,\dot{y}^2 -$$

$$- 3\,g(\rho - \sigma)\,y = 0. \tag{3}$$

The other Lagrangean equation is obtained by interchanging in (3) x and y and also ρ and σ in all terms except the last. Instead of using the two Lagrangean equations, we may, however, use equation (3) and the energy equation

$$T + U = 0. \tag{4}$$

The total energy is taken to be zero because we assume that the system starts with zero velocity and with a flat horizontal interface. By a suitable change of the scales of x, y, and t, it is possible to write the equations (3) and (4) in a form in which ρ, σ and g do not appear. This is done by the following transformations

$$\xi = \rho x \qquad \eta = \sigma y \qquad \tau = \sqrt{3g(\rho - \sigma)} \ t. \qquad (5)$$

With these new coordinates, the equations (4) and (3) become

$$\left(1 + \frac{\eta}{\xi}\right) \frac{\dot{\xi}^2}{\xi} + \left(1 + \frac{\xi}{\eta}\right) \frac{\dot{\eta}^2}{\eta} + \left(\frac{1}{\eta} + \frac{1}{\xi}\right) \dot{\xi} \dot{\eta} = 1 \qquad (6)$$

$$\left(2 + 2 \frac{\eta}{\xi}\right) \ddot{\xi} + \left(1 + \frac{\xi}{\eta}\right) \ddot{\eta} - \frac{\eta \dot{\xi}^2}{\xi^2} + \left(\frac{2}{\eta} + \frac{4}{\xi}\right) \dot{\xi} \dot{\eta} - 2 \frac{\xi \dot{\eta}^2}{\eta^2} = 1. \qquad (7)$$

The dots represent in these equations derivatives with respect to τ. By making use of the similarity properties of these equations, they can be reduced to the first order. The appropriate transformations are the following

$$\left. \begin{array}{ll} \xi = e^{2s+2q} & \eta = e^{2s-2q} \\[2mm] 4p = \dfrac{\dot{\xi}}{\sqrt{\xi}} + \dfrac{\dot{\eta}}{\sqrt{\eta}} & r = \dfrac{d\ q}{d\ s} \end{array} \right\} \qquad (8)$$

By substitution one obtains the following equation of the first order

$$r \frac{d\ r}{d\ q} = (3 + r^2)\left(\operatorname{tgh} 2q - \frac{5}{3} r\right), \qquad (9)$$

and also the additional equations

$$\left. \dot{s} = \frac{e^{-s}}{\sqrt{8(3 + r^2)\cosh 2q}} \quad ; \quad \frac{\dot{\xi}}{\sqrt{\xi}} = \frac{e^{q}(1 + r)}{\sqrt{2(3 + r^2)\cosh 2q}} \right.$$

$$\left. \frac{\dot{\eta}}{\sqrt{\eta}} = \frac{e^{-q}(1 - r)}{\sqrt{2(3 + r^2)\cosh 2q}} \right\} \qquad (10)$$

which can be used in passing from the solution of equation (9) to the solution of our physical problem.

In selecting the solution of (9) corresponding to the actual case, one needs the following initial values of q and r. These are obtained as follows. As long as the disturbance has very low amplitude, it is known that the wave is of sinusoidal shape and exponentially increasing amplitude. This phase of the phenomenon is not represented by our treatment which describes only the late phase of the motion. In fact, we may assume that the proper initial conditions for our problem correspond to the time when the exponential solution of the early phase breaks down. At this moment we have approximately x = y and dx/dt = dy/dt. Making use of equation (5) and equation (8), this situation corresponds to

$$q = \tfrac{1}{4} \ln \tfrac{\rho}{\sigma}; \quad r = 0. \tag{11}$$

In Figure 2 the shape of the solution of equation (9) corresponding to these initial conditions is outlined.
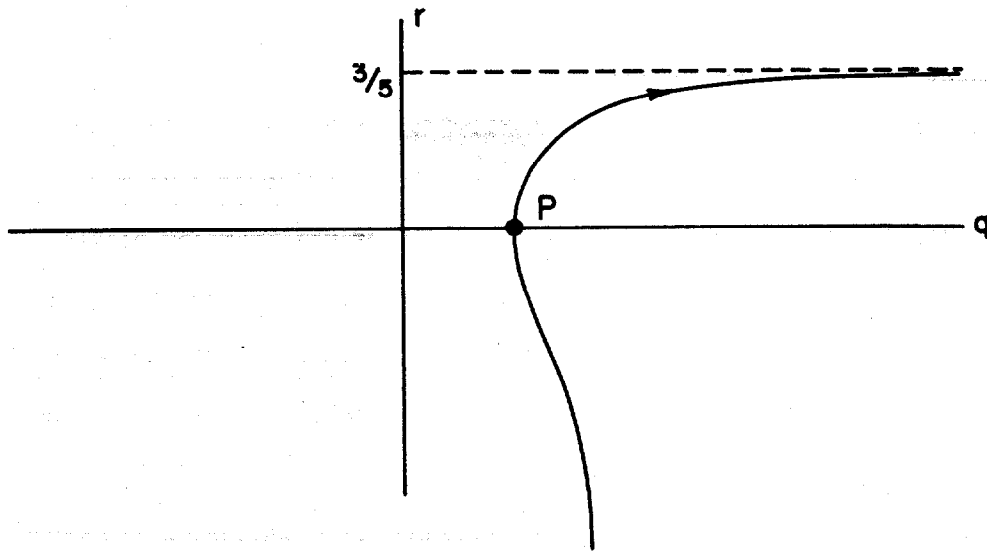

Figure 2

The initial point is P and the arrow indicates the direction of increasing time. As time increases q becomes positive infinite and r converges to the value 3/5. One can now find without trouble the following asymptotic expressions

$$x \longrightarrow \left. \begin{array}{l} \frac{1}{2} \frac{8}{7} g \frac{\rho - \sigma}{\rho} t^2 \\ y \sim t^{1/2} \end{array} \right\} . \tag{12}$$

The first of these equations indicates that the heavy fluid moves into the light fluid with uniformly accelerated motion, as was found to be the case when $\sigma = 0$. The acceleration is $8/7 \, g \, (\rho - \sigma)/\rho$. Presumably the factor $8/7$ in front of the expression should not be there in a more correct theory because the same factor was obtained also when $\sigma = 0$, in which case one would expect a free fall with acceleration g. We may, therefore, conclude tentatively that the heavy liquid should penetrate the light liquid with an acceleration

$$g \, \frac{\rho - \sigma}{\rho} . \tag{13}$$

Again we find that a bubble of the light liquid rises much more slowly into the heavy liquid. The fact that the height of this bubble is proportional to $\sqrt{t}$ and not to t is presumably due to the inaccuracy of the model.

Conclusions

The present discussion makes it appear likely that the features of the Taylor instability at the interface between two liquids of different density are similar to those corresponding to the case of the boundary between a liquid and a vacuum. The main difference is that according to formula (13) the acceleration describing the fall of the heavy into the light liquid is reduced by the factor $(\rho - \sigma)/\rho$. There is, of course, another phenomenon that has been here

entirely neglected and which may in some cases play a very important role.

All along the line BB' in Figure 1 one might expect Helmholtz instability to

develop because the heavy liquid moves downwards on one side of the boundary

and the light liquid moves upwards on the opposite side. This instability

will presumably further contribute to the mixing and may, in particular,

break up the spike of heavy liquid as soon as it becomes sufficiently thin.

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