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Reconstruction of vector and tensor fields from sampled discrete data

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Abstract. We construct atomic spaces $S \subset L_2(\mathbb{R}^n, T_1^1)$ that are appropriate for the representation and processing of discrete tensor field data. We give conditions for these spaces to be well defined, atomic subspaces of the Wiener amalgam space $W(C, L_2(\mathbb{R}^n, T_1^1))$ which is locally continuous and globally L_2. We show that the sampling or discretization operator R from S to $l_2(\mathbb{Z}^n, T_1^1)$ is a bounded linear operator. We introduce the dilated spaces $S_\Delta = \mathrm{D}_\Delta S$ parametrized by the coarseness Δ, and show that the discretization operator is also bounded with a bounded inverse for any $\Delta \in \mathbb{Z}^n$. This allows us to represent discrete tensor field data in terms of continuous tensor fields in S_Δ, and to obtain continous representations with fast filtering algorithms.

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1. Introduction

Modern imaging systems, (e.g., Magnetic resonance image scanner) acquire discrete sets of data and store them as arrays of numbers. In many new imaging modalities, the acquired images are no longer a set of scalar values representing the gray levels voxels (spatial positions on some three dimensional lattice). Instead, the images are vector or tensor-valued functions. Prototypical example (and the motivation behind this mathematical development) is Diffusion Tensor Magnetic Resonance Imaging (DT-MRI) which provides a measurement of the effective diffusion tensor of water in each voxel of an image volume (see Figure 1). These tensor images can be used to elucidate the three dimensional fiber architectural features of anisotropic fibrous tissues and organs in vivo, and provide microstructural information noninvasively and nondestructively in materials sciences applications [**4, 3**]. However, the measured tensor in each voxel is inherently a noisy, discrete, and volume-averaged quantity. Thus, one goal of this work is to develop mathematical methods to ameliorate these problems. More generally, we are interested in developing a general mathematical framework that enables us to analyze, process and compress these data sets. We show we can do this by constructing a smooth, continuous representation of the diffusion tensor field. Moreover, the algorithms that

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implement the mappings between the tensor data and their continuous representations must be fast and efficient, since the size of a typical tensor images in a single experiment may be large - on the order of 16,777,000 tensors - which represents a monumental data processing effort.

To address the aforementioned problems, we propose a mathematical framework for representing discrete tensors that is suited to the post-processing applications, such as pattern recognition, registration and geometric transformations in general. In Section (2), we introduce the atomic tensor-field spaces S that we use to approximate these tensor fields. We give the necessary and sufficient conditions for these spaces to be well defined subspaces of the Wiener amalgam space $W(C, L_2(\mathbb{R}^n, T_1^1))$ which is locally continuous and globally L_2. We show that the sampling (or restriction) operator R from S to $l_2(\mathbb{Z}^n, T_1^1)$ is a bounded linear operator. We then introduce the dilated spaces $S_\Delta = \mathrm{D}_\Delta S$ parametrized by the coarseness Δ, and give a condition on S that guaranties that the discretization operators R_Δ (which takes tensor fields from S_Δ defined on $\mathbb{R}^n$ and restrict them on $\mathbb{Z}^n$) are bounded with bounded inverse for any $\Delta \in \mathbb{Z}^n$. This allows us to generate discrete tensor field spaces S_Δ^d that are useful for the representing and approximating tensor field data. In Section 3, we study the problem of approximating a set of discrete tensor data Φ by a continuous tensor field $G_\Delta^{\approx} \in S_\Delta$ that minimizes the l_2-error between the data and its sampled approximation. We then show the link between the approximation problems and the filtering paradigm in signal and image processing, , which leads to fast filtering implementations.

2. The atomic Wiener amalgam spaces

2.1. Atomic tensor fields spaces. The collected data that we consider consists of a set of discrete tensors $\{\Phi(k)\}_{k\in\mathbb{Z}^n}$ that are obtained by sampling a tensor field $F(\cdot)$ defined on $\mathbb{R}^n$. The sampling grid is a regular lattice that, without loss of generality, we identify with $\mathbb{Z}^n$. Our first goal is to consider tensor spaces defined on $\mathbb{R}^n$ that will be used to approximate the original tensor field $F(\cdot)$ from the knowledge of its samples $\Phi = F|_{\mathbb{Z}^n}$. For this purpose, we consider atomic tensor field spaces of the form [**1**]

$$S(B) = \left\{ G(x) = \sum_{j\in\Lambda} \sum_{k\in\mathbb{Z}^n} c^j(k) B^j(x-k) : c \in l_2^{(r)} \right\} \tag{2.1}$$

where $\Lambda = \{1, \dots, r\}$ is an index set, $c := (c^j)_{j\in\Lambda}$ is a vector of sequences, $l_2^{(r)} := l_2 \oplus \cdots \oplus l_2$ is the direct sum of r copies of l_2, and is endowed with the natural norm, and $B = (B^j)_{j\in\Lambda}$ is vector of tensor functions (which will also be considered as a rectangular matrix of functions. For the space S to be well defined, the *generating tensors* $\{B^j; j \in \Lambda\}$ cannot be chosen arbitrarily. In particular, if the tensors in S are square integrable, then the generating tensors $\{B^j; j \in \Lambda\}$ must also be square integrable:

$$\left\| B^j \right\|_{L_2(\mathbb{R}^n, T_1^1)} = \int_{\mathbb{R}^n} ||B^j(x)||^2 dx, \tag{2.2}$$

where $||\cdot||$ is the Frobenius pointwise norm of the finite dimensional space of tensors T_1^1 generated by the Frobenius natural inner product (also called Euclidian or Schur inner product)

$$\operatorname{trace} CD^* = \sum_{i=1}^{m}\sum_{j=1}^{m} C_{i,j}\overline{D_{j,i}}, \tag{2.3}$$

where C and D are $m \times m$ tensors' matrices, with components $C_{i,j}$, and $D_{j,i}$ ($m \times m$ is the dimension of the tensor space T_1^1), and where $\overline{z}$ denotes the complex conjugate of z[1]. Using the Frobenius inner product in T_1^1, the inner product in $L_2(\mathbb{R}^n, T_1^1)$ associated with the norm (2.2) is defined by

$$\langle F, G\rangle = \int\limits_{\mathbb{R}^n} \operatorname{trace}(F(x)G^*(x))dx. \tag{2.4}$$

