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Communication

Analytically exact correction scheme for signal extraction from noisy magnitude MR signals

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Abstract

An analytically exact method is proposed to extract the signal intensity and the noise variance simultaneously from noisy magnitude MR signals. This method relies on a fixed point formula of signal-to-noise ratio (SNR) and a correction factor. The correction factor, which is a function of SNR, establishes a fundamental link between the variance of the magnitude MR signal and the variance of the underlying Gaussian noise in the two quadrature channels. A more general but very similar method is developed for parallel signal acquisitions with multiple receiver coils. In the context of MR imaging, the proposed method can be carried out on a pixel-by-pixel basis if the mean and the standard deviation of the magnitude signal are available.
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Keywords: Magnitude MR noise; Rician; Noise; Gaussian; Signal-to-noise; SNR fixed point

1. Introduction

The basic formulation of noise in magnitude MR images and its related numerical computation was first investigated by Henkelman [1], who provided a numerical look-up table for his correction scheme. Pure Johnson noise in magnitude images can be described by the Rayleigh distribution. The Rayleigh distribution, a special case of the Rician distribution, was used by Edelstein et al. [2] in the context of signal-to-noise (SNR) calibration for NMR imaging systems. Later, a more general framework of noise analysis and detectability of signals in MR images using the Rician distribution was investigated by Bernstein et al. [3] whose results relied on the first two moments of the Rician distribution. These moments of Rician distribution, which will be very important to the present study, were derived by Rice in his seminal work on mathematical analysis of random noise [4]. Besides the mathematical development of noise in MRI, the physical principle of noise in MRI is also of great relevance. This aspect of noise in MRI was investigated by Macovski [5].

Several correction schemes have been proposed to extract or estimate the signal intensity from the magnitude MR signal [6–11]. These schemes provide different approaches to estimating the signal intensity. An unbiased estimator of the signal intensity that depends on the second moment of the magnitude signal was developed simultaneously by McGibney and Smith [6] and Miller and Joseph [7]. However, these methods require prior knowledge of the noise variance. Gudbjartsson and Patz [8] reviewed the Rician distribution of noise in MR images and provided another correction scheme to extract the signal intensity. This correction scheme is similar to that of McGibney and Smith [6] and Miller and Joseph [7], differing from the other two methods only by a small numerical factor and one additional absolute value operation. The method of Gudbjartsson and Patz [8] can be shown to be a special case of our proposed correction scheme at high SNR.

Signal intensity estimation is intricately tied to the noise variance estimation. Simultaneous estimation of the noise variance and the signal intensity from MR magnitude images based on the maximum-likelihood framework has

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been investigated by Sijbers et al. [9–11]. However, this approach requires nonlinear optimization of a 2D function on each pixel. Karlsen et al. [12] proposed a similar framework for T_1 and perfusion measurements.

In this study, we propose a fresh approach for estimating the signal intensity and the noise variance simultaneously from the magnitude MR signals. This correction scheme is to analytically exact in the sense that the correction scheme is derived from two mathematical identities. The first identity is the correction factor that relates the variance of the magnitude signal and the variance of the underlying Gaussian noise in the two quadrature channels. The second identity, a fixed point formula of SNR, connects the first two moments of the Gaussian distribution and those of the Rician distribution, which, in effect, establishes a one-to-one correspondence between the signal-to-noise ratio, $\mathrm{SNR} \equiv \frac{\eta}{\sigma_g}$, and the magnitude signal-to-noise ratio, $\frac{\langle M \rangle}{\sigma_M}$, where η, σ_g, $\langle M \rangle$, and σ_M are the signal intensity, the standard deviation (SD) of noise, the mean of the magnitude MR signal and the SD of the magnitude MR signal, respectively.

In general, fixed point formulae are elegant and can be used to better understand the problem of interest both mathematically and conceptually. But, each fixed point formula is different and, therefore, requires further investigation on its numerical stability. In this study, we discovered that the proposed fixed point formula of SNR may not be optimal for numerical computation at low SNR due to slow convergence. Fortunately, a simple modification, moving from fixed point searching to root finding, can help increase the rate of convergence by at least fivefold at low SNR. This modification is based on the Newton's method of root finding together with simple but important improvements.

