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AND PLATEAU PRODUCTION FUNCTIONS

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Abstract

The S-shaped three-stage production function, popular in teaching economics and farm management, is often motivated in the context of yield response to fertilization and illustrated using a third degree polynomial specification (cubic). Indeed, the cubic model is used as basis for fertilization recommendations and norm setting in Denmark. Yet, contemporary agricultural economic and agronomic knowledge favors non-linear response with plateau (NLRP) specifications. An analysis, based on fitting the cubic to yield/nitrogen data sets from 40 Danish winter wheat field trials, did not in a single case reveal a three-stage response pattern. It is interesting, however, that in the majority of 40 estimations, the cubic exhibited or strongly suggested an NLRP-like pattern within the domain of measured nitrogen application. Thus, the flexible cubic often works well for the purpose of norm setting when care is taken to avoid extrapolation outside the data domain.

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THE FLEXIBLE CUBIC POLYNOMIAL AND PLATEAU PRODUCTION FUNCTIONS

Introduction

In a recent article in this *Journal*, we (Mortensen and Beattie, 2003) raised the question of whether choice of functional form matters in setting maximum allowable N-application rates in Danish agriculture. Based on a review of both agricultural economic and agronomic literature, it was argued that contemporary knowledge favors non-linear response with plateau (NLRP) models rather than cubic polynomials as presently used in setting Danish N-application norms (Plantedirektoratet, 2003).

We stand by that conclusion. However, we also noted that besides its normally good fit to observed data, the cubic model more often than not yields parameter estimates that closely emulate an NLRP specification. In our estimations for 84 individual data sets from Danish field trials of N on winter wheat, spring barley, and winter barley, we found that in 62 cases, when a cubic polynomial was fitted, the coefficient on the cubic term was positive rather than negative. Only a negative sign would enable revelation of a classical three-stage production function commonly presented in economic textbooks. Specifically, we commented: "Interestingly, for 62 of the 84 data sets the cubic-coefficient was positive rather than negative, giving rise to a function looking more like a variable cost curve than a production function. We think the data 'cry out' for a plateau and the only way a non-plateau third-degree polynomial can attempt that is with a 'reversal' of the usual curvature pattern. The cubic seems to want to place the inflection point so that it is approached horizontally rather than vertically and so that it occurs 'in the vicinity of the true' maximum yield." (p. 346)

This paper amplifies on this curious result by providing further detail including the graphs of the cubic response functions obtained specifically for the winter wheat trails (40 of the 84 data sets). It is interesting we think that in not a single case for winter wheat was a “classic three-stage response function pattern” revealed. In spite of (or perhaps, because of) this, it is interesting how well the cubic polynomial works for the purpose of setting N-fertilizer norms if one is careful to disregard the ill-behaved response outside the domain of the data. When using the cubic model to estimate economically optimal N-application, the Danish Agricultural Advisory Service actually precludes inadequate use of the cubic, and also sets limits for extrapolations outside the data domain.

Economists and Three Production Stages

In teaching production economics and farm management, agricultural economists are fond of three-stage production functions like that depicted in Figure 1. Motivation is often provided (unfortunately) in the context of the fertilization problem – using, say nitrogen (N), as the example variable input applied to an acre of land and other fixed factors on the horizontal axis and with per acre yield of the crop, say wheat (Y), on the vertical axis. To demonstrate the properties of the three-stage model a cubic polynomial is typically used, sans the intercept and linear terms so that the total physical product (TPP), average physical product (APP), and marginal physical productivity (MPP) curves emanate from the origin as in Figure 1.

This paper addresses two questions: 1) How well does Figure 1 stand up in the real world of plant response to fertilization? And, 2) what happens when a cubic polynomial is used in attempting to estimate a fertilizer response function that the researcher believes should look like Figure 1? The answers are, in part, surprising. Not, however, in the first case: As argued in our earlier paper, the three-stage textbook model has serious shortcomings for the applied fertilizer response modeler. The considerable conceptual and empirical work of many response researchers, notably Quirino Paris and co-workers (Ackello-Ogutu, Paris, and Williams,1985; Grimm, Paris, and Williams, 1987; Paris and Knapp, 1989; Paris, 1992) as well as the plant science literature point to the von Liebig hypothesis as being theoretically and empirically superior for fertilizer response modeling.

