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Abstract
Response research during the last couple of decades tends to support agronomic knowledge pointing to decreasing marginal yield response up to a point where additional nitrogen does not result in further yield increase but rather a yield plateau. In contrast to a non-linear response with plateau model (NLRP) a cubic polynomial specification is generally used by those responsible for recommending/setting nitrogen norms in compliance with Danish environmental policy. The flexibility of the cubic polynomial ensures a relatively good fit to observed nitrogen/yield data and, surprisingly, often renders a close emulation of the NLRP model. Unfortunately, the cubic also often produces unrealistic optimal nitrogen application estimates. In those cases, Danish authorities use a simple quadratic specification as a fallback. We argue that an NLRP specification would be a better first choice model than the cubic – among other things, avoiding the need for a fallback model. Further progress in response research and choice of proper model specification could possibly be achieved by accounting for total nitrogen supply by including nitrogen provided in the form of mineralized nitrogen from soil sources.

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Introduction

Despite much progress in plant and soil science, economics and econometrics, and computational capability, response researchers still have not succeeded in identifying *the* mathematical specification that best represents observed data on crop yield for various levels of nitrogen (N) application. However, response research during the last couple of decades tends to support received agronomic knowledge pointing to concave yield (decreasing marginal product) up to a point where further N application does not result in yield increase but rather a yield plateau (non-linear response with plateau – NLRP).

In contrast, a cubic specification is generally used by those responsible for recommending/setting N norms in Denmark. The cubic yields at least as good a statistical fit to observed data as other plausible contending models due to its flexibility – often rendering a surprisingly close emulation of an NLRP specification. Unfortunately, the cubic also often produces unreasonably high optimal N estimates, in which cases a fallback specification, namely a simple quadratic, is used. Given contemporary agronomic knowledge it could be argued that an NLRP model would be a better fallback. Moreover, since the cubic more often than not supports the NLRP formulation (at least for our work with wheat and barley), the latter should perhaps be adopted as the preferred specification for norm setting purposes. If so, a quadratic with plateau specification seems a good choice considering that it fits most data sets well and is relatively simple to use in practice.

Further progress in response research seems especially important when estimated functions provide the basis for public policy implementation as with N-norm setting in Denmark. In particular, there seems need to account not only for applied N but also for

considerable and fluctuating plant-available N released through mineralization from soil sources of organic N during the growing season. Unaccounted for mineralized N can surely wreak havoc with response function estimation.

Profit Maximization and Implementation of Maximum N-Norms

Prediction of profit maximizing fertilizer application has long been the goal of applied response research in agricultural economics and agronomy. Such research also became important in deliberations about how to meet increasing global food demand (Pinstrup-Andersen). More recently, concern about plant nutrients as a source of environmental pollution has underscored the need for reliable information about plant utilization of available nutrients, notably N (FAO/IAEA, Rude and Frederiksen).

In Denmark, concern about nitrate pollution from crop fertilization came to a head in the early 1980s. A host of measures in pursuance of Aquatic Environment Action Program I resulted in better utilization of N in livestock manure during the 1980s and 1990s, a corresponding significant decrease in applied fertilizer-N per ha, and reduced estimated loss of N to the environment. In 1998, the Danish Parliament enacted maximum N-norms by crops and types of soil in accordance with stiffened rules in Aquatic Environment Action Program II. Based on 1997-98 crop acreage, the norms represented a 10 percent countrywide reduction compared with profit maximizing norms as recommended by The Danish Farm Organizations' Agricultural Advisory Service. The Danish Plant Directorate publishes N-norms for more than 100 non-irrigated crops for four soil types plus norms for irrigated crops. Farmers must prepare annual crop rotation plans and detailed nitrogen budgets and enroll in the public exemption register to avoid the tax of DKR 5 per kg purchased N-fertilizer. The individual farmer incurs sizeable

fines if his mandatory end-of-season N-balance account shows N-application in excess of the maximum allowable rate.

Given the mandatory "90-percent rule", we ask the question: "To what extent does the *protocol* for determining the fertilizer norm for a particular crop and soil type affect the actual mandated N-application rate?" As with all public policy, it is not just the policy provisions per se, but the implementation protocol, that matter. In particular, "How sensitive is profit-maximizing N, and thus the 90-percent maximum allowable N-application rate, to the choice of the response function form/specification from which profit-maximizing N levels are determined?"