We first develop the proposed method for a single receiver coil by demonstrating the connection between the Rician distribution and the formulation provided by Henkelman [1]. Using the first two moments of the Rician distribution, we establish the correction factor. This correction factor underlies much of the present work and will be given due attention in this note. From this relation, the correction scheme can be constructed based on the fixed point formula of SNR. The proposed correction scheme for a single receiver coil is then generalized to multiple receiver coils suitable for parallel MR signal acquisitions.

2. Methods

2.1. Review on formulation of MR noise problem

According to Henkelman [1], the joint probability density of the noise from two quadrature channels can be expressed as

$$p_1(n_r, n_i) = \frac{1}{2\pi\sigma_g^2} \exp\left(-\frac{n_r^2 + n_i^2}{2\sigma_g^2}\right), \tag{1}$$

where n_r and n_i are the noise from the real and complex MR signals with assumed Gaussian distribution of mean zero and standard deviation σ_g. The mean of the magnitude MR signal is

$$\langle M \rangle_{p_1} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{(\eta + n_r)^2 + n_i^2}\, p_1(n_r, n_i)\, \mathrm{d}n_r\, \mathrm{d}n_i, \tag{2}$$

where η is the signal intensity and $M = \sqrt{(\eta + n_r)^2 + n_i^2}$ [1]. The signal intensity is assumed real through a rotation of the quadrature detector [1]. Note that the notation adopted here is different from that of Henkelman. By two simple changes of variables, a linear shift ($n = \eta + n_r$), and a polar coordinate transformation, Eq. (2) can be written as

$$\langle M \rangle_{p_1} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{n^2 + n_i^2}\, \frac{1}{2\pi\sigma_g^2} \exp\left(-\frac{(n-\eta)^2 + n_i^2}{2\sigma_g^2}\right) \mathrm{d}n\, \mathrm{d}n_i \tag{3}$$

$$= \int_0^{\infty} M \underbrace{\frac{M}{\sigma_g^2} \exp\left(-\frac{M^2 + \eta^2}{2\sigma_g^2}\right) I_0\left(\frac{\eta M}{\sigma_g^2}\right)}_{p_r} \mathrm{d}M = \langle M \rangle_{p_r}, \tag{4}$$

where p_r is the Rician probability density and I_0 is the zeroth order Modified Bessel function [4,3,6,14].

2.2. The correction factor ξ

According to Rice [4], the first and second moments of M, $\langle M \rangle_{p_r}$ and $\langle M^2 \rangle_{p_r}$, can be expressed analytically as follows [4,11–13,15]:

$$\langle M \rangle_{p_r} = \frac{1}{2\sigma_g^2}\left(\exp\left(-\frac{\eta^2}{4\sigma_g^2}\right)\sqrt{\frac{\pi}{2}}\sigma_g\left[(\eta^2 + 2\sigma_g^2) I_0\left(\frac{\eta^2}{4\sigma_g^2}\right) + \eta^2 I_1\left(\frac{\eta^2}{4\sigma_g^2}\right)\right]\right) \tag{5}$$

and

$$\langle M^2 \rangle_{p_r} = 2\sigma_g^2 + \eta^2, \tag{6}$$

where I_1 is the first order modified Bessel function. Based on Eqs. (5) and (6), the variance of the magnitude signal M, $\sigma_r^2 \equiv \langle M^2 \rangle_{p_r} - \langle M \rangle_{p_r}^2$, can be expressed simply as

$$\sigma_r^2 = \xi(\theta)\sigma_g^2 \tag{7}$$

by factoring out σ_g^2 and by the substitution of $\theta \equiv \frac{\eta}{\sigma_g} \equiv \mathrm{SNR}$. The correction factor ξ is defined as

$$\xi(\theta) = 2 + \theta^2 - \frac{\pi}{8} \times \exp\left(-\frac{\theta^2}{2}\right)\left((2 + \theta^2) I_0\left(\frac{\theta^2}{4}\right) + \theta^2 I_1\left(\frac{\theta^2}{4}\right)\right)^2. \tag{8}$$

As mentioned in Section 1, the relation between the variance of the magnitude signal and the variance of the Gaussian noise, which is shown in Eq. (7), is very important to the present work for it underlies most of the results established later. Note that the correction factor $\xi(\theta)$ is an increasing function of $\mathrm{SNR} \equiv \theta$, Fig. 1.