The surprising answer involves the second question. We find that a cubic model quite often works well as a single-nutrient fertilizer response model – if the analyst is careful – but not because it is a good approximation of the classic three-stage production function. When confronted with fertilizer-yield trial data, the cubic model often reveals a graph not like Figure 1, but rather one resembling a total cost function. That is, a curve with a positive Y-axis intercept and that increases first at a decreasing rate and then becomes nearly horizontal (rather than more vertical) as it passes through its inflection point near the outer limits of the data set. When this happens the estimated cubic response function exhibits diminishing marginal returns followed by something much like a yield plateau (i.e. a Liebig-like model) *within the domain of the data*. Only somewhere beyond the domain of the data does the estimated cubic function "harmlessly" rise, which is why the "if-the-analyst-is-careful" caveat is important.

The remainder of the paper is organized as follows: First, we briefly review the single-variable-factor geometry of the NLRP model as well as the cubic approximation. The complication that plants obtain N in unknown quantity from soil sources in addition to fertilizer sources of N is recognized. Next, based on extensive N-fertilization trials on winter wheat in Denmark, we present findings for the cubic approximation of NLRP. Lastly, the final section summarizes and concludes.

The NLRP in Contrast to the Classic Three-Stage Model

A single-variable-nutrient depiction of an NLRP model is presented in Figure 2. In contrast to Figure 1, there is no Stage I (no area of increasing APP) and TPP does not immediately begin to decline upon reaching its maximum (no Stage III).[1] The function exhibits diminishing MPP up to the point of maximum yield and becomes horizontal (zero) thereafter. If the specification of the growth segment curvature is quadratic, then the MPP will be linear with negative slope for $N < N^{max}$ and zero thereafter; if the growth segment curvature is modeled as Mitscherlich or Cobb-Douglas, for example, then the MPP will decline at a diminishing rate up to N^{max}.

In our version of Figure 2, we include a positive Y-axis intercept and an N-axis intercept to the left of NF = 0. The logic is that the crop acquires *plant available* N from two sources – fertilizer N (NF) represented on the positive segment of the N-axis and soil N (NS) represented by the negative segment of the N-axis beginning at the N-axis intercept. That is, when the unobserved soil N (NS), as well as NF, is presumed to be zero, then yield would also be zero, and when NF = 0, yield would be positive owing to NS. Most often applied fertilizer represents only a partial picture of total plant-available N during the course of the growing season. While we sometimes know NS at planting time from soil tests, we never know the additional NS that becomes available from mineralization of fixated N during the growing season.

In sum, we (like Paris and others) argue that Figure 2, more so than Figure 1, complies with contemporary plant science and fertilizer response knowledge. Further, if the flexible cubic is specified with the intention of enabling a classic three-stage textbook response, interesting things happen. Suppose our empirical cubic specification allows for a Y-axis intercept (as it

[1] Although uninteresting from an economic perspective, the possibility of declining yield could be incorporated into an NLRP model by allowing TPP to decline following the yield plateau when excessive application of N becomes damaging to the crop.

should given that there is always unaccounted for NS), and a linear effect, as well as quadratic and cubic terms, i.e.

(1) $Y = \beta_0 + \beta_1 NF + \beta_2 NF^2 + \beta_3 NF^3$

(error term suppressed). Then what often manifests upon estimation is not Figure 1 but rather Figure 3 (either Panel A or B). That is, an amazing likeness to Figure 2 *within the data domain* of plant available NF.

A cubic model when it looks like Figure 3 (Panel A or B) has one undesirable and one desirable global property as a production function. First, the fact that the implied MPP function is U-shaped rather than an inverted U-shape is undesirable. While the left-hand wing of the U is fine, the notion that TPP might again rise at some point following a local yield maximum (or plateau-like maximum) is clearly untenable. On the other hand, a revealed negative NF-axis intercept and positive Y-axis intercept is entirely plausible – in fact, appealing as discussed above. In the former case, we can take comfort in the fact that the undesirable property is likely only manifest outside the domain of the data. In the latter case, we can smile confidently that we are surely on to something in spite of the fact that we have no N-yield observations in the NS domain. In both cases, we can invoke the usual complaint about the deficient experimental designs, viz. insufficient data points at low and high N levels.