Figure 1 with crop yield on the vertical axis (Y) and plant nutrient, in particular N, on the horizontal axis illustrates the economic story of short-run profit maximizing fertilization. Figure 1 reflects conventional (and we argue later, appropriate) assumptions of concavity (diminishing marginal returns) of the response relationship and price-taking behavior in the product and input (nitrogen) markets. Given the single-input production function (crop yield response to nitrogen) labeled TPP (Total Physical Product) and the ratio of the nitrogen price to the crop price, P_N/P_Y, the profit-maximizing N-application rate occurs at N*, where the P_N/P_Y ratio is tangent to the production function.[1] That is, the profit-maximizing producer will apply N up to, but not beyond, the point where the nitrogen-yield price ratio is equal to the slope of the production function.

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[1] If short-run profit is defined as $\pi = P_Y\, Y - P_N\, N$ and $Y = f(N)$, then the first first-order necessary condition for maximization of π is $d\pi/dN = P_Y\, dY/dN - P_N = 0$, the familiar requirement that the value of the marginal product ($P_Y\, dY/dN$) should equal the marginal input cost (P_N). Rearranging terms, $dY/dN = P_N/P_Y$, or the slope of the response function (called marginal physical product) must equal the input-product price ratio.

Two additional features of Figure 1 will be useful later: 1) There is a positive yield-axis intercept reflecting the fact that fertilizer N is not the sole source of plant-available N (there is soil N available, e.g. as previous year carry over and from in-season mineralization). And, 2) in accordance with widely accepted fertilizer response theory, there is a maximum attainable yield from the application of N. It occurs over an extended plateau beginning at N^{max} rather than at a single N level, as commonly represented in economics textbooks and as is generally imposed when a polynomial response specification (e.g., quadratic or cubic) is hypothesized.

Those responsible for determining N* in order to set N norms generally have quite reliable data on which to make good guesses about the P_N/P_Y ratio. The tricky part is choosing a plausible mathematical specification of the N-response relationship and then successfully estimating the model parameters. To implement the Danish N-norm policy, those charged with determining N* for various crops and soil types utilize the historical N trial data of the Danish Agricultural Advisory Service, Crop Production. Field trial data have long been the platform for the farm organization's extension services in their recommendations of economically optimal fertilization for various crops and soil types. Recommendations are generally based on fitting a third-degree polynomial to the data. When results of the statistical fit of the cubic are deemed problematic, a quadratic specification is used as a fallback. The mandatory N-norms rest on these N* estimates submitted to the Danish Plant Directorate by the Advisory Service.

As noted, a central problem of applied response research is zeroing in on specification of the N-response function. Albeit difficult and yet an unsettled issue, economists and agronomist have made considerable progress over the years.

Evolution of Response Research – from von Liebig to Q. Paris

Beginning in the 19th century, advances in chemistry, soil science, and plant physiology led to better understanding of vegetation principles. Scientific fact gradually replaced the theory of direct plant assimilation of organic carbon compounds as well as other nourishment from soil humus. The humus theory suffered a severe blow when, around 1840, the German chemist, Justus von Liebig suggested that carbon used in formation of organic plant material was atmospheric carbonic acid absorbed through leaves and stems (von Liebig). Wild offers a comprehensive historical review of development in plant and soil science.

Response Modeling

Most often, response researchers have summarized their interpretation of von Liebig's "law of the minimum" in three central assumptions: Absence of nutrient substitution, linear yield response to increases in quantity of the limiting factor, and a plateau maximum yield. von Liebig's theory was long considered *the* law of response. Objections did not arise until the first quarter of the 20th century, when Mitscherlich and, some years later, Spillman, independently proposed mathematical specifications of a "law of diminishing return". Mitscherlich and Spillman embraced somewhat Liebig's plateau yield in that their models included an asymptotic yield maximum. However, they not only deviated from von Liebig's law with regard to curvature of the growth segment, but also opposed the idea of non-substitution.

During the mid-1950s, Heady and co-workers started a new distinct phase of applied response research focusing on polynomial models, especially the quadratic, depicting a smooth increasing curve with a unique yield maximum followed by

decreasing yield. Continuously decreasing marginal productivity, easy statistical handling, and readily deduced optimal fertilization rates were merited features. Contrary to von Liebig, substitution among plant nutrients was explicit in the polynomial models that were found to provide good fits to empirical field research data.