Clearly, choosing tensors $\{B^j; j \in \Lambda\}$ that are square integrable is not sufficient for $S(B)$ to be a subspace of $L_2(\mathbb{R}^n, T_1^1)$. In fact, we need even more restrictions on the tensors $\{B^j; j \in \Lambda\}$, since we want $S(B)$ to satisfy other properties. In particular, since we will be interested in using the space $S(B)$ as an approximation space for our tensor data, we want to choose $\{B^j; j \in \Lambda\}$ so that S is closed. Moreover, we want to compute and describe the approximations in $S(B)$ using efficient stable algorithms. One way of enforcing this requirement is to choose the set $\left\{B^j(x-k) : \ j \in \Lambda, \ k \in \mathbb{Z}^n\right\}$ to be a Riesz basis of $S(B)$, i.e., we want

$$a_1 \left\|c\right\|^2_{l_2^{(r)}} \le \left\|\sum_{j\in\Lambda}\sum_{k\in\mathbb{Z}^n} c^j(k)B^j(x-k)\right\|^2_{L_2(\mathbb{R}^n, T_1^1)} \le a_2 \left\|c\right\|^2_{l_2^{(r)}} \quad \forall c \in l_2^{(r)} \tag{2.5}$$

for some positive constants $0 < a_1 \le a_2$. To achieve all the requirements above, we use the following theorem by S. L. Lee and W. S. Tang [**13**].

Theorem 2.1. *Let* $\mathrm{U} = (\mathrm{U}_1, \dots, \mathrm{U}_n)$*, be an ordered pair of n-tuple of distinct commuting unitary operators on a complex Hilbert space* $\mathcal{H}$*, and let* $Y = \{y_1, \dots, y_r\}$ *be a finite subset of* $\mathcal{H}$*.*
Consider the set $\mathrm{U}^{\mathbb{Z}^n}(Y) = \left\{\mathrm{U}^k\, y_j : k \in \mathbb{Z}^n, j \in \Lambda\right\}$*, and let* $\hat{A}(\nu)$ *be the* $r\times r$ *matrix function defined by*

$$\hat{A}_{p,q}(\nu) = \sum_{k\in\mathbb{Z}^n}\left\langle y_p, \mathrm{U}^k\, y_q\right\rangle e^{i2\pi\nu\cdot k}, \quad (p,q) \in \Lambda\times\Lambda \tag{2.6}$$

then the following conditions are equivalent

1. $\mathrm{U}^{\mathbb{Z}^n}(Y)$ *is a Riesz basis for its closed linear span.*
2. *There exist positive constants* a_1 *and* a_2 *such that*

$$a_1\mathbf{I}_r \le \hat{A}(\nu) \le a_2\mathbf{I}_r. \tag{2.7}$$

Special cases of this theorem can also be found in [**1, 11, 14**].

Let $\mathcal{H} = L_2(\mathbb{R}^n, T_1^1)$ be the space of square integrable tensor fields on $\mathbb{R}^n$, with the inner product defined by (2.4). Clearly, the shift operator

$$\mathrm{U}_i(G)(x_1, \dots, x_i, \dots, x_n) = G(x_1, \dots, x_i - 1, \dots, x_n)$$

is a unitary operator, and U_i commute with U_j for any i, j. Hence, as a corollary of Theorem 2.1 we get

[1]For real valued symmetric tensors, the Euclidian or Schur inner product reduces to the familiar tensor double dot product (see for example [**16**]), which have been used previously as an inner product for discrete diffusion tensors [**5, 3**]

Corollary 2.2. *The space S is a well-defined, closed subspace of $L_2(\mathbb{Z}^n, T_1^1)$ with $\left\{B^j(x-k) :\ j \in \Lambda,\ k \in \mathbb{Z}^n\right\}$ as its Riesz basis if and only if, for almost all $\nu \in \mathcal{Q} = [0,1]^n$ there exist two positive constants $0 < a_1 \leq a_2 < \infty$ such that the smallest and largest eigenvalues $\lambda_{\min}\left(\hat{A}(\nu)\right)$ and $\lambda_{\max}\left(\hat{A}(\nu)\right)$ of the $r \times r$ positive self-adjoint matrix $\hat{A}(\nu)$ satisfy:*

$$a_1 \leq \operatorname*{ess\,inf}_{\nu \in \mathcal{Q}} \left(\lambda_{\min}\left(\hat{A}(\nu)\right)\right) \leq \operatorname*{ess\,sup}_{\nu \in \mathcal{Q}} \left(\lambda_{\max}\left(\hat{A}(\nu)\right)\right) \leq a_2. \tag{2.8}$$

Remark 2.3. The entries $\hat{A}_{i,j}(\nu)$ of $\hat{A}(\nu)$ in (2.6) are the multivariate Fourier series of the discrete functions $\alpha(k) = \left\langle y_i, \mathrm{U}^k\, y_j \right\rangle$. In particular, for the space $S(B)$, the matrix-function $\hat{A}(\nu)$ is the Fourier series of the matrix-sequence

$$\left(A_{i,j}\right)(k) := \left(\left\langle B^i(x), B^j(x-k)\right\rangle\right).$$

Let D_Δ be the dilation operator by a factor $\Delta = (\Delta_1, \Delta_2, \ldots, \Delta_n)$, where $\Delta_i > 0,\ i = 1, \ldots, n$:

$$(\mathrm{D}_\Delta\, G)(x) = (\Delta_1 \Delta_2 \cdots \Delta_n)^{1/2} G\left(\frac{x}{\Delta}\right) \quad \forall G \in L_2(\mathbb{R}^n, T_1^1),$$

where $x = (x_1, \ldots, x_n)$, and $\frac{x}{\Delta} = \left(\frac{x_1}{\Delta_1}, \ldots, \frac{x_n}{\Delta_n}\right)$. Using the dilation operator, we define the space $S_\Delta = \mathrm{D}_\Delta\, S$ which can also be described as

$$S_\Delta(B) = \left\{ G(x) = \sum_{j \in \Lambda} \sum_{k \in \mathbb{Z}^n} c^j(k) B^j\left(\frac{x}{\Delta} - k\right) : c \in l_2^{(r)} \right\} \tag{2.9}$$

where, $B_\Delta = (B_\Delta^j)$, and $B_\Delta^j = \mathrm{D}_\Delta\, B^j$. Since, the dilation operator is unitary (recall that $\Delta_i \neq 0$ for $i = 1, \ldots, n$), the space S_Δ is isomorphic to the space S. Thus, if S is closed, then S_Δ is also closed. Moreover, if $\left\{B^j(x-k) :\ j \in \Lambda,\ k \in \mathbb{Z}^n\right\}$ is a Riesz basis of S, then the set $\left\{B^j(x/\Delta - k) :\ j \in \Lambda,\ k \in \mathbb{Z}^n\right\}$ is a Riesz basis of S_Δ.