Empirical Findings

Mortensen (2000) reports findings on goodness-of-fit tests for 10 alternative cereal grain/N response functions (winter wheat, winter barley, and spring barley) based on field trial data provided by the Danish Agricultural Advisory Service, Crop Production (1999). The fitted functions included several single-nutrient non-plateau models (including a cubic model) and counterpart plateau models (except for the cubic).

To control as much as possible for exogenous factors, Mortensen selected only 84 of more than 1,200 cereal grain trials for study. Only trials with cereals as the preceding year crop and on soil types JB#6 and JB#7 with high clay content (and high water retaining capacity) were selected. Further, trials were excluded when manure or slurry had been applied to the trial site in previous years. As noted previously, for purposes of this paper, we limit our consideration to the 40 winter wheat trials, all of which had six levels of N-application spanning 0-250 kg of applied N per ha at 50-kg intervals. In one trial there were six replications of the N-application regime, while 27 trials had five replications and 12 trials had four replications, resulting in a total of 1134 nitrogen-yield observations. All trials were treated so that no other growth factor would become limiting in the relevant N-domain, and state-of-the-art production technique was applied in all cases. Trial design and the generally accepted assumption of non-substitution among plant nutrients justify the use of a single-variable-input model specification.

For the 40 winter wheat trials, the cubic model, while not significantly better in terms of goodness-of-fit, did have the lowest average mean squared error of the 10 fitted models. Parameter estimates for the constant, linear, quadratic, and cubic terms and associated significance levels are provided in Table 1 for each of the 40 trials/estimations. In spite of multicollinearity inherent in polynomial models, the constant ($\hat{\beta}_0$) and linear ($\hat{\beta}_1$) coefficients were all positive, as anticipated, and significantly different from zero at $\alpha = 0.01$ or better. All quadratic coefficients ($\hat{\beta}_2$) were negative indicating decreasing marginal product for all positive N-values with all but 13 significant at a 5 percent level or better. As required for the cubic model to emulate a plateau, the cubic coefficients ($\hat{\beta}_3$) were very small, ranging from - 0.0000006

(Trial 1) to + 0.000007 (Trial 40). Not surprising, in most cases the cubic coefficient was not significantly different from zero.

Interestingly, and no doubt surprising to some, not in a single case did the estimated cubic function support the classic three-stage hypothesis – never was there a Stage I (increasing APP) revealed for observed NF or plausible NS levels. In 34 of the 40 cases the estimate of β_3 (the cubic effect) had a positive sign rather than the required negative sign for revelation of a three-stage model. Closer examination of these 34 cases is revealing. With $\beta_3 > 0$, there are two possibilities: 1) If $\beta_2^2 < 3\beta_1\beta_3$, then there are no turning points and the function is monotonically increasing and looks very much like a total cost curve (Panel A in Figure 3). 2) If $\beta_2^2 > 3\beta_1\beta_3$, then there are two stationary values with the local maximum of TPP preceding the local minimum (Panel B). The function slope at the inflection point in all cases is $\beta_1 - \beta_2^2/3\beta_3$. With $\beta_3 > 0$, the closer the inflection-point slope to zero, when $\beta_2^2 = 3\beta_1\beta_3$, the more nicely the model emulates a plateau.

Graphs of all 40 estimated functions, including six with a negative β_3 coefficient, are provided in the six panels of Figure 4. On each graph the inflection point is noted by an open circle and the local maximum by an open square (when they exist within 50 kg of the lowest or highest NF treatment level). The vertical dashed lines at 250 kg represent the highest NF treatment level (the right-hand boundary of the data). NF equal to zero, of course, represents the lowest treatment level.

The panel titles on Figure 4 describe the suggested overall function shape: A. Strongly NLRP (9 cases), B. Likely NLRP (3 cases), C. Plateau Seeking – either NLRP or asymptotic plateau – with revealed inflection point at < 250 kg NF (8 cases), D. Plateau Seeking with revealed inflection point at > 250 kg NF (7 cases), E. Possibly Plateau Seeking (8 cases), or F. Two Stage (5 cases). Again, the most telling observation is that there is no panel titled “Classic Three Stage”, as there were no cases. In the 9 cases claimed “Strongly NLRP” (Panel A) the inflection-point slopes ranged between negative 0.05 and zero. For the three “Likely NLRP” cases, the positive inflection-point slopes are all < 0.03. In Panel C the inflection points are all positively sloped and revealed within the data domain (< 250 kg NF) – suggesting the likelihood of either an asymptotic plateau or NLRP. In the companion Panel D, the inflection point (revealed at an NF level > 250 kg) is positively sloped in three cases; in two cases the inflection-point slope is negative (- 0.02 and - 0.06); and in the two other cases the inflection point occurs harmlessly in the negative (N and Y) quadrant (both cases where the β_3 coefficient was negative). The graphs in the category of “Possibly Plateau Seeking” (Panel E) seem mostly silent on the issue of plateau versus Stage III (including two cases with negative β_3 coefficients). Only in Panel F is a Stage III clearly suggested (three cases with a positive, and two with a negative, β_3 coefficient).