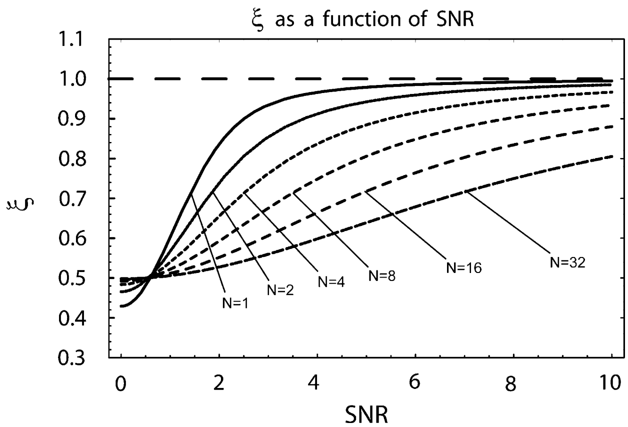

Fig. 1. The correction factor ξ as a function of SNR for different values of N (number of combined channels).

2.3. Fixed point formula of SNR

In this section, we construct a fixed point formula of SNR from the relation established in Eq. (7). Due to experimental constraints, we will assume that the mean and variance of the magnitude signal can measured and that the unknowns are η and σ_g.

We begin by equating Eq. (7) with the definition of the variance of the magnitude signal by writing out the second moment of M

$$2\sigma_g^2 + \eta^2 - \langle M \rangle_{p_r}^2 = \xi(\theta)\sigma_g^2. \tag{9}$$

By factoring out σ_g^2 on both sides of the equation and by substituting η^2/σ_g^2 with θ^2, we arrive at

$$2 + \theta^2 - \langle M \rangle_{p_r}^2/\sigma_g^2 = \xi(\theta). \tag{10}$$

If we replace the remaining σ_g^2 term in Eq. (10) by $\sigma_g^2 = \sigma_r^2/\xi(\theta)$, we arrive at the fixed point formula of SNR:

$$\theta = \sqrt{\xi(\theta)\left[1 + \frac{\langle M \rangle_{p_r}^2}{\sigma_r^2}\right] - 2}. \tag{11}$$

This fixed point formula of SNR has a unique solution for all values of $\frac{\langle M \rangle_{p_r}}{\sigma_r}$ such that $\frac{\langle M \rangle_{p_r}}{\sigma_r} \geqslant \sqrt{\frac{\pi}{4-\pi}} = 1.9130$; this lower bound is the limit at $\theta = 0$. Given $\langle M \rangle_{p_r}$ and σ_r, we can determine the values of SNR and of the correction factor $\xi(\theta)$ simultaneously by mapping an initial guess of θ, θ_0, iteratively using the right-hand side formula of Eq. (11). For example, define $g(\theta) \equiv \sqrt{\xi(\theta)\left[1 + \frac{\langle M \rangle_{p_r}^2}{\sigma_r^2}\right] - 2}$, this iterative map will always converge in the sense of $|g^i(\theta_0) - \theta_{i-1}| \leqslant \varepsilon$ for some nonnegative integer i and a fixed positive number ε, i.e., $\varepsilon = 1.0 \times 10^{-8}$; g^i denotes composition of function, i.e., $g^m(\theta) \equiv \underbrace{g(\ldots g(g(\theta)))}_{m\ \text{terms}}$. Although this iterative map is simple to understand and implement, it is not optimal due to slow convergence at low SNR. We have implemented an algorithm based on the Newton's method of root finding to speed up convergence. The algorithm is shown in Appendix A. In this algorithm, we have made a few simple but important improvements:

(1) we use a known lower bound in our selection of the initial guess;
(2) we provide an analytical expression for the derivative of the iterative function.