The space S_Δ can be viewed as a copy of S at a different resolution. Given a continuous tensor field, we can approximate it by a tensor field $G_\Delta^{\approx} \in S_\Delta$, and control the resolution of the approximation by controlling the parameter Δ. However, the collected data $\Phi \in l_2(\mathbb{Z}^n, T_1^1)$ consists of samples of a noisy version of the tensor field F. For this purpose, we introduce the discretization operator R_Δ which takes a tensor field defined on $\mathbb{R}^n$ and restricts it onto the lattice $\mathbb{Z}^n$ (we will use R for the discretization operator on S). Using the discretization operator, we define the space

$$S_\Delta^d = \mathrm{R}_\Delta\, S_\Delta \tag{2.10}$$

which is obtained by sampling the space S_Δ on the regular grid $\mathbb{Z}^n$. However, for the definition above to be meaningful, we need extra conditions to ensure that the discretization is well defined. In particular, the space S_Δ should consist of continuous tensor fields. Moreover, the space S_Δ should be sufficiently small for the discretization operator R_Δ to be bounded. These requirements can be achieved by choosing generating tensor fields $\{B^j, j \in \Lambda\}$ that are continuous and are sufficiently regular. Specifically, we will require the entries $B_{p,q}^j$ of the generating tensors B^j to belong to the Wiener amalgam space $W(C, L^1)$ [**9**].

A continuous function g belongs to the Wiener amalgam space $W_p = W(C, L_p)$ if its localization

$$F_h(g)(x) = \sup_{\xi \in \mathbb{R}^n} |h(\xi - x) g(\xi)|, \tag{2.11}$$

by the window function $h(\xi - x) \in \mathcal{D}(\mathbb{R}^n)$ is globally L_p (Here, $\mathcal{D}(\mathbb{R}^n)$ is the usual space of compactly supported, C^∞ test functions):

$$\|g\|_{W_p}^p = \int_{\mathbb{R}^n} |F_h(g)(x)|^p \, dx < \infty. \tag{2.12}$$

The spaces $W_p = W(C, L_p)$ is a proper subspace of $C \cap L_p$.

It turns out that an equivalent characterization of Wiener amalgam spaces can be obtained by a discrete form of (2.12) and gives the equivalent norm

$$\|g\|_{W_p}^p \approx \sum_{l \in \mathbb{Z}^n} |F_h(g)(l)|^p, \tag{2.13}$$

e.g., if the set $\{h(x-l)\}_{l \in \mathbb{Z}^n}$ forms a Bounded Uniform Partition of Unity (BUPU), i.e., $\sum_l h(x - l) = 1$ [**8**]. An important feature of both descriptions of these spaces is the fact that they do not depend on the choice of the window function $h \in \mathcal{D}(\mathbb{R}^n)$, i.e., changing h will simply change the norm of the spaces to an equivalent norm [**9**].

Using the definition of the scalar Wiener amalgam space W_p, we can define Wiener amalgam spaces $W(C, L_p(\mathbb{R}^n, T_1^1))$ of tensors as follows: A continuous tensor field G belongs to $W(C, L_p(\mathbb{R}^n, T_1^1))$ if

$$\|G\|_{W(C, L_p(\mathbb{R}^n, T_1^1))} = \int_{\mathbb{R}^n} |F_h(G)(x)|^p \, dx < \infty, \tag{2.14}$$

with

$$F_h(G)(x) = \sup_{\xi \in \mathbb{R}^n} |h(\xi - x)| \, \|G(\xi)\| \,, \tag{2.15}$$

where $\|\cdot\|$ is the Frobenius norm of the finite dimensional space T_1^1. The expected relation between a tensor field $G \in W(C, L_p(\mathbb{R}^n, T_1^1))$ and its components $G_{i,j}$ is stated in the following Lemma below.

Remark 2.4.
1. The Frobenius norm for T_1^1 used in (2.15) can be replaced by any norm for T_1^1 since, for finite dimensional spaces, all norms are equivalent.
2. We abuse notation and use the same symbol $F_h(\cdot)$ to describe the operator in (2.11) and the operator in (2.15). However, the distinction should be clear from the context.

Lemma 2.5. *A tensor field G belongs to $W(C, L_p(\mathbb{R}^n, T_1^1))$ if and only if its components $G_{i,j}$ belong to the scalar Wiener spaces $W_p = W(C, L_p)$, and we have*

$$\|G_{i,j}\|_{W_p} \le \|G\|_{W(C, L_p(\mathbb{R}^n, T_1^1))} \le \sum_{i,j} \|G_{i,j}\|_{W_p} \,. \tag{2.16}$$

Moreover, a tensor field G belongs to $L_p(\mathbb{R}^n, T_1^1)$ (a function h from $\mathbb{Z}^n$ to T_1^1 belongs to $l_p(\mathbb{Z}^n, T_1^1)$) if and only if its components $G_{i,j}$ ($h_{i,j}$) belong to $L_p(\mathbb{R}^n)$

($l_p(\mathbb{Z}^n)$), and we have

$$\|G_{i,j}\|_{L_p} \le \|G\|_{L_p(\mathbb{R}^n,T_1^1)} \le \sum_{i,j} \|G_{i,j}\|_{L_p}. \tag{2.17}$$

$$\|h_{i,j}\|_{l_p(\mathbb{Z}^n)} \le \|h\|_{l_p(\mathbb{Z}^n,T_1^1)} \le \sum_{i,j} \|h_{i,j}\|_{l_p(\mathbb{Z}^n)}. \tag{2.18}$$

Proof. For any $\xi \in \mathbb{R}^n$, we have the pointwise estimate

$$|G_{i,j}(\xi)| \le \|G(\xi)\| \le \sum_{i,j} |G_{i,j}(\xi)|.$$

Thus, we get the pointwise estimate

$$F_h(G_{i,j})(x) \le F_h(G)(x) \le \sum_{i,j} F_h(G_{i,j})(x),$$

from which we get (2.16). The proof of the second part of the Lemma is similar. □

We are now ready to state the conditions under which our atomic space S defined by (2.1) consists of continuous tensor fields.