We submit that in 12 of the 40 cases, the picture strongly suggests a plateau – specifically the graphs in Panels A and B (disregarding the increasing TPP phase associated with three Panel B graphs). We also argue that in 15 additional cases (Panels C and D) the graphs are appropriately labeled “Plateau Seeking” (either NLRP or asymptotic plateau), and in 8 cases a plateau (either NLRP or asymptotic) is neither convincingly suggested nor ruled out. In only 5 of 40 cases is a Stage III revealed. Finally, *in not a single case* is a classic three-stage pattern revealed or suggested.

Conclusion

A cubic response function model is used in Denmark to set maximum nitrogen application rates for various crops. It is interesting that the cubic model performs as well as it does for this task in view of contemporary wisdom in agricultural economics and agronomy favoring non-linear with plateau (NLRP) models. This is doubtless due to the considerable flexibility of the cubic model. In the majority of 40 cubic model estimations of winter wheat response to N trial data reported in this paper, an NLRP-like pattern was exhibited or strongly suggested *within the domain of measured fertilizer N*.

Recent literature clearly favors NLRP models on both theoretical and empirical grounds for fertilizer response modeling. It would seem wise to adopt such superior models for policy implementation (as well as other) purposes. That notwithstanding, it is interesting that the cubic model so often performs well for purposes of N-norm setting when analysts are careful, as it seems those responsible for setting N-norms in Denmark are.

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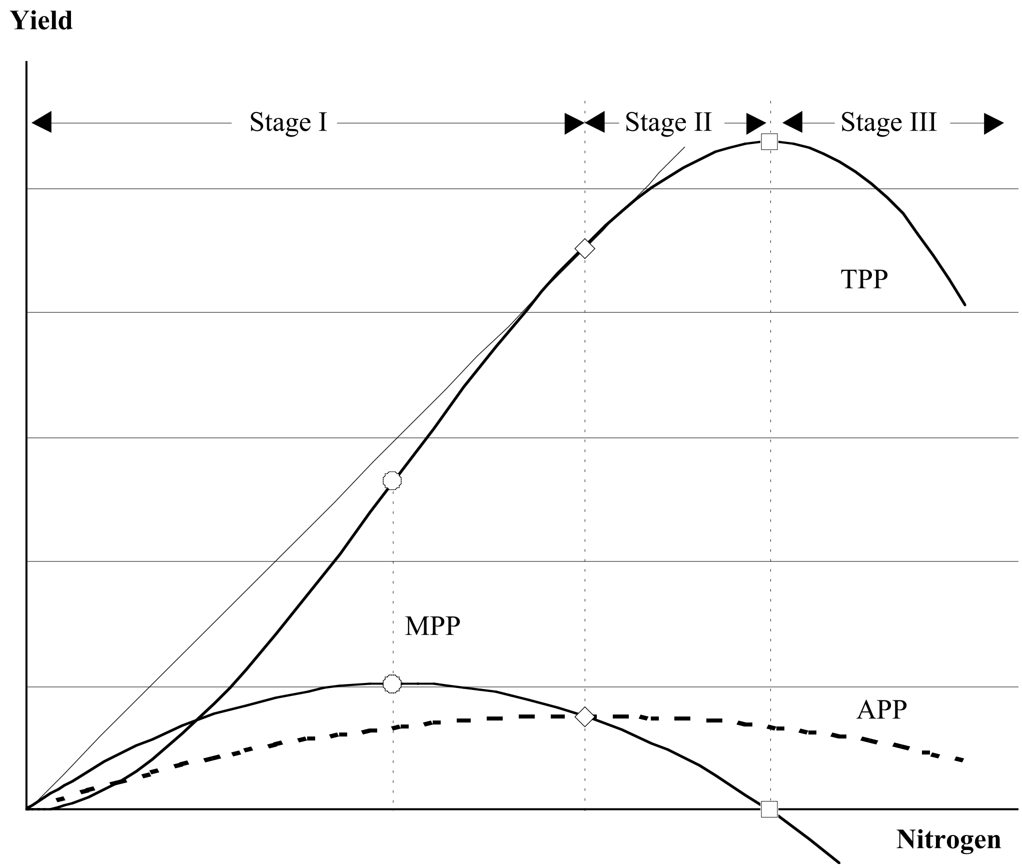