Once the correction factor is determined, we can estimate the signal intensity η from the following expressions:

$$\eta^2 = \langle M \rangle_{p_r}^2 + (\xi(\theta) - 2)\sigma_g^2, \tag{12}$$

or

$$\eta^2 = \langle M \rangle_{p_r}^2 + (1 - 2/\xi(\theta))\sigma_r^2. \tag{13}$$

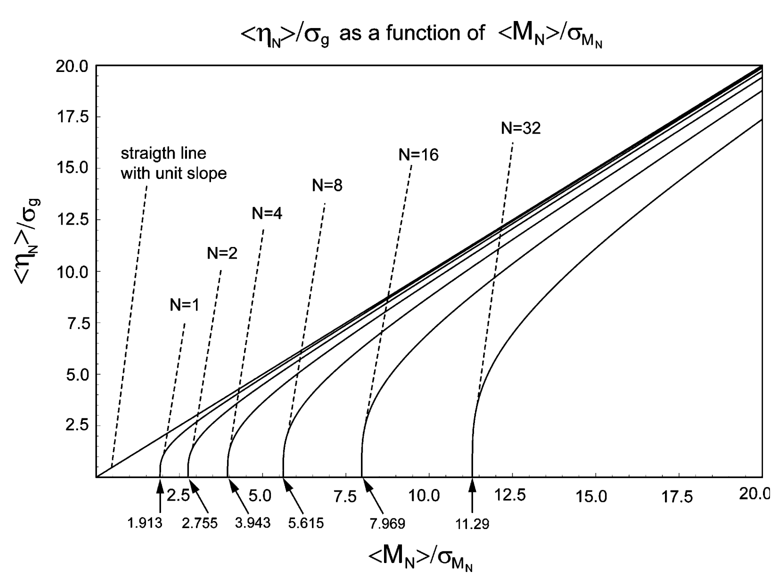

Fig. 2. The estimated SNR as a function of magnitude SNR, $\frac{\langle M_N \rangle}{\sigma_{M_N}}$, for different values of N (number of combined channels).

Table 1
Some statistical properties of the exact correction scheme for various SNR levels

η/σ_g	$\langle\eta\rangle/\sigma_g$	$\sigma_{\langle\eta\rangle}/\sigma_g$
0.	1.49×10^{-8} (1.03)	1.0 (0.35)
0.5	0.5 (1.10)	1.0 (0.42)
1.	1. (1.30)	1.0 (0.59)
1.5	1.5 (1.61)	1.0 (0.79)
2.	2. (2.03)	1.0 (0.96)
2.5	2.5 (2.50)	1.0 (1.04)
3.	3. (3.00)	1.0 (1.07)

The numerical values in parentheses are taken from the approximate scheme of (5).

Fig. 2 shows the estimated signal intensity η in units of σ_g (or SNR) as a function of $\frac{\langle M\rangle_{p_r}}{\sigma_r}$. Comparing Eq. (12) with the approximate correction scheme proposed by Gudbjartsson and Patz [8], $\eta = \sqrt{|\langle M\rangle^2_{p_r} - \sigma^2_g|}$, one can see that the correction scheme of Gudbjartsson et al. is the limiting case of the present approach at high SNR because $\xi \to 1$ for large θ. Therefore, this scheme overestimates the true signal intensity at low SNR because $\xi(\theta)$ is an increasing function of SNR.

It should be noted that as long as $\langle M\rangle_{p_r}$ and σ^2_r can be computed reliably with sufficient precision, the signal intensity can be estimated at low SNR (Table 1, Fig. 2). The results taken from Gudbjartsson and Patz [8] on the mean of the estimated signal intensity for various SNR are compared with the proposed scheme in Table 1. It shows that the proposed scheme better estimates the signal intensity than the scheme by Gudbjartsson and Patz [8].