Theorem 2.6. *If $B^j \in W(C, L_1(\mathbb{R}^n, T_1^1))$ for $j \in \Lambda$, and Condition (2.8) is satisfied, then the space S consists of continuous tensor fields and the $L_2(\mathbb{R}^n, T_1^1)$-norm is equivalent to the $W(C, L_2(\mathbb{R}^n, T_1^1))$-norm, i.e., there exist positive constants $0 < a_1 \le a_2$ such that*

$$a_1 \|G\|_{W(C,L_2(\mathbb{R}^n,T_1^1))} \le \|G\|_{L_2(\mathbb{R}^n,T_1^1)} \le a_2 \|G\|_{W(C,L_2(\mathbb{R}^n,T_1^1))} \quad \forall G \in S \tag{2.19}$$

Remark 2.7. If $B^j \in W(C, L_1(\mathbb{R}^n, T_1^1))$ for $j \in \Lambda$, then, using Lemma 2.5, each of its components $B^j_{p,q}$ belongs $W_1 = W(C, L_1)$. Therefore, each component $B^j_{p,q}$ also belongs to $W_2 = W(C, L_2)$ [**8**]. We conclude, using Lemma 2.5, that $B^j \in W(C, L_2(\mathbb{R}^n, T_1^1))$ for $j \in \Lambda$.

Proof of Theorem 2.6. (*Right inequality.*) For a scalar functions $g \in W_2 = W(C, L_2)$, it is always true that $\|g\|_{L_2} \le a_2 \|g\|_{W_2}$ for some constant a_2 [**8**]. Using Lemma 2.5 it follows that each component $G_{p,q}$ of a tensor field $G \in W(C, L_2(\mathbb{R}^n, T_1^1))$ satisfies a similar bound. Thus, using Lemma 2.5 once more, the right inequality follows.

(*Left inequality.*) Let $\mathcal{D}(\mathbb{R}^n)$ denotes the usual Schwartz space of compactly supported, $C^\infty(\mathbb{R}^n)$ test functions. We choose a bounded uniform partition of unity $\{h(x-l)\}_{l\in\mathbb{Z}^n}$ generated by a function $h \in \mathcal{D}(\mathbb{R}^n)$ such that $|h(0)| = 1$, $|h(x)| \le |h(0)|, \forall x \in \mathbb{R}^n$, and $h(x) = 0$ for $|x| \ge \rho$. For a function $G(x) = \sum_{j\in\Lambda} \sum_k c^j(k) B^j(x-k)$, and any $l \in \mathbb{Z}^n$, we have that

$$\begin{aligned} F_h(G_{p,q})(l) &= \sup_{x\in\mathbb{R}^n} |h(x-l) G_{p,q}(x)| \\ &\le \sum_{k\in\mathbb{Z}^n} \sum_{j=1}^{r} \left|c^j(k)\right| \sup_{x\in\mathbb{R}^n} \left|h(x-l) B^j_{p,q}(x-k)\right| \\ &\le \sum_{k\in\mathbb{Z}^n} \sum_{j=1}^{r} \left|c^j(k)\right| F_h\left(B^j_{p,q}\right)(l-k). \end{aligned} \tag{2.20}$$

Since each component $B^j_{p,q}$ belongs to W_1 for $j \in \Lambda$, we use the norms equivalence (2.13) of the Wiener spaces to conclude that the sequence $F_h\left(B^j_{p,q}\right)(k)$ belongs to the sequence space l_1, and that $\left\|B^j_{p,q}\right\|_{W_1} \approx \left\|F_h\left(B^j_{p,q}\right)\right\|_{l_1}$. From this fact, Eq. (2.13), the last inequality of (2.20), Young's inequality for convolutions, and Lemma 2.5 we obtain

$$\|G\|^2_{W(C,L_2(\mathbb{R}^n,T^1_1))} \leq a_1 \max_{j\in\Lambda}\left(\left\|B^j\right\|_{W(C,L_1(\mathbb{R}^n,T^1_1))}\right)\sum_{j=1}^{r}\left\|c^j\right\|^2_{l_2}. \tag{2.21}$$

This last inequality together with the left inequality of (2.5) imply that S is continuously embedded in $W(C, L_2(\mathbb{R}^n, T^1_1))$, which completes the proof. □

Under the conditions of the theorem above, the tensor fields in the space S are continuous. Therefore, they can be sampled. In fact, if we sample $G \in S$, we obtain a discrete tensor field $\Gamma = G\,|_{\mathbb{Z}^n}$ that belongs to $l_2(\mathbb{Z}^n, T^1_1)$. Moreover, the samples depend continuously on G:

THEOREM 2.8. *Let B^j belong to $W(C, L_1(\mathbb{R}^n, T^1_1))$ for each $j \in \Lambda$. Furthermore, assume that the set $\left\{B^j(x-k) : \, j \in \Lambda, \; k \in \mathbb{Z}^n\right\}$ satisfies Condition (2.8). Then, for any $\Delta \in \mathbb{N}^n$, the discretization operator $\mathrm{R}_\Delta : S_\Delta \leftarrow l_2(\mathbb{Z}^n, T^1_1)$ (also called sampling operator) is a bounded linear operator.*

Recall that in the theorem above $\Delta = (\Delta_1, \ldots, \Delta_n)$ must also satisfy $\Delta_i > 0$ for $i = 1, \ldots, n$.

REMARK 2.9. In general, sampling a continuous tensor field belonging to $L_2(\mathbb{R}^n, T^1_1)$ does not even guaranty that the samples belong to $l_2(\mathbb{Z}^n, T^1_1)$, even if the continuous tensor G belongs to a well defined space S of the form (2.1).

PROOF. We choose a bounded uniform partition of unity $\{h(x-l)\}_{l\in\mathbb{Z}^n}$ generated by a function $h \in \mathcal{D}(\mathbb{R}^n)$ such that $|h(0)| = 1$, $|h(x)| \leq |h(0)|, \forall x \in \mathbb{R}^n$, and $h(x) = 0$ for $|x| \geq \rho$. For $G \in W(C, L_2(\mathbb{R}^n, T^1_1))$ we have the pointwise estimate

$$\|G(k)\| = |h(0)|\,\|G(k)\| \leq \sup_{x\in\mathbb{R}^n} |h(x-k)|\,\|G(x)\| = F_h(G)(k), \tag{2.22}$$

which implies that $\sum\limits_k \|G(k)\|^2 \leq \sum\limits_k |F_h(G)(k)|^2$. Using the discrete norm in W_2 defined by (2.13) for each component $G_{p,q}$ together with Lemma 2.5, we get that $\sum\limits_k \|G(k)\|^2 \leq a_1 \|G\|_{W(C,L_2(\mathbb{R}^n,T^1_1))}$ for some constant a_1 independent of G. Finally, using the equivalence of the $L_2(\mathbb{R}^n, T^1_1)$ and $W(C, L_2(\mathbb{R}^n, T^1_1))$ norms (2.19) of Theorem 2.6, we immediately obtain