Figure 1. Three-Stage Production Function, Cubic Specification

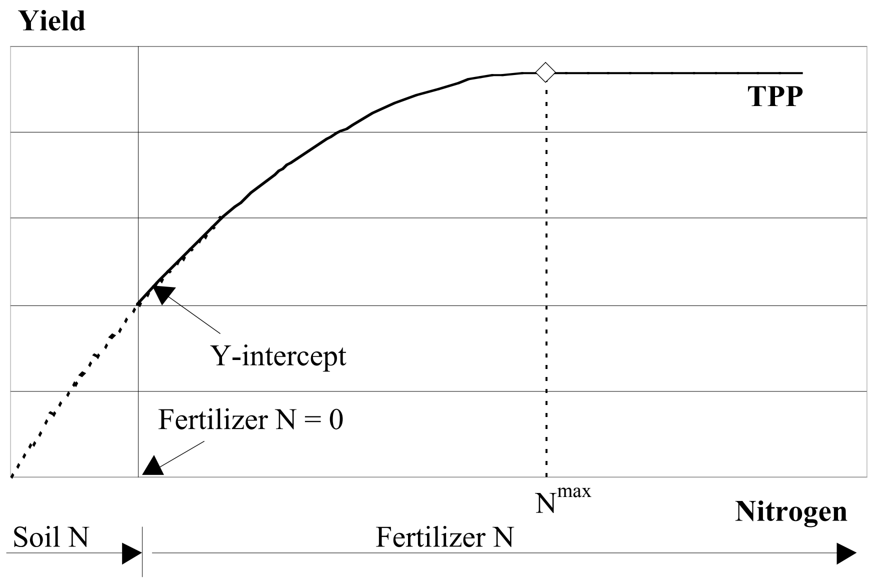
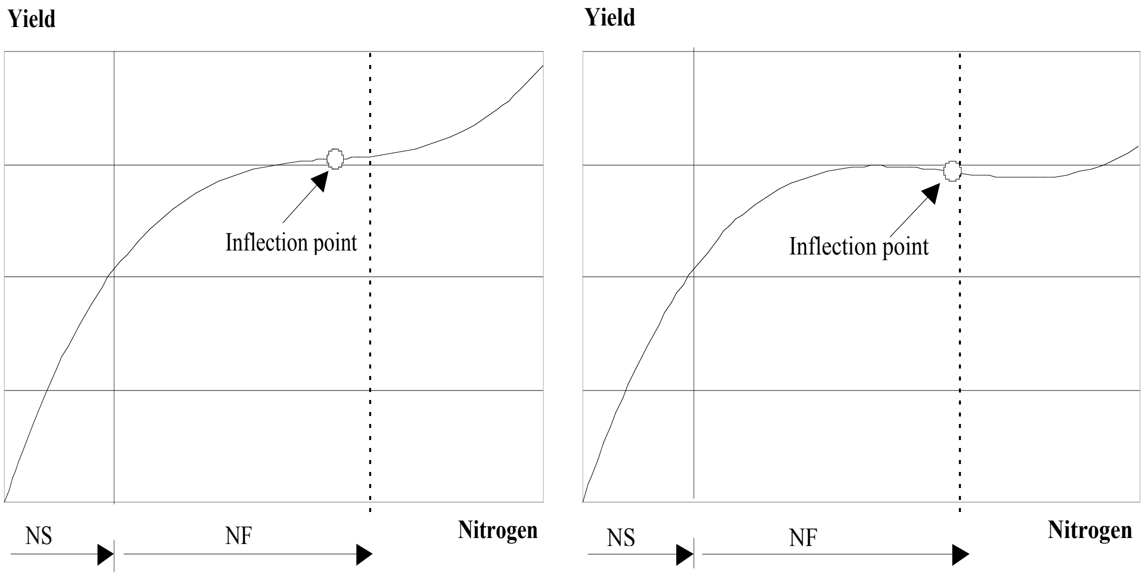