2.4. Extension to parallel signal acquisitions

In this section, we will generalize the fixed point formula of SNR and the correction scheme to parallel signal acquisitions using multiple receiver coils in a phased array system where the sum-of-squares algorithm is used [16]. The general formulation of the noise problem in multiple channels is well known in communication theory [17] and was introduced to MR imaging by Constantinides et al. [13].

Let the composite magnitude signal be $M_N = \sqrt{\sum_{j=1}^{N}[M^2_{jr} + M^2_{ji}]}$ where N denotes the number of receiver coils, and $M_{jr} = \eta_{jr} + n_{jr}$ and $M_{ji} = \eta_{ji} + n_{ji}$ denote the observed real and imaginary signals reconstructed from the jth receiver coil, respectively. Further, the signal intensities contributing to the real and the imaginary signals of the jth receiver coil are denoted, respectively, by η_{jr} and η_{ji}. The noise from both the real and imaginary parts of the magnitude signal for all j, n_{jr} and n_{ji}, are assumed to have the same standard deviation σ_g. The goal of the correction scheme is to obtain the combined signal intensity $\eta_N \equiv \sqrt{\sum_{j=1}^{N}[\eta^2_{jr} + \eta^2_{ji}]}$.

The probability density of M_N, also known as the non-central Chi distribution, can be written as [13,17]:

$$p(M_N) = \frac{\eta_N}{\sigma^2_g}\left(\frac{M_N}{\eta_N}\right)^N \mathrm{Exp}\left(-\frac{(\eta^2_N + M^2_N)}{2\sigma^2_g}\right) I_{N-1}\left(\frac{M_N\eta_N}{\sigma^2_g}\right). \tag{14}$$

The first and the second moments of M_N are [13,15]

$$\langle M_N\rangle = \sqrt{\frac{\pi}{2}}\frac{(2N-1)!!}{2^{N-1}(N-1)!}\,{}_1F_1\left(-\frac{1}{2}, N, -\frac{\eta^2_N}{2\sigma^2_g}\right)\sigma_g, \tag{15}$$

and

$$\langle M^2_N\rangle = 2N\sigma^2_g + \eta^2_N, \tag{16}$$

respectively, where ${}_1F_1$ is the confluent hypergeometric function and the double factorial is defined as follows: $n!! = n(n-2)(n-4)\times\cdots$ [18,19].

Let $\beta_N = \sqrt{\frac{\pi}{2}}\frac{(2N-1)!!}{2^{N-1}(N-1)!}$ and SNR $\equiv \theta \equiv \frac{\eta_N}{\sigma_g}$, the correction factor ξ is now a function of both the SNR and the number of receiver coils. This correction factor $\xi(\theta, N)$ can be derived from the variance of M_N, $\sigma^2_{M_N}$:

$$\sigma^2_{M_N} \equiv \langle M^2_N\rangle - \langle M_N\rangle^2 = \xi(\theta, N)\sigma^2_g, \tag{17}$$

where

$$\xi(\theta, N) = 2N + \theta^2 - \beta^2_N\left[{}_1F_1\left(-\frac{1}{2}, N, -\frac{\theta^2}{2}\right)\right]^2. \tag{18}$$

The correction factor for different numbers of receiver coils is plotted in Fig. 1.

Based on Eqs. (17) and (18) again, the fixed point formula of SNR can be established as follows:

$$\begin{aligned}
&\langle M^2_N\rangle - \langle M_N\rangle^2 = \xi(\theta, N)\sigma^2_g \\
&\iff 2N\sigma^2_g + \eta^2_N - \langle M_N\rangle^2 = \xi(\theta, N)\sigma^2_g \\
&\iff 2N + \theta^2 - \frac{\langle M_N\rangle^2}{\sigma^2_g} = \xi(\theta, N) \\
&\iff 2N + \theta^2 = \xi(\theta, N)\left(1 + \frac{\langle M_N\rangle^2}{\sigma^2_{M_N}}\right) \\
&\iff \theta = \sqrt{\xi(\theta, N)\left(1 + \frac{\langle M_N\rangle^2}{\sigma^2_{M_N}}\right) - 2N}.
\end{aligned} \tag{19}$$