In the 1970s, polynomial models came under criticism. Scientists had noted that field experiment data often tended to reveal a flat-topped/plateau image and they argued that some polynomials yielded fertilizer recommendations greater than typical application in practical farming. They obtained better statistical fit to experimental data with models involving intersecting straight lines. This revival of von Liebig's thesis was given considerable impetus owing to studies during the 1980s and 1990s by Paris and co-authors. Although Paris' early work presumed linear pre-plateau response, he later rigorously argued for a non-linear specification as did Frank, Beattie, and Embleton.

Plant Science Evidence Also Supports Non-Linear Plateau Models

Plant scientists recognize that N influences cereal yield via a multitude of physiological processes during all stages of plant growth. Adequate N availability early in the growing season is essential for vegetative growth and yield. Early development of leaves with high photosynthetic activity furthers formation of carbon assimilates, which are important for branching. Also, essential plant hormones are positively influenced by high N concentrations in the plant. Gregory suggests considering grain yield as a function of three components – head-bearing straws, kernels per head, and kernel weight. The influence of nitrogen in all three areas suggests NLRP.

On the matter of curvature, Spiertz and Ellen, among others, found that increasing amounts of N enhances the tillering capacity, and thereby, the number of heads per plant

and per unit area in winter wheat. The authors found a positive, concave relationship between the number of kernels per head and N. They also reported increasing average kernel weight with increasing N at lower levels of N and stability at higher N levels. Combined, these findings indicate increasing yield at a decreasing rate as N increases.

Plant science literature also supports the idea of a yield plateau. Photosynthetic assimilation is positively related to green leaf area, which in turn depends on the available amount of N. When the leaf index, i.e. the ratio between total leaf area and planted area, exceeds an optimal level, upper leaves will shadow lower leaves where the formation of carbon assimilates will cease so that yield no longer increases (Mengel). Further, N in excess of that needed for yield maximization can be inactivated and stored or volatilized from the plant, which is generally tolerant to high N concentrations. This supports the idea that, after a yield maximum has been achieved, the generation of dry matter in the plants can stabilize over a rather extended plateau as N supply increases further.

These plant science ideas lend support to the NLRP model as advanced over the last two decades by response researchers. Unfortunately, numerous alternative model specifications could embody curvature and plateau yield. Importantly, model choice might well matter in terms of the implied optimal N norm. (For further discussion of the plant science support for NLRP, see Mortensen; for further discussion of agricultural economics response-research literature, see Mortensen and Holloway and Paris.)

Comparative Analysis of Selected Response Models

To evaluate the adequacy of different general response hypotheses, in particular polynomial specifications versus NLRP models, we analyzed an extensive field

experiment data set graciously made available by the Danish Agricultural Advisory Service (Mortensen). Like those charged with setting the Danish N-norms, we limited our analysis to the relationship between crop yield and applied fertilizer N. Such a single-product, single-variable-nutrient approach is appropriate given the generally accepted assumption that no substitution takes place between N and other plant nutrients. It is, of course, a requirement that supplies of non-N nutrients are available in amounts that do not limit yield. The data generating Danish field trials comply with this condition.

The Data Set

For the purpose of this study, data from about 1,200 field experiments for three major cereal crops – winter and spring barley and winter wheat – were considered. The trials were executed in different regions of Denmark and spanned the 11-year period, 1987 to 1998. To maintain as much homogeneity as possible regarding growth factors other than N, trials were excluded when manure and slurry had been applied to the trial site in earlier years. Only trials with cereals as the preceding crop were included. Further, the sample was restricted to include only experiments conducted on soil types with high clay content and high water retaining capacity (soil types JB#6 and #7). The trial sites did not receive artificial irrigation.

This selection resulted in 84 data sets, half of which were from the most recent six years, 1993-98, comprising 40 winter wheat trials and 36 and 8 spring barley and winter barley trials, respectively. All trials had five or six levels of N application, ranging from zero to 200 or 250 kg N per ha. Most trials were executed with five replications (20 trials involved 4 replications and 2 trials had six replications). In total the 84 trials produced 2,259 individual yield-N observations. The standard size of individual trial plots was 30

m^2. Yield was defined in terms of hkg of harvested grain per hectare, standardized for moisture content but not for protein content and disregarding straw. For the latter half of the period, soil samples at the trial sites offer information about plant available N at the beginning of the growing season, before application of fertilizer.