$$\sum_k \|G(k)\|^2 \leq a_2 \|G\|_{L_2(\mathbb{R}^n,T^1_1)}\,. \tag{2.23}$$

If $G_\Delta \in S_\Delta = \mathrm{D}_\Delta\, S$, then we can replace G by G_Δ in the first part of this proof to obtain $\sum\limits_k \|G_\Delta(k)\|^2 \leq a_3 \|G_\Delta\|_{W(C,L_2(\mathbb{R}^n,T^1_1))}$. Using the fact that the dilation operator is an isomorphism from $W(C, L_2(\mathbb{R}^n, T^1_1))$ into itself, we conclude that for any $G_\Delta \in S_\Delta = \mathrm{D}_\Delta\, S$, we have

$$\begin{aligned}\sum_k \|G_\Delta(k)\|^2 &\le a_3 \|G_\Delta\|_{W(C,L_2(\mathbb{R}^n,T_1^1))}\\ &\le a_4 \|G\|_{W(C,L_2(\mathbb{R}^n,T_1^1))}\\ &\le a_5 \|G\|_{L_2(\mathbb{R}^n,T_1^1)}\\ &\le a_6 \|G_\Delta\|_{L_2(\mathbb{R}^n,T_1^1)}\end{aligned}$$

for some constant a_6 independent of G_Δ. □

The previous theorem states that, under the appropriate conditions, the discretization operator produces discrete tensor fields of $l_2(\mathbb{Z}^n, T_1^1)$. Thus, the range of the discretization operator, $S_\Delta^d = \mathit{Range}(\mathrm{R}_\Delta\, \mathrm{D}_\Delta)$ is a linear subspace of $l_2(\mathbb{Z}^n, T_1^1)$. The topological properties of this space and its structure are inherited from the space S as well as from the properties of the discretization operator. In particular, under the appropriate conditions, the discrete linear space is closed, and has an atomic structure generated by a Riesz basis. Specifically, defining the tensor sequences

$$\Theta_\Delta^j(\cdot) := \mathrm{R}_\Delta\, B_\Delta^j(\cdot) = \mathrm{R}_\Delta\, B^j\left(\frac{\cdot}{\Delta}\right), \tag{2.24}$$

we get:

Theorem 2.10. *If the discretization operator* R *from* S *to* $l_2(\mathbb{Z}^n, T_1^1)$ *is injective and its range is closed, then for any* $\Delta \in \mathbb{N}^n$ *the discrete tensor space* S_Δ^d *is isomorphic to* S.

Remark 2.11. The previous theorem states that if the space S^d produced by sampling the space S is isomorphic to S, then any regular refinement of the sampling also produces an isomorphic space S_Δ^d.

As a corollary of the previous theorem we immediately get

Corollary 2.12. *If the discretization operator* R *from* S *to* $l_2(\mathbb{Z}^n, T_1^1)$ *is injective and its range is closed, then for any* $\Delta \in \mathbb{N}^n$ *the discrete tensor space* S_Δ^d *is closed and the set* $\left\{\Theta_\Delta^j(l - \Delta k) :\ j \in \Lambda, k \in \mathbb{Z}^n\right\}$ *is a Riesz basis of* S_Δ^d.

As before, we have adopted the product convention $\Delta k = (\Delta_1 k_1, \dots, \Delta_n k_n)$.

Proof of Theorem 2.10. Clearly, if the operator R has an inverse R^{-1} from $\mathit{Range}(\mathrm{R}) \subset l_2(\mathbb{Z}^n, T_1^1)$ into S, and if the range is closed, then by the closed graph theorem, the inverse R^{-1} is a bounded operator. Hence R is an isomorphism between the $\mathit{Range}(\mathrm{R})$ and S.

For an element $\Gamma_\Delta = \mathrm{R}_\Delta\, G_\Delta \in S_\Delta^d$ where $G_\Delta \in S_\Delta$ (recall that R_Δ denotes the sampling operator on S_Δ), there exists a unique element $\Gamma = \mathrm{R}\, G \in S^d$ ($G \in S$) such that $\Gamma_\Delta = \Gamma$ on the sublattice $\Delta\mathbb{Z}^n$ ($G_\Delta = \mathrm{D}_\Delta\, G$). In fact, Γ is the restriction of Γ_Δ on $\Delta\mathbb{Z}^n$, i.e., $\Gamma(l) = \Gamma_\Delta(\Delta l)$, $\forall l \in \mathbb{Z}^n$. Thus we have that

$$\|\Gamma\|_{l_2(\mathbb{Z}^n,T_1^1)} \le \|\Gamma_\Delta\|_{l_2(\mathbb{Z}^n,T_1^1)}. \tag{2.25}$$

Moreover, if $\mathrm{R}_\Delta\, G_\Delta = \Gamma_\Delta$, then $G_\Delta = \mathrm{D}_\Delta\, \mathrm{R}^{-1}(\Gamma)$. Thus, we get

$$\begin{aligned}\|G_\Delta\|_{W(C,L_2(\mathbb{R}^n,T_1^1))} = \left\|\mathrm{D}_\Delta\, \mathrm{R}^{-1}\,\Gamma\right\|_{W(C,L_2(\mathbb{R}^n,T_1^1))} &\le a_1 \left\|\mathrm{R}^{-1}\,\Gamma\right\|_{W(C,L_2(\mathbb{R}^n,T_1^1))}\\ &\le a_1 \left\|\mathrm{R}^{-1}\right\| \|\Gamma\|_{l_2(\mathbb{Z}^n,T_1^1)}.\end{aligned}$$

From the previous set of inequalities and (2.25), we immediately conclude that

$$\|G_\Delta\|_{W(C,L_2(\mathbb{R}^n,T_1^1))} \leq \left\|\mathrm{R}^{-1}\right\| \|\Gamma\|_{l_2(\mathbb{Z}^n,T_1^1)} \leq \left\|\mathrm{R}^{-1}\right\| \|\Gamma_\Delta\|_{l_2(\mathbb{Z}^n,T_1^1)}$$

Since Γ is unique and R is injective, we conclude that R_Δ is injective, and–using the last inequality–it has a bounded inverse from $Range(\mathrm{R}_\Delta)$. Thus, by the closed graph theorem, $Range(\mathrm{R}_\Delta)$ is closed. Therefore $S_\Delta^d = Range(\mathrm{R}_\Delta)$ is an isomorphism from $S_\Delta = \mathrm{D}_\Delta\, S$ to S_Δ^d, and hence from S to S_Δ^d which concludes the proof. □

As a corollary we immediately obtain:

Corollary 2.13. *If the discretization operator* R *from* S *to* $l_2(\mathbb{Z}^n, T_1^1)$ *is bijective, then for any* $\Delta \in \mathbb{N}^n$ *we have the equivalence of norms*

$$a_1 \|\mathrm{R}_\Delta\, \mathrm{D}_\Delta\, G\|_{l_2(\mathbb{Z}^n,T_1^1)} \leq \|G\|_{L_2(\mathbb{R}^n,T_1^1)} \leq a_2 \|\mathrm{R}_\Delta\, \mathrm{D}_\Delta\, G\|_{l_2(\mathbb{Z}^n,T_1^1)}\,,$$

i.e., $0 < a_1 \leq a_2$ *are positive constants independent of* G.