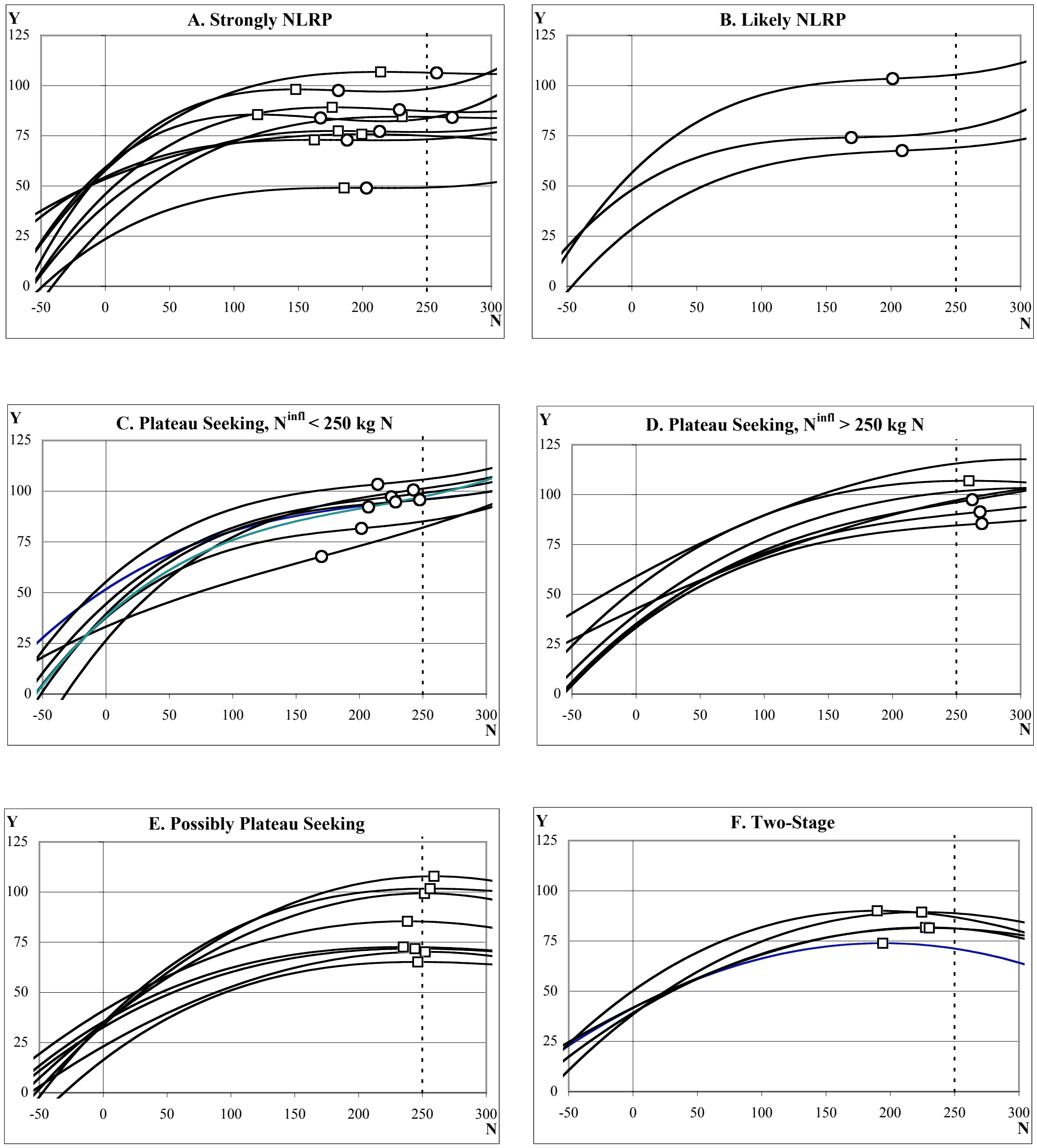
Figure 2. NLRP Representation of Cereals Response to Nitrogen



