Unified Mathematical Model of q-Space & Diffusion Tensor Imaging

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Background

- Random H_2O movement in the presence of magnetic field gradients $B_z = B_0 + \mathbf{x}\cdot\mathbf{G}(t)$ causes signal decay (due to dephasing)
- **Diffusion Tensor Imaging**
  - Models the distribution of random movements by non-isotropic Gaussian probability density / point spread function (PSF)
    - $P(\mathbf{x},t;\mathbf{D}) = \exp[-\mathbf{x}\cdot\mathbf{D}^{-1}\cdot\mathbf{x}/4t] \big/ [(4\pi t)^{3/2}\det\mathbf{D}]$
  - Can analytically calculate signal decay E for any gradient history [$d\mathbf{q}/dt = \gamma\mathbf{G}(t)$] from this model: $\ln(E) = -\int_0^\Delta \mathbf{q}(t)\cdot\mathbf{D}\cdot\mathbf{q}(t)\,dt$
    - $\mathbf{q}$ denotes spatial frequency during diffusion encoding phase of imaging, distinct from $\mathbf{k}$ during image encoding and readout
    - Δ is the duration of the diffusion encoding phase
  - DTI then estimates $\mathbf{D}$ by measuring $E\{\mathbf{q}(t)\}$ for a number of different trajectories through q-space, and then fitting the 6 parameters in $\mathbf{D}$ to the set of E values
- ***q*-Space Imaging**
  - General PSF model for distribution of random movements of magnetization [arbitrary $P(\mathbf{x},\Delta)$]
  - **Drawback**: Can only analytically calculate signal decay E for impulsively large gradients [$q(t) \equiv$ square wave]
  - Need more data to reconstruct general $P(\mathbf{x},\Delta)$ for each voxel
  - Asymptotically as $\mathbf{q}(t)\to\mathbf{0}$, q-space imaging reproduces DTI [1]
- **Goal**: unify q-space and DTI formalisms further, to allow systematic extensions of DTI-type processing to more general H_2O random transport models

Mathematical Analysis

- $M(\mathbf{x},t) =$ transverse magnetization
- $\hat{M}(\mathbf{q},t) =$ spatial Fourier transform $= \Im_{\mathbf{x}\to\mathbf{q}}[M(\mathbf{x},t)]$
- $P(\mathbf{x},t) =$ point spread function at time t
- $M(\mathbf{x},t) = M(\mathbf{x},0) * P(\mathbf{x},t) =$ transport of magnetization in real space
- $\hat{M}(\mathbf{q},t) = \hat{M}(\mathbf{q},0)\cdot\hat{P}(\mathbf{q},t) =$ transport of magnetization in q-space
- Differentiating: $\dfrac{\partial\hat{M}(\mathbf{q},t)}{\partial t} = \dfrac{\partial\hat{P}(\mathbf{q},t)/\partial t}{\hat{P}(\mathbf{q},t)}\cdot\hat{M}(\mathbf{q},t)$
- Define: $\hat{P}(\mathbf{q},t) = \exp[-u(\mathbf{q},t)]$ so $\dfrac{\partial\hat{P}(\mathbf{q},t)/\partial t}{\hat{P}(\mathbf{q},t)} = -\dfrac{\partial u(\mathbf{q},t)}{\partial t} \equiv -u_t(\mathbf{q},t)$
- Magnetization transport in the presence of gradients (generalization of Bloch-Torrey-Stejskal equation for magnetization transport with diffusion):

$$\frac{\partial M(\mathbf{x},t)}{dt} = \underbrace{-i\frac{d\mathbf{q}}{dt}\cdot\mathbf{x}\,M(x,t)}_{\text{phase change: effect of gradients}} + \underbrace{\Im^{-1}_{\mathbf{q}\to\mathbf{x}}\left[-u_t(\mathbf{q},t)\hat{M}(\mathbf{q},t)\right]}_{\text{effect of spatial transport}}$$