When $N = 1$, Eq. (19) reduces to Eq. (11). This fixed point formula has a unique solution for all $\frac{\langle M_N\rangle}{\sigma_{M_N}} \geqslant \sqrt{\frac{2N}{\xi(0,N)} - 1}$. This lower bound is derived by setting Eq. (19) to zero, Fig. 2. Please refer to Appendix A for Newton's method of root finding together with specific implementation details. Once the SNR is determined, the composite signal intensity can be obtained by solving the following equation:

$$\eta^2_N = \langle M_N\rangle^2 + \left(1 - \frac{2N}{\xi(\theta, N)}\right)\sigma^2_{M_N}. \tag{20}$$

3. Discussion and conclusion

We have provided an analytically exact method for simultaneous estimation of the signal intensity and the noise variance from noisy magnitude MR signals. The proposed method depends on the fixed point formula of SNR and the correction factor ξ. In the case of $N = 1$, the qualitative behavior and limitation of other approximate correction schemes were described in light of the proposed scheme. We have also provided a specific implementation of a root-finding algorithm to speed up convergence.

It should be emphasized here that the method of averaging used in $\langle M \rangle_{p_r}$ or $\langle M_N \rangle$ is an ensemble average over the noise fluctuations, which involves repeated measurements. This important issue was correctly pointed out by Andersen [20]. In the context of MR imaging, if an ensemble of M is available, then, in principle, we can compute SNR estimate on a pixel-by-pixel basis. If the assumption is not valid then spatial (or ROI-based) averaging may be used. But, ROI-based averaging should be used with great caution because it is only applicable to objects having homogeneous signals within the region of interest.

Finally, this correction scheme will be useful to applications in MR spectroscopic signal processing and MR image processing as well as MR system calibration where magnitude signals are processed and analyzed [2,21–27]. Specifically, the proposed method provides a foundation for quantitative comparison of parallel imaging systems with different receiver coils. This is a topic of interest that is under investigation.

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Appendix A

In this appendix, we will provide a specific algorithm for finding the fixed point of SNR based on the Newton's method of root finding.

Define $r \equiv \frac{\langle M_N \rangle}{\sigma_{M_N}}$ and assume that r and N are known. Let $\theta = g(\theta) = \sqrt{\xi(\theta, N)\left(1 + \frac{\langle M_N \rangle^2}{\sigma^2_{M_N}}\right) - 2N}$ and $f(\theta) = g(\theta) - \theta = 0$. The goal of this algorithm is to find the root of f. The Newton's method of finding the root of $f(\theta)$ begins with the following iterations:

$$\theta_{n+1} = \theta_n - \frac{f(\theta_n)}{f'(\theta_n)} \quad \text{for } n \geqslant 0, \tag{A.1}$$

where f' denotes the derivative of f with respect to θ. The analytical expression of f' can be obtained quite easily due to the following relation ${}_1F_1'(a, b, c\theta^m) = \frac{acm\theta^{m-1}}{b}\,{}_1F_1(a+1, b+1, c\theta^m)$ [15,18,19]. With some algebraic manipulation, Eq. (A.1) can be written as

$$\theta_{n+1} = \theta_n - \frac{g(\theta_n)(g(\theta_n) - \theta_n)}{\theta_n(1+r^2)\left[1 - \frac{\beta(N)^2}{2N}\,{}_1F_1\left(-\frac{1}{2}, N, -\frac{\theta^2}{2}\right){}_1F_1\left(+\frac{1}{2}, N+1, -\frac{\theta^2}{2}\right)\right] - g(\theta_n)}. \tag{A.2}$$