Examined Response Model Specifications

Ten alternative response models were considered (Mortensen). Discussion here is limited to the five models presented in Table 1 – the quadratic (Q) and cubic (C) polynomial, which are the two specifications used in implementing the Danish N-norm policy, the Cobb-Douglas (CD), and two alternative NLRP models, viz. a quadratic with plateau (QP)[2] and Cobb-Douglas with plateau (CDP). For all models, Y denotes yield and N is the level of applied nitrogen; for the plateau models, N^{max} is the lowest amount of N that maximizes yield. The error term (deviation between observed and model predicted yield) is suppressed in the formulas shown in Table 1.

All specifications allow for a yield-axis intercepts where fertilizer N application is zero, see Figure 1. The intercept denotes yield generated exclusively by assimilated N from soil sources. Soil N can be conceived as the sum of plant available N in spring at the beginning of the growing season (can be determined in soil samples) plus the generally unknown net mineralization from organic N compounds in soil during the growing season. In Figure 1, both applied fertilizer N and (unknown) soil N are represented in the hypothesized total yield/nitrogen response curve.

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[2] The quadratic with plateau model reported here ties the plateau to the parabola top in the estimation procedure. We also considered a more flexible quadratic with plateau model (reported in Mortensen) where the knot point could be to the left of the parabola top. While we prefer, on theoretical grounds, the more flexible model, the additional degree of freedom required in estimation generally resulted in a somewhat "poorer" statistical fit. There were only small differences between parameter estimates in the two QP versions.

Comparison of Statistical Fit

The alternative functional forms were fitted to each trial data set (84) with minimization of the sum of squared residuals as the fitting criterion. The goodness-of-fit was expressed in terms of mean squared error (MSE). Each functional form was fitted to the 84 data sets and an MSE value was calculated. The average of MSE values over all trial data sets for each function is presented in Table 2 by crop. The relatively small (within column) differences in average MSE values for the various functional forms are striking. In fact, when we conducted a Tukey simultaneous test of MSE averages for all 10 functions, controlling for the probability of making type I errors (Devore), we found no significant MSE differences among the most plausible contending models, i.e. among those model specifications permitting non-linear yield response with or without a plateau (Mortensen). Thus, little importance can be attached to the fact that the order of functions according to average MSE varies among crops in Table 2. The upshot is that no one function can be claimed superior on the basis of statistical fit for the 84 trial data sets that we examined. Having found that, it is interesting that the different estimated functions have quite different implications for maximum yield and for the profit-maximizing N level, i.e. for N^{max} and N^* – a subject to which we now turn.

Does Choice of Function Matter?

With non-decisive statistical differences among alternative model specifications, it would be tempting to conclude that any of the examined functions – and possibly a myriad of other plausible specifications – could equally well (or poorly) serve as a basis for setting fertilization norms. To address this issue, we considered, as a case study, pooled experimental data for 27 winter wheat trials from the most recent six years of the data set.

As revealed in Table 3, all the functions predict a yield close to 40 hkg per ha when applied N is zero (corresponds to the Y-intercept in Figure 1). In contrast, the estimated N application that maximizes wheat yield (N^{max}) varies widely among functions – from about 177 kg N per ha for the CDP to more than 210 kg N for the other models. The CD, of course, has no maximum. Nor, does, in our case, the third-degree polynomial (C), which is somewhat disconcerting – a point to which we return later.

Like N^{max} the deduced profit maximizing N application (N*) varies considerably among the functional forms. For the CDP, N^{max} and N* coincide at the knot point where the growth segment of the function joins the plateau.[3] The QP has a calculated N* of 190 kg N per ha, which is 20 kg shy of N^{max}. The polynomial without plateau specifications (Q and C) suggest N* figures in the range 200-215 kg per ha. The CD fails to produce a plausible N*.

We believe the data in Table 3 suggest that, in spite of the fact that there are no statistically significant differences in goodness-of-fit among plausible response functions, choice of functional form indeed has important practical implications for both farm profitability and environmental quality. It is evident that a wrong choice of fertilization norm within the wide range allowable under Table 3 would entail farm-level costs because both too low and too high N applications are sub-optimal. Likewise, misspecification – leading to excessive use of N – gives rise to societal costs in the form of negative environmental impacts. So, the important practical question remains: Which

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[3] This was also the case, of course, for the classical linear response with plateau (LRP) model included among the ten examined functions by Mortensen. For the LRP, the implied N* value was as low as 128 kg per ha.

functional form among plausible model specifications should be used in setting N-norms given the available information and the uncertainties involved.