Thus, under the conditions of Corollary 2.13, the $L_2(\mathbb{R}^n, T_1^1)$-norm of any tensor field $G \in S$ is equivalent to the discrete $l_2(\mathbb{Z}^n, T_1^1)$-norms of any sampling that is a regular refinement of $\mathbb{Z}^n$.

Theorem 2.10, allows us to conclude that, for any $\Delta \in \mathbb{N}^n$, the space S_Δ^d have the right structure to be an atomic space, without computing the associated autocorrelation matrix

(2.26)
$$[(A_\Delta)_{i,j}](l) = \left\langle \Theta_\Delta^i\,(k)\,, \Theta_\Delta^j\,(k-\Delta l)\right\rangle_d = \sum_{k\in\mathbb{Z}^n} \mathrm{trace}\left(\Theta_\Delta^i\,(k)\,(\Theta_\Delta^j)^*\,(k-\Delta l)\right),$$

or showing that its Fourier transform $\hat{A}_\Delta$ satisfies the requirement (2.7) of Theorem 2.1. In fact, all we need to know is that requirement (2.7) is satisfied for $\hat{A}_1$ (here the subscript 1 denotes $\Delta = (1, 1, \ldots, 1)$) to conclude that it is satisfied for any $\hat{A}_\Delta$ with $\Delta \in \mathbb{N}^n$ (see also Theorem 3.1):

Corollary 2.14. *If the discretization operator* R *from* S *to* $l_2(\mathbb{Z}^n, T_1^1)$ *is bijective (equivalently, if* $\hat{A}_1$ *satisfies (2.7)), then for any* $\Delta \in \mathbb{N}^n$ *the autocorrelation matrix function defined by (2.26) satisfies*

$$a_1 \leq \operatorname*{ess\,inf}_{\nu\in\mathcal{Q}} \left(\lambda_{\min}\left(\widehat{A}_\Delta(\nu)\right)\right) \leq \operatorname*{ess\,sup}_{\nu\in\mathcal{Q}} \left(\lambda_{\max}\left(\widehat{A}_\Delta(\nu)\right)\right) \leq a_2 \qquad a.e. \tag{2.27}$$

for some positive constants $0 < a_1 \leq a_2$ *independent of* ν.

Remark 2.15. The fact that $\hat{A}_\Delta$ satisfies (2.27) is crucial for computing continuous representations of discrete tensor data as will be discussed in the next section.

Remark 2.16. We note that, all our results are also valid for more general tensor field spaces, such as $L_2(\mathbb{R}^n, T_p^q)$. In particular, the results are valid for vector fields defined on $\mathbb{R}^n$.

In summary, in this section we have shown how to construct atomic tensor spaces that consist of contiuonus tensor fields. We have also given conditions on the generating tensors that guaranty that the discrete tensor field spaces generated by any regular sampling of the continuous spaces are atomic and are isomorphic to the continuous spaces.

3. Representation and approximation

If the discretization operator R from $L_2(\mathbb{R}^n, T_1^1)$ to $l_2(\mathbb{Z}^n, T_1^1)$ is bijective, then the interpolation problem is solvable. In particular, given a set of discrete tensors $\{\Phi(k) \ : k \in \mathbb{Z}^n\} \in l_2(\mathbb{Z}^n, T_1^1)$, there exists a tensor field $G^I \in S$ that interpolates $\Phi(k)$, i.e., we can find coefficients $c \in l_2^{(r)}$ such that

$$(G^I(x) \mid_{\mathbb{Z}^n})(l) = \sum_{j=1}^{r} \sum_{k \in \mathbb{Z}^n} c^j(k) B^j(l-k).$$

However, the collected data Φ is often perturbed by noise. Interpolating the data to obtain a continuous representation would not be appropriate, since we would just be finding a continuous representation of the noisy tensor data, rather than of the underlying tensor field. Instead, we should approximate the data by a continuous tensor field $G^{\approx}$ in such a way as to reduce the noise. One way of doing this is by approximating the data by tensor fields in spaces that are coarser than S. In particular, for $\Delta \in \mathbb{N}^n$, the space S_Δ is coarser than S. In fact, since S_Δ is a dilation of S by a factor Δ, we should be able to reduce the variance of an added white noise by a factor proportional to the volume $|\Delta|$ of Δ ($|\Delta| = \Delta_1 \Delta_2 \cdots \Delta_r$). Since we collect discrete data, the norm measuring the error of approximation between the data Φ and the continuous approximation $G_\Delta^{\approx}$ should be measured in terms of the difference between the data and the discretized version (on the data's lattice) of the continuous representation:

P1 <u>Approximation by a continuous tensor field:</u>
Find $G_\Delta^{\approx} \in S_\Delta$ such that the error

$$\mathrm{E}(k) = (\mathrm{R}_\Delta \, G_\Delta^{\approx} - \Phi)(k)$$

is minimum in the $l_2(\mathbb{Z}^n, T_1^1)$-norm.

As before, R_Δ denotes the sampling operator on S_Δ. A related problem is given by:

P2 <u>Approximation by a discrete tensor field:</u>
Find the best approximation $\Phi^{\approx}$ of Φ in the space S_Δ^d.