For simplicity, define

$$k(\theta_n, N, r) \equiv \theta_n - \frac{g(\theta_n)(g(\theta_n) - \theta_n)}{\theta_n(1+r^2)\left[1 - \frac{\beta(N)^2}{2N}\,{}_1F_1\left(-\frac{1}{2}, N, -\frac{\theta^2}{2}\right){}_1F_1\left(+\frac{1}{2}, N+1, -\frac{\theta^2}{2}\right)\right] - g(\theta_n)} \tag{A.3}$$

and

$$LowerBound\ (N) \equiv \sqrt{\frac{2N}{\xi(0, N)} - 1}. \tag{A.4}$$

The algorithm for finding the root of f is shown below:

Newton's method of root finding:

```
RootFinder (r, N) {    r and N are inputs
   if (r ⩽ LowerBound (N)) return 0.0;

   max = 500;    maximum iteration
   ε = 1.0 × 10^-8;    tolerance
   t0 = r − LowerBound (N);    initial guess
   t1 = k(t0,N,r);

   while(|t1 − t0| > ε){
      t0 = t1;
      t1 = k(t0,N,r);
      max = max − 1;
      if (max < 0) break;
   }
   return t1;    return the computed result
}
```

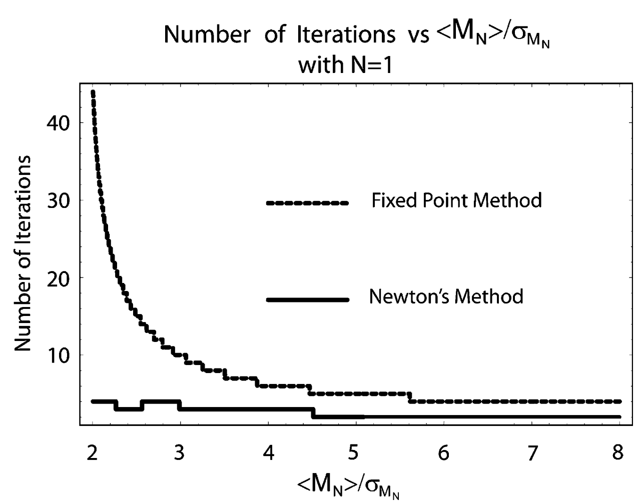

Fig. 3. The number of iterations needed to reach convergence for both the Newton's method and the fixed point method as a function of magnitude SNR, $\frac{\langle M_N \rangle}{\sigma_{M_N}}$, for $N = 1$. This plot shows that the fixed point method, which uses the iterative function g shown in Appendix A, may not be appropriate for finding the fixed point at low magnitude SNR. With a simple modification, the Newton's method of root finding can be constructed to overcome this issue of slow convergence. The plot shows that the Newton's method is uniformly better than the fixed point method.

In fact, the iterative function k in the algorithm can be replaced by the iterative function g for carrying out the fixed point estimation but the number of iterations needed to reach convergence will be much higher at low SNR. Fig. 3 shows the comparison between the Newton's method, which uses the iterative function k, and the fixed point formula, which uses the iterative function g. This comparison is carried out at $N = 1$ with the same numerical tolerance as shown in the algorithm. Fig. 3 shows that the Newton's method is uniformly better than the fixed point method.