Implications

In thinking about the foregoing question, we are perplexed for two reasons. First, we are pessimistic regarding the likelihood of finding a "best" functional form given existing data. Yet, from Table 3 we know that functional from matters in the practice of setting obligatory N norms. So, the question cannot be dodged. In thinking about the question, we find a two-pronged approach useful. First, what makes the most sense if we presume that analysts are stuck with existing data and knowledge; the second approach, involves an idea for improving our knowledge/data base. We begin with the first approach.

Presuming No Improvement in Knowledge or Data Base

Under a "we 'gotta' do something with what we got" scenario, a common sense conclusion would be to reject the CD (without plateau) model from further consideration. The CD often tends to produce unrealistically high N* estimates because it never attains (even asymptotically) a maximum yield. We also believe the cubic polynomial can be problematic, although it is used carefully – and in certain cases is replaced with a quadratic specification – by those setting N norms in Denmark. In our judgment the cubic is simply too flexible and such flexibility is unwarranted in terms of received evidence and theory regarding plant response to N. Many agricultural economists (and economists) like the cubic specification. It allows for the possibility of an increasing marginal returns phase, followed by and diminishing, and ultimately a negative, marginal returns phase in the response relationship, i.e. the classic three-stage textbook response. There is hardly support for the three-stage yield/N response in plant

science literature, and it is perplexing that upon estimation only in 22 cases out of 84 did we get a negative coefficient on the third-degree term giving rise to the possibility of a classic three-stage textbook function. For 11 of these 22 cases the N-coordinate of the inflection point was strongly negative and a three-stage function could not be revealed inside an appropriate N domain, even when sensible values for soil N were considered. That is, the function displayed decreasing marginal productivity over the entirety of the plausible total N domain. For the other 11 cases the inflection point was within the limits of an appropriate N domain, but for seven of those the yield was positive at the local function minimum so there was no intercept with the N-axis in the soil N domain. For three of the remaining four cases with appropriate negative N-axis intercepts, the inflection point appeared between the two negative intercepts so that the marginal productivity was decreasing everywhere in the relevant N domain. Only in one case did the inflection point appear between the relevant N-axis intercept and the local maximum. The upshot is that the classic three-stage production function was only revealed in one case when fitting the cubic function to the 84 trial data sets.

Interestingly, for 62 of the 84 data sets the cubic-coefficient was positive rather than negative, giving rise to a function looking more like a variable cost curve than a production function. We think the data "cry out" for a plateau and the only way a non-plateau third-degree polynomial can attempt that is with a "reversal" of the usual curvature pattern. The cubic seems to want to place the infection point so that it is approached horizontally rather than vertically and so that it occurs "in the vicinity of the true" maximum yield. That is, in most cases (again, 62 of 84) the onset of the increasing marginal returns phase, rather than being at low input levels, is revealed near or well

beyond the largest N-treatment level where it raises minimal or no havoc with the minimization of squared error. Moreover, as noted earlier, this pattern was also exhibited in our pooled-data winter wheat application (Table 3).

We believe the interesting results for the cubic model suggest a different/better model, viz. a concave growth stage followed by a yield plateau, an NLRP model. Received agronomic evidence suggests that von Liebig type models are more promising – in particular plateau models exhibiting diminishing marginal returns in the growth (pre-plateau) phase. We like the quadratic plateau construct for its simplicity and for the fact that, while not significantly better than the CDP model, it does tend to consistently fit most data sets well. As a potential model for setting N-norms in Denmark it represents the least change from the present approach, which uses a quadratic (without plateau) specification as a fall back when the implied N-norm from the cubic specification is deemed unreasonable. Further, the QP model is relatively simple to apply compared with several other specifications that allow for non-linear pre-plateau growth. The QP model points to an N norm of about 190 kg per ha, a figure that reasonably well concerts with practical farm experience for the period and circumstances considered in our study.

Presuming Possibility of Improved Data Base – A Suggestion for Further Research
As suggested earlier, it is known that soil sources provide a significant share of total N available to plants during the growing season. Olesen reported that annual mineralization could be as much as two percent of typically 5-7,000 kg of soil N per ha in Denmark. Jarvis reported Scottish experiments showing about half of total N-uptake in cereal crops was from soil sources, part of which was already in plant-available form at the beginning of the growing season. All indications are that during-season mineralization plays an

important role in total N supply. However, at present, mineralization data do not exist in a form that can be integrated with field experiment data on yield and applied fertilizer N.