If they exist, then the solutions to problems P1 and P2 are related by

$$\Phi^{\approx} = \mathrm{R}_\Delta \, G_\Delta^{\approx}.$$

If the discretization operator R is a bijection between S and $l_2(\mathbb{Z}^n, T_1^1)$, then by Theorem 2.12 the space S_Δ^d is closed for any $\Delta \in \mathbb{N}^n$. Thus the best approximation in this space is well defined. In particular, the error $\mathrm{E} = \Phi^{\approx} - \Phi$ must be orthogonal to the basis vectors $\left\{\Theta_\Delta^j(l - \Delta k) : \ j \in \Lambda, \ k \in \mathbb{Z}^n\right\}$ (recall that $\Theta_\Delta^j(l) = B^j(x/\Delta)\big|_{x=l}$):

$$\left\langle (\Phi^{\approx} - \Phi)(\cdot), \Theta_\Delta^j(\cdot - \Delta k) \right\rangle_d = 0, \tag{3.1}$$

where, $\langle \cdot, \cdot \rangle_d$ is the inner product in $l_2(\mathbb{Z}^n, T_1^1)$ defined by

$$\langle \Phi(\cdot), \Psi(\cdot) \rangle_d = \sum_{k \in \mathbb{Z}^n} \mathrm{trace}(\Phi \Psi^*)(k).$$

Since $\Phi^{\approx}$ belongs to S^d_Δ, it must be of the form

$$\Phi^{\approx}(l) = \sum_{j=1}^{r} \sum_{k\in\mathbb{Z}^n} c^j(k)\Theta^j_\Delta(l-\Delta k).$$

Combining this fact with equation (3.1), we get

$$\sum_{p\in\mathbb{Z}^n} c^i(p) \left\langle \Theta^i_\Delta\left(\cdot-\Delta p\right), \Theta^j_\Delta\left(\cdot-\Delta l\right)\right\rangle = \Gamma^j(l), \tag{3.2}$$

where the right hand side $\Gamma^j(l) = \left\langle (\Phi)(\cdot), \Theta^j_\Delta(\cdot-\Delta l)\right\rangle_d$. Using the $r\times r$ autocorrelation matrix-sequence A_Δ defined by (2.26) we can rewrite (3.2) as the generalized multivariate convolution equation

$$A_\Delta *_n c = \Gamma. \tag{3.3}$$

where $c = (c_1, \dots, c_r)^T$, and $\Gamma = (\Gamma_1, \dots, \Gamma_r)^T$, and where the generalized multivariate convolution equation in (3.3) is defined to be

$$\sum_{q=1}^{r} \sum_{k\in\mathbb{Z}^n} (A_\Delta)_{p,q}(l-k)c^q(k) = \Gamma_p(l).$$

A solution c of (3.3) exists and is unique if and only if the convolution inverse A_Δ^{-1} exists, and it is a bounded operator from $l_2^{(r)}$ into itself, i.e., if there exists a matrix sequence A_Δ^{-1} such that

$$(A_\Delta^{-1} *_n A_\Delta)_{p,q}(l) = \sum_{i=1}^{r} \sum_{k\in\mathbb{Z}^n} (A_\Delta)_{p,i}(l-k)(A_\Delta)_{i,q}(k) = \delta_0(l)\mathbf{I},$$

and if $A_\Delta^{-1} *_n \bullet$ defines a bounded operator on $l_2^{(r)}$ (here $\mathbf{I}$ is the $r\times r$ identity matrix, and $\delta_0(0) = 1,\ \delta_0(k) = 0$ for $k \neq 0$). Such a convolution inverse exists if and only if the Fourier transform of A_Δ is bounded and has a bounded inverse [**1**, Theorem 2.2]. This is precisely the condition (2.27) of Corollary 2.14. Thus we have

THEOREM 3.1. *Let B^j belong to $W(C, L_1(\mathbb{R}^n, T^1_1))$, $j \in \Lambda$, and assume that the set $\left\{B^j(x-k) :\ j\in\Lambda\ , k\in\mathbb{Z}^n\right\}$ satisfies condition (2.8) of Corollary (2.2). If the discretization operator (or sampling operator)* R *from S to $l_2(\mathbb{Z}^n, T^1_1)$ is bijective (equivalently, if $\hat{A}_1$ satisfies (2.7)), then for each $\Delta\in\mathbb{N}^n$, the operator A_Δ has a convolution inverse A_Δ^{-1} that defines a bounded linear operator from $l_2^{(r)}$ to $l_2^{(r)}$, and the approximation problem in S^d_Δ has a unique solution.*

The coefficients $\{c^j(k) : j \in \Lambda, k \in \mathbb{Z}^n\}$ which are solutions to Eq. (3.3) are the coefficients of the basis $\left\{\Theta^j_\Delta(l-\Delta k) :\ j\in\Lambda,\ k\in\mathbb{Z}^n\right\}$ that give the best approximation of the discrete data Φ by a tensor sequence $\Phi^{\approx}$ in the space S^d_Δ. Thus, they give the unique solution to Problem P2. Hence, from our previous discussion, these coefficients are also the coefficients for the continuous tensor field

$$G^{\approx}_\Delta(x) = \sum_{j\in\Lambda} \sum_{k\in\mathbb{Z}^n} c^j(k)B^j(\frac{x}{\Delta}-k)$$

that solves Problem P1. Thus, we have

Corollary 3.2. *Let B^j belong to $W(C, L_1(\mathbb{R}^n, T_1^1))$, $j \in \Lambda$, and assume that the set $\left\{B^j(x-k) :\ j \in \Lambda\ , k \in \mathbb{Z}^n\right\}$ satisfies condition (2.8) of Corollary (2.2). If the discretization operator (or sampling operator) R from S to $l_2(\mathbb{Z}^n, T_1^1)$ is bijective (equivalently, if $\hat{A}_1$ satisfies (2.7)), then for each $\Delta \in \mathbb{N}^n$, the solution to problem P1 (or P2) exists and is unique.*

Remark 3.3.

1. If we change the discrete inner product $\langle \cdot, \cdot \rangle_d$ in $l_2(\mathbb{Z}^n, T_1^1)$ to a new shift invariant inner product $\langle \Phi(\cdot), \Psi(\cdot) \rangle_d^{\sim}$ (i.e., $\langle \Phi(\cdot - l), \Psi(\cdot - l) \rangle_d^{\sim} = \langle \Phi(\cdot), \Psi(\cdot) \rangle_d^{\sim}$) that induces an equivalent norm, then all the results are still valid, but the matrix sequence A_Δ in (3.3) must be replaced by the matrix sequence $\tilde{A}_\Delta$ that reflects the new discrete norm.
2. If instead we change the discrete inner product to a spatialy-weighted inner product (for this case the inner product is not shift invariant, in general), then all the results are still valid and problems P1 (or P2) have a unique solution obtained by solving (3.2) directly (instead of (3.3) which is no longer valid). Thus, for this case, the computational complexity for finding the solution may be higher.