- Fourier transforming gives a linear first order (advection) PDE in $(\mathbf{q},t)$ space: $\dfrac{\partial\hat{M}(\mathbf{q},t)}{\partial t} - \dfrac{d\mathbf{q}}{dt}\cdot\nabla_{\mathbf{q}}\hat{M}(\mathbf{q},t) = -u_t(\mathbf{q},t)\hat{M}(\mathbf{q},t)$
- Solution by method of characteristics:

$$\hat{M}\left(\mathbf{q}_0 - \mathbf{q}(t),t\right) = e^{-\int_0^t u_t(\mathbf{q}(\tau),\tau)\,d\tau}\,\hat{M}(\mathbf{q}_0,0) \text{ where } \mathbf{q}_0 \text{ is arbitrary}$$

- Imaging: rewind $\mathbf{q}(\Delta)$ to $\mathbf{q}=\mathbf{0}$ before k-space readout, so magnetization that will be imaged has been attenuated by

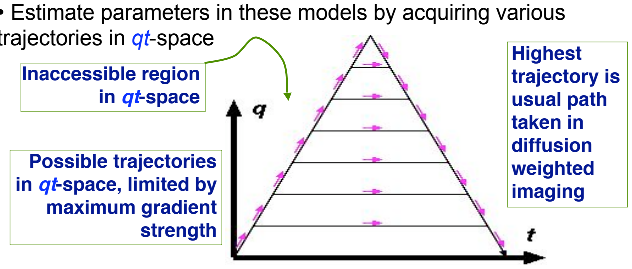
$$\ln(E) = -\int_0^\Delta u_t(\mathbf{q}(t),t)\,dt = \int_0^\Delta\left[\frac{\partial\hat{P}(\mathbf{q}(t),t)/\partial t}{\hat{P}(\mathbf{q}(t),t)}\right]dt$$

Connections

- **Diffusion Tensor Imaging**
  - $\hat{P}(\mathbf{q},t;\mathbf{D}) = \exp[-t\cdot\mathbf{q}\cdot\mathbf{D}\cdot\mathbf{q}]$ so $u_t(\mathbf{q}(t),t;\mathbf{D}) = \mathbf{q}(t)\cdot\mathbf{D}\cdot\mathbf{q}(t)$
  - $\Rightarrow E\{\mathbf{q}(t)\} = \exp\left[-\int_0^\Delta \mathbf{q}(t)\cdot\mathbf{D}\cdot\mathbf{q}(t)\right]dt$
  - This quadratic dependence on $\mathbf{q}(t)$ is the standard result for DTI
- ***q*-Space Imaging**
  - PGSE with $\delta << \Delta$: $\mathbf{q}(t) = \begin{cases}\gamma\delta\mathbf{G} = \text{const} & 0 < t < \Delta \\ \mathbf{0} & \text{otherwise}\end{cases}$
  - $\Rightarrow u(\mathbf{q}(t),t) = -\ln\hat{P}(\gamma\delta\mathbf{G},t)$
  - $\Rightarrow E = e^{-\int_0^\Delta u_t(\gamma\delta\mathbf{G},t)\,dt} = e^{-u(\gamma\delta\mathbf{G},\Delta)} = \hat{P}(\gamma\delta\mathbf{G},\Delta)$
  - The standard PGSE q-space result: depends only on the final PSF and not on the intermediate ($t<\Delta$) $P(\mathbf{x},t)$ functions
- **Generically**
  - $E\{\mathbf{q}(t)\}$ is a tomographic integral through qt-space of the PSF for water transport, depending on $P(\mathbf{x},t)$ at all times $0 < t < \Delta$

Future Potential

- Develop parametrized models for stochastic transport that are more advanced than unrestricted diffusion; for example, [2] and [3]
- Estimate parameters in these models by acquiring various trajectories in qt-space



References

[1] **Basser PJ**. Relationships between diffusion tensor and q-space MRI. *Magn Reson Med* **47**:392-397 (2002).

[2] **Özarslan E**, **Mareci TH**. Generalized diffusion tensor imaging and analytical relationships between diffusion tensor imaging and high angular resolution diffusion imaging. *Magn Reson Med* **50**:955-965 (2003).

[3] **Sato K-I**. *Levy Processes and Infinitely Divisible Distributions.* Cambridge University Press. 1999.