Soil N (denoted NS in the expanded list of acronyms used henceforth in this section) can be thought of as the sum of measurable plant-available N in the soil at the beginning of the growing season (NS1) and net mineralization of organic soil N during the growing season (NS2). While response modelers commonly have good data on fertilizer N (NF) and may know NS1, they/we do not have data on, or a decent method for estimating, NS2. While the total N supply (NT) is

$$NT = NS + NF = NS1 + NS2 + NF$$

in response modeling just NF is generally used as the regressor, leaving the critical NS2 and hence NT unaccounted for.

Because plants assimilate and utilize available NS equally well as NF, the NS curve segment of Figure 1 (denoted Soil N in Figure 1) should be consistent with (connected to) the estimated response relationship in the NF domain (denoted Fertilizer N in Figure 1). Given that NS2 fluctuates from plot-to-plot and year-to-year, among other things due to varying soil climate (notably soil temperature and moisture content), the NF segment may be revealed at different locations along the entire response curve as sketched in Figure 2. A specific NF-value data point (e.g.200 kg NF per ha) will represent different levels of NT supply depending on soil type and soil climatic conditions. When the regressor is NF as is generally the case, we clearly obtain an incomplete and misleading picture of the true relationship between yield and NT (NS1 + NS2 +NF). That is, our estimation of the response relationship is less reliable (perhaps

much more so) than what is needed to accurately determine the true profit-maximizing N level.

Figure 2 also reflects that variation in climate factors – not necessarily identical to those influencing mineralization – have an impact on the efficiency with which growing plants utilize available N, indicated by varying slopes of the response curve for different climate conditions. Again, failure to account for this factor in the curve fitting process hampers statistical distinction between and among alternative response specifications.

To come to grips with these missing data problems, one could imagine a series of specific field trials, which – in addition to producing traditional information on yield for different levels of NF – would attempt measurement of net mineralization and simultaneously monitor pertinent soil environmental and aerial climate conditions that affect mineralization and productivity. Improving data along these lines would provide a basis for estimating NS2 and thus NT as a function of the important determining variables as well as the influence of important aerial climate variables on growth. It is our hypothesis that incorporation of these elements would enhance response researchers' ability to find statistically superior response models,[4] and thereby enhance our ability to better serve the policy process – both policy making and implementation processes.

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[4] Using simulated NS2 data together with available NS1 data for the different trial sites, Mortensen re-fit the original 10 functional forms using "NT" rather than NF as the regressor. The upshot was an apparent separation of the NLRP and the cubic functions from the other functions – something that was not possible when estimation was based solely on the NF data.