3.1. Fast filtering implementation. Because of the particular structure of the spaces S_Δ, and S_Δ^d, computing the solution to the approximation and interpolation problems consists only of simple convolution and addition operations. These linear operations have been studied extensively in signal and image processing because of their fast and efficient implementation and their satisfying interpretation using linear systems theory. In particular, for data of length L the complexity of the algorithms are of order L. Moreover, by using the present framework for representing and approximating tensor data, many signal and image processing operations can be performed digitally using fast filtering algorithms with a complexity of Order L, e.g., rotation translation, dilation, affine transformation, and geometric transformation in general [**1, 2, 17, 18**].

Remark 3.4.

1. An alternative method for approximating the noisy data $\{\Phi(k)\ \ : k \in \mathbb{Z}^n\} \in l_2(\mathbb{Z}^n, T_1^1)$ is to first interpolate it by a tensor field $G^I \in S(B^I)$, and then project G^I onto a coarse space $S_\Delta(B)$ to obtain the representation $G_\Delta^{\approx}$ (note that the generating tensor fields B^I and B need not be related). This method also produces a multivariate convolution equation similar to (3.3), and therefore can also be solved with fast filtering implementations.
2. We can also use a regularization operator to control the approximation $G_\Delta^{\approx}$. For example, we can require the tensor field $G_\Delta^{\approx}$ to minimize the functional
$$\left\|\mathrm{R}_\Delta\, G_\Delta^{\approx} - \Phi\right\|^2 + \lambda \int\limits_{\mathbb{R}^n} \left\|L G_\Delta^{\approx}(x)\right\|^2 dx$$
where L is a linear differential operator. For this case, and under the appropriate conditions, we also obtain multivariate convolution equation similar to (3.3), and solutions that can be implemented with fast filtering algorithms.

3.1.1. *Choice of vector and tensor bases.* As mentioned in Remark 2.16, our results are valid for any tensor field space $L_2(\mathbb{R}^n, T_p^q)$. In particular, the results are valid for vector fields defined on $\mathbb{R}^n$, including discrete measurements of velocity fields, and chemical or optical spectra [**7, 10, 6, 15**]. For these cases, the generating vectors can be chosen to be of the form $B^1(x) = b^1(x)(1, 0, \dots, 0)^T$,

$B^2(x) = b^2(x)(0,1,0,\dots,0)^T$, ..., $B^n(x) = b^n(x)(0,\dots,0,1)^T$, where $b^j(x)$ are scalar functions.

For DT-MR images, the representation spaces must consist of symmetric tensor fields. For this case, we can choose generating tensor functions of the form

$$B^1(x) = b^1(x) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B^2(x) = b^2(x) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B^3(x) = b^3(x) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, B^4(x) = b^4(x) \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B^5(x) = b^5(x) \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, B^6(x) = b^6(x) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

where, again, $b^i(x)$ are scalar valued functions.

For the choices of generating functions above, the matrix sequence $A_\Delta(k)$ in (3.3) is a sequence of diagonal matrices. Thus, the convolution operator $A_\Delta *_n \bullet$ reduces to r independent scalar convolution operators. Hence, for these cases, the approximation problems for vectors or tensors reduce to r scalar independent approximation problems.

A further simplification is possible if we can use separable functions for the $b^i(x)$ so that $b^i(x) = b^i(x_1)\ b^i(x_2)\ \dots\ b^i(x_n)$. Then, the approximation problem for vectors or tensors can be decoupled for each spatial dimension.

Choosing $b^i(x)$ to have compact support or at least exponential decay, is useful for the digital implementation of image processing algorithms, e.g., polynomial splines of order n. For example, the infinite sum

$$G^{\approx}(x_0) = \sum_{j\in\Lambda}\sum_{k\in\mathbb{Z}^n} c^j(k)B^j(x_0-k)$$

for the evaluation of $G^{\approx}$ at an arbitrary sampling point $x_0 \in \mathbb{R}^n$ becomes finite.

In summary, we have constructed Wiener amalgam tensor spaces $S_\Delta \subset L_2(\mathbb{R}^n, T_1^1)$ that are appropriate for multiresolution representation and processing of discrete tensor and vector field data. The practical importance of this work is that it enables one to represent and analyze an entire class of discrete data sets to which these powerful methods have not been applied, and do so using fast linear filtering algorithms.

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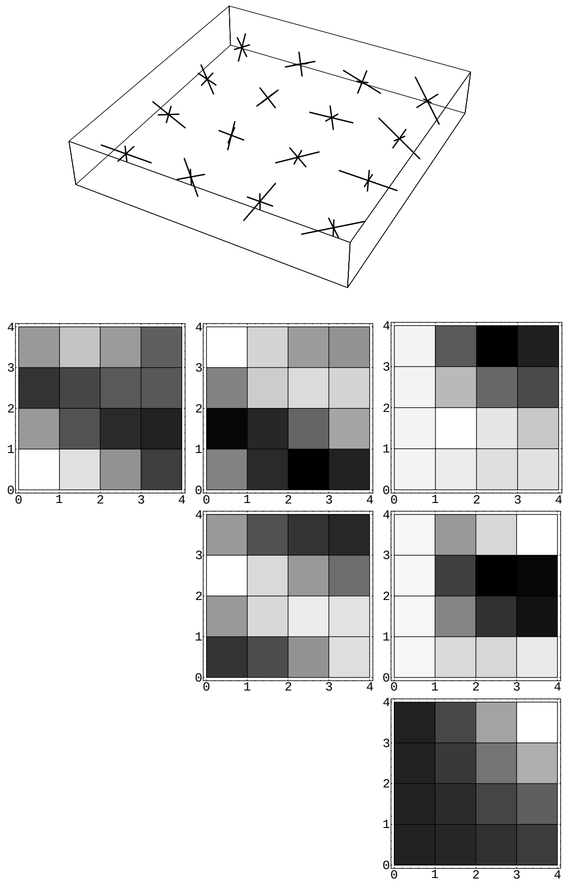

FIGURE 1. Synthetic tensor image: (Top) A 4 x 4 pixel image intended to illustrate important features of diffusion tensor MRI data sets, which are intended to characterize water diffusion in heterogeneous, anisotropic media. The orientation and length of the three principal axes of the diffusion ellipsoid are depicted in each voxel. (Bottom) The corresponding 4 x 4 pixel images displaying the six independent components of the 3 x 3 diffusion tensor, which were used to construct the diffusion ellipsoid image depicted in the top. These six computed scalar grayscale images represent numerical values of the components of the diffusion tensor. Since the diffusion tensor can be represented as a symmetric matrix, only six of the nine components need to be displayed. For off-diagonal elements, which can be positive or negative, white corresponds to 5 and black to -5. For the diagonal elements, whose values are always positive, white corresponds to 10 and black corresponds to 0.

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