References

[1] R.M. Henkelman, Measurement of signal intensities in the presence of noise in MR images, Med. Phys. 12 (2) (1985) 232–233, Erratum in 13 (1986) 544.
[2] W.A. Edelstein, P.A. Bottomley, L.M. Pfeifer, A signal-to-noise calibration procedure for NMR imaging systems, Med. Phys. 11 (2) (1984) 180–185.
[3] M.A. Bernstein, D.M. Thomasson, W.H. Perman, Improved detectability in low signal-to-noise ratio magnetic resonance images by means of a phase-corrected real reconstruction, Med. Phys. 15 (5) (1989) 813–817.
[4] S.O. Rice, Mathematical Analysis of Random Noise. Bell System Technical Journal, 1944, vols. 23 and 24. (Reprinted by Wax N. "Selected Papers on Noise and Stochastic Processes", Dover Publications 1954).
[5] A. Macovski, Noise in MRI, Magn. Reson. Med. 36 (1996) 494–497.
[6] G. McGibney, M.R. Smith, An unbiased signal-to-noise ratio measure for magnitude resonance images, Med. Phys. 20 (4) (1993) 1077–1078.
[7] A.J. Miller, P.M. Joseph, The use of power images to perform quantitative analysis on low SNR MR images, Magn. Reson. Imag. 11 (1993) 1051–1056.
[8] H. Gudbjartsson, S. Patz, The Rician distribution of noisy MRI data, Magn. Reson. Med. 34 (1995) 910–914, Erratum in Magn. Reson. Med. 36(2) (1996) 332–333.
[9] J. Sijbers, A.J. Den Dekker, P. Scheunders, D. Van Dyck, Maximum-likelihood estimation of Rician distribution parameters, IEEE Trans. Med. Imag. 17 (3) (1998) 357–361.
[10] J. Sijbers, A.J. Den Dekker, J.V. Van Audekerke, M. Verhoye, D. Van Dyck, Estimation of the noise in magnitude MR images, Magn. Reson. Imag. 16 (1) (1998) 87–90.
[11] J. Sijbers, A.J. Dekker, Maximum likelihood estimation of signal amplitude and noise variance from MR data, Magn. Reson. Med. 51 (2004) 586–594.
[12] O.T. Karlsen, R. Verhagen, M.M.J. Bovée, Parameter estimation from Rician-distributed data sets using a maximum likelihood estimator: application to T_1 and perfusion measurements, Magn. Reson. Med. 41 (1999) 614–623.
[13] C.D. Constantinides, E. Atalar, E.R. McVeigh, Signal-to-noise measurements in magnitude images from NMR phased arrays, Magn. Reson. Med. 38 (1997) 852–857, Erratum in Magn. Reson. Med. 52 (2004) 219.
[14] A. Papoulis, Probability, Random Variables, and Stochastic Processes, McGraw-Hill, New York, 1965.
[15] S. Wolfram, The Mathematica Book, fifth ed., Wolfram Media, 2003.
[16] P.B. Roemer, W.A. Edelstein, C.E. Hayes, S.P. Souza, O.M. Mueller, The NMR phased array, Magn. Reson. Med. 16 (1990) 192–225.
[17] A.D. Whalen, Detection of Signals in Noise, Academic Press, New York, 1971.
[18] G. Arfken, Mathematical Methods for Physicists, second ed., Academic Press, New York, 1985.
[19] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions, Dover Publications, New York, 1974.
[20] A.H. Andersen, On the Rician distribution of noisy MRI data, Mag. Reson. Med. 36 (2) (1996) 331–332.
[21] L.C. Chang, G.K. Rohde, C. Pierpaoli, An automatic method for estimating noise-induced signal variance in magnitude-reconstructed magnitude resonance images, Prog. Biomed. Opt. Imaging 6 (No. 24) (2005) 1136–1142.
[22] P.T. Callaghan, D. MacGowan, K.J. Packer, F.O. Zelaya, J. Magn. Reson. 90 (1990) 177–182.
[23] D. LeBihan, Diffusion NMR imaging, Magn. Reson. Q. 7 (1991) 1–30.
[24] P.J. Basser, J. Mattiello, D. LeBihan, MR diffusion tensor and imaging, Biophys. J. 66 (1994) 259–267.
[25] K.P. Pruessmann, M. Weiger, M.B. Scheidegger, P. Boesiger, SENSE: sentivity encoding for fast MRI, Magn. Reson. Med. 42 (1999) 952–962.
[26] W.E. Kyriakos, L.P. Panych, D.F. Kacher, C.F. Westin, S.M. Bao, R.V. Mulkern, F.A. Jolesz, Sensitivity profiles from an array of coils for encoding and reconstruction in parallel (SPACE RIP), Magn. Reson. Med. 44 (2000) 301–308.
[27] P. Kellman, E.R. McVeigh, Image reconstruction in SNR units: a general method for SNR measurement, Magn. Reson. Med. 54 (2005) 439–447.