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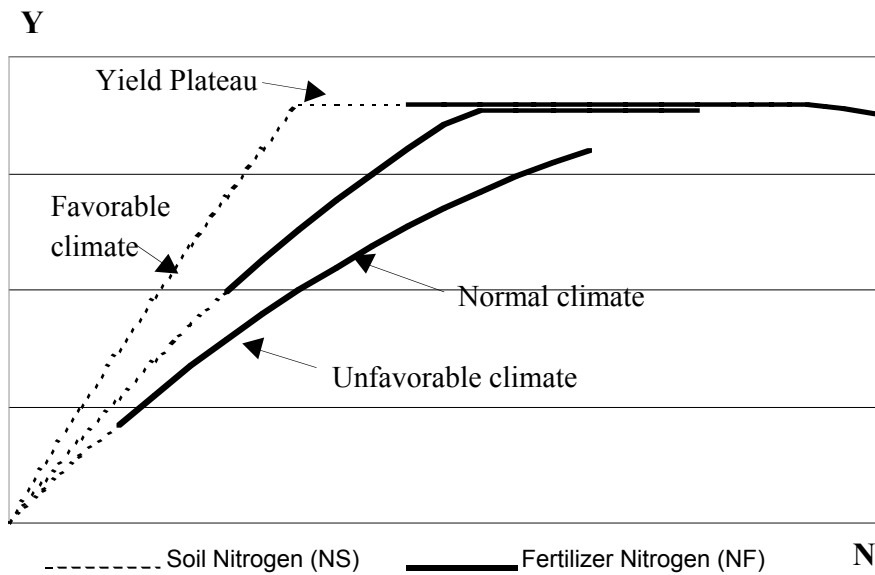
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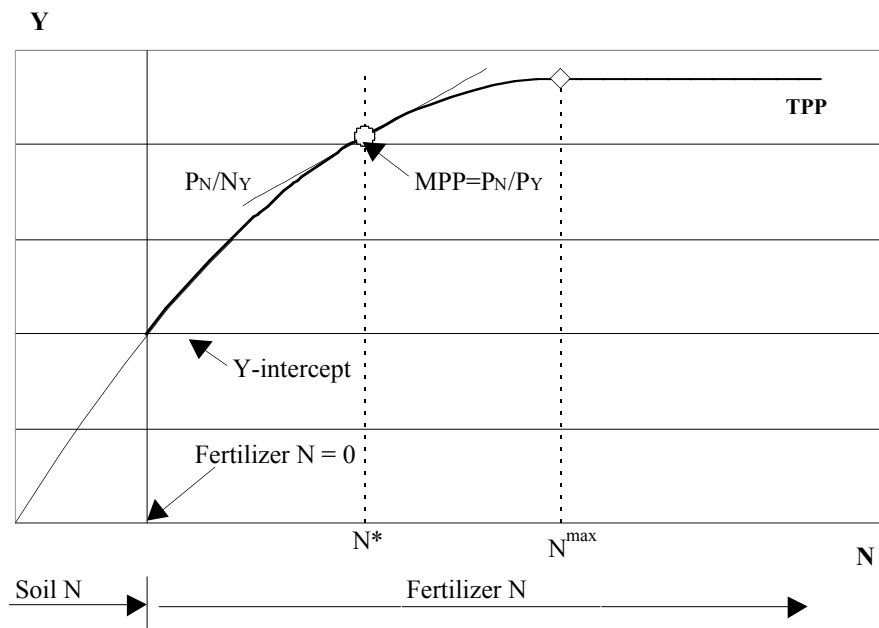
Figure 1. Crop Yield (Y) Response to N-Fertilization

Figure 2. Climate Impacts on N-Mineralization and Yield

Table 1. Response Functions Fitted to Trial Data

Model	Functional Form
Quadratic (Q)	$Y = \alpha + \beta_1 N + \beta_2 N^2$
Cubic (CB)	$Y = \alpha + \beta_1 N + \beta_2 N^2 + \beta_3 N^3$
Cobb-Douglas (CD)*	$Y = \beta_1(\beta_2 + N)^{\beta_3}$
Quadratic with plateau (QP)**	$Y = \alpha + \beta_1 N + \beta_2 N^2$ for $N < N^{max}$ and
	$= \alpha + \beta_1 N^{max} + \beta_2(N^{max})^2$ for $N \geq N^{max}$
	s.t. $N^{max} = -\beta_1/2\beta_2$ which is N-coordinate
Cobb-Douglas w/ plateau (CDP)*	$Y = \beta_1(\beta_2 + N)^{\beta_3}$ for $N < N^{max}$ and
	$= \beta_1(\beta_2 + N^{max})^{\beta_3}$ for $N \geq N^{max}$

* The extra parameter, β_2, was inserted to enable positive Y intercept.
**Two versions of the quadratic with plateau were examined: one where the onset of the plateau, as here, is tied to the top of the parabola, and the other where the onset can be at or to the left of the parabola top, i.e. s.t. $NF^{max} \leq -\beta_1/2\beta_2$.

Table 2. Average MSE by Models and Crops, 84 Trials

Spring barley, 36 trials		Winter barley, 8 trials		Winter wheat, 40 trials	
Function	MSE	Function	MSE	Function	MSE
CD	9.81	CD	16.70	CD	16.80
CDP	8.48	CDP	16.26	Q	14.63
Q	8.47	C	14.06	CDP	14.22
QP	8.31	Q	14.05	QP	13.56
C	8.08	QP	13.94	C	13.53

Table 3. Key Figures for 27 Pooled Winter Wheat Trials, 1993-98

	Yield, hkg per ha,	N-application, kg per ha, for	
Function:	for applied NF = 0	Max. yield	Max. profit*
Q	40.3	222.6	200.2
C	39.5	-	214.1
CD	39.5	-	(632)
QP	40.0	210.5	190.2
CDP	39.5	176.8	176.8

* Assuming DKR 85 per hkg of wheat and DKR 3.50 per kg N in fertilizer.