Panel A
Positive Slope at Inflection Point

Panel B
Negative Slope at Inflection Point

Figure 3. Cubic Approximation to NLRP



O Inflection point   □ Yield maximum

Figure 4. Cubic Fitted to 40 Winter Wheat Trial Data Sets by Behavioral Pattern

Table 1. Parameter Estimates for Cubic Model
40 Danish Winter Wheat Trials

Trial	$\hat{\beta}_0$		$\hat{\beta}_1$		$\hat{\beta}_2$		$\hat{\beta}_3$	
1	41.792053	***	0.3193865	***	-0.0004888	n.s.	-6.257E-07	n.s.
2	58.945581	***	0.3498481	***	-0.0003895	n.s.	-4.11E-07	n.s.
3	34.548883	***	0.4994118	***	-0.0008977	**	-2.515E-07	n.s.
4	42.742545	***	0.2982354	***	-0.0002813	n.s.	-1.607E-07	n.s.
5	23.060321	***	0.3655865	***	-0.0006745	n.s.	-1.33E-07	n.s.
6	41.667563	***	0.3311182	***	-0.0008444	n.s.	-2.922E-08	n.s.
7	33.699816	***	0.5778178	***	-0.0011449	n.s.	7.629E-08	n.s.
8	40.868681	***	0.3852423	***	-0.0008777	n.s.	1.931E-07	n.s.
9	39.18603	***	0.3871836	***	-0.0009693	n.s.	3.722E-07	n.s.
10	38.474108	***	0.4878051	***	-0.0013144	*	6.755E-07	n.s.
11	32.610154	***	0.3764163	***	-0.0011088	n.s.	9.227E-07	n.s.
12	50.228931	***	0.4530407	***	-0.0014572	n.s.	9.266E-07	n.s.
13	53.438459	***	0.2649507	**	-0.000971	n.s.	1.025E-06	n.s.
14	35.305335	***	0.375497	***	-0.0011682	n.s.	1.047E-06	n.s.
15	39.897723	***	0.505191	***	-0.001307	*	1.092E-06	n.s.
16	33.208762	***	0.2687407	**	-0.0005747	n.s.	1.127E-06	n.s.
17	52.895686	***	0.498527	***	-0.0014336	*	1.216E-06	n.s.
18	16.2938	***	0.4839334	***	-0.0015117	*	1.433E-06	n.s.
19	34.749569	***	0.6354013	***	-0.0018973	***	1.709E-06	*
20	51.677043	***	0.4063113	***	-0.0013825	*	1.862E-06	n.s.
21	34.588308	***	0.5085461	***	-0.0015375	*	1.954E-06	n.s.
22	35.15183	***	0.4966157	***	-0.0016052	**	1.994E-06	n.s.
23	33.373173	***	0.4908741	***	-0.0016554	***	2.046E-06	n.s.
24	54.454452	***	0.3052649	***	-0.0016547	**	2.935E-06	n.s.
25	30.120336	***	0.6282992	***	-0.0023794	***	2.941E-06	*
26	57.735426	***	0.6016446	***	-0.0024053	***	3.114E-06	**
27	23.535618	***	0.3821751	***	-0.0018976	**	3.118E-06	n.s.
28	44.379813	***	0.5624839	***	-0.0021837	*	3.234E-06	n.s.
29	38.100911	***	0.5593884	***	-0.0021598	**	3.477E-06	n.s.
30	55.31815	***	0.5537641	***	-0.0023048	***	3.584E-06	***
31	28.538089	***	0.5026384	***	-0.002268	*	3.625E-06	n.s.
32	39.376771	***	0.6276894	***	-0.0025315	***	3.694E-06	*
33	26.454676	***	0.7423214	***	-0.0027006	***	3.713E-06	*
34	37.706581	***	0.5322242	***	-0.0023371	**	3.865E-06	n.s.
35	40.184024	***	0.5456626	***	-0.0026193	***	4.096E-06	**
36	45.887561	***	0.6210614	***	-0.0028666	***	4.179E-06	**
37	48.043923	***	0.4328397	***	-0.0024686	**	4.86E-06	*
38	56.579403	***	0.6421646	***	-0.0030493	***	5.053E-06	**
39	59.197281	***	0.545296	***	-0.0035643	***	7.101E-06	***
40	58.549617	***	0.6948218	***	-0.0039697	***	7.307E-06	**
Significance level:								
*** 0.1%		40		38		12		2
** 1%		0		2		7		5
* 5%		0		0		8		5
n.s. not significant		0		0		13		28
All trials		40		40		40		40