

# Scale and Statistics in Variable Conductance Diffusion

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## Introduction

Non-linear diffusion or Variable Conductance Diffusion (VCD) processes can be used as to filter away noise while preserving boundary information. However, adjustment of the control parameters is not well understood. By approaching the control of VCD systems through a multiscale statistical analysis, information about the image is revealed regarding the strength of boundaries and their influence with regard to the local geometry of the object. This information can then be used to monitor and guide the diffusion process.

## Background

Several approaches to computer vision begin with a set of governing constraints and equations, and construct the mathematics and superstructure for vision based upon these axiomatic principles. One such approach toward vision is based upon diffusion as a plausible model of vision. Grossberg describes a model that includes a boundary detection system and a system by which feature values undergo a process of "filling-in" [Grossberg, 1985]. The properties of diffusion in such a context of visual processing are particularly well principled, and fit this construction well. In particular, the Gaussian curve and its derivatives are solutions to the heat equation, an equation which governs diffusion behavior. The treatment of images through multiple scales has been modeled using Gaussian operators (and Gaussian derivatives) of increasing spatial variance to better understand the nature of images at differing resolution or measurement apertures [Romeny, 1991ab][Florack, 1993].

An extension beyond isotropic diffusion as an analytical tool is the notion of multiscale non-linear or Variable Conductance Diffusion (VCD)[Whitaker, 1993ab]. The work of Whitaker is preceded by the concepts of "anisotropic diffusion," an analytical process where images are often treated at pixel resolution [Perona, 1988]. The fixed inherent scale of this process generates instabilities arising from noise with high spatial frequency. The multiscale properties of VCD allow it to perform smoothing operations in the presence of noise, while preserving boundary information inherent in the image.

The practical aspects of VCD in medicine are being explored at many institutions, where non-linear diffusion filters often serve a pre-processing role before traditional Statistical Pattern Recognition (SPR) classifiers are applied. In particular, filtering mechanisms provided by Guido Gerig [Gerig, 1991ab] are in use at the Harvard Brigham and Women's Surgical Planning Laboratory as part of a classification procedure for the processing of MR and X-ray CT data.

However, the control of VCD systems has often been difficult; the nature of the diffusion parameters are only partly understood. Seemingly insignificant changes in control constants can drastically modify the behavior of the image as the diffusion process progresses. The VCD process as well as the properties of the noise contained within the image can be studied through statistical pattern recognition. Traditional statistical pattern recognition is performed at the maximum outer scale of the image, where histograms and clustering are analyzed across the entire range of observed pixels [Duda, 1973]. A multiscale statistical analysis illuminates elements of the relationship between scale and object shape. By using critical values revealed in the analysis, images may be reconstructed at "natural" boundaries, accounting for the size and shapes of the objects within the image. This approach begins with an adaptation of traditional measurements of distributions of random variables [Olkin, 1980] to a multiscale view of image processing.

## Sampling and Scale

Consider an array of observed values,  $I(x)$ , where for purposes of discussion the location specified by parameter  $x$  is in  $\mathcal{R}^1$ , but can easily be generalized to  $\mathcal{R}^n$ . The values may be sampled using a neighborhood function,  $h(x)$ , using the convolution operation,  $I(x) \otimes h(x)$ , where

$$I(x) \otimes h(x) = \int_{-\infty}^{\infty} h(\tau) I(x - \tau) d\tau$$

We choose our sampling function with care, specifying a function that is invariant with respect to spatial translation and spatial rotation (i.e. is independent of the value and orientation of  $x$ ) [Florack, 1993]. One such family of

functions is the Gaussian (or Normal) distribution and its derivatives. Therefore, let

$$h(x) = G(x, s) = \frac{1}{\sqrt{2\pi}s^2} e^{-\frac{1}{2} \frac{x^2}{s^2}}$$

where the parameter  $s$  represents the width of the sampling aperture, or put another way,  $s$  represents the scale at which measurements of  $I(x)$  are made. One way of interpreting the ideas presented above is through the language of statistics. Specifically, let us consider the measurement made using the Gaussian sampling kernel to be the average or expected value of  $I(x)$  over the neighborhood defined by the aperture value  $s$ . This implies that the local mean over this neighborhood can be defined by

Local Mean:

$$\mu_I(x, s) = E\langle I(x), s \rangle = \sum_{\text{neighborhood}(s)} P(I(x)) I(x) = \int G(x-y, s) I(y) dy$$

where  $E\langle I(x), s \rangle$  is read as the expected value of  $I(x)$  measured with aperture  $s$  and  $P(I(x))$  is the probability of observing the value  $I(x)$ .

Beyond the calculation of a local mean, it is straightforward to calculate a value for the local variance and the related standard deviation over the neighborhood specified by the scale  $s$ .

Local Variance:

$$\begin{aligned} \mu_I^{(2)}(x, s) &= E\left\langle \left( I(x) - \mu_I(x, s) \right)^2, s \right\rangle \\ &= \sum_{\text{neighborhood}(s)} P\left( \left( I(x) - \mu_I(x, s) \right)^2 \right) \left( I(x) - \mu_I(x, s) \right)^2 \\ &= \int G(x-y, s) \left( I(y) - \mu_I(x, s) \right)^2 dy \\ &= \int G(x-y, s) I(y)^2 dy - \left( \mu_I(x, s) \right)^2 \end{aligned}$$

$$\text{Local Standard Deviation: } \sigma_I(x, s) = \sqrt{\mu_I^{(2)}(x, s)}$$

**Other Central Moments:** Observe that the third and fourth local central moments are easily calculated in a similar fashion.

$$\begin{aligned} \mu_I^{(3)}(x, s) &= \int G(x-y, s) \left( I(y) - \mu_I(x, s) \right)^3 dy \\ &= \int G(x-y, s) I(y)^3 dy - 3\mu_I(x, s) \int G(x-y, s) I(y)^2 dy \\ &\quad + 3\left( \mu_I(x, s) \right)^2 \int G(x-y, s) I(y) dy - \left( \mu_I(x, s) \right)^3 \\ &= \int G(x-y, s) I(y)^3 dy - 3\mu_I(x, s) \mu_I^{(2)}(x, s) - \left( \mu_I(x, s) \right)^3 \end{aligned}$$

$$\begin{aligned} \mu_I^{(4)}(x, s) &= \int G(x-y, s) \left( I(y) - \mu_I(x, s) \right)^4 dy \\ &= \int G(x-y, s) I(y)^4 dy \\ &\quad - 4\mu_I(x, s) \int G(x-y, s) I(y)^3 dy \\ &\quad + 6\left( \mu_I(x, s) \right)^2 \int G(x-y, s) I(y)^2 dy \\ &\quad - 4\left( \mu_I(x, s) \right)^3 \int G(x-y, s) I(y) dy \\ &\quad + \left( \mu_I(x, s) \right)^4 \int G(x-y, s) dy \\ &= \int G(x-y, s) I(y)^4 dy - 4\mu_I(x, s) \mu_I^{(3)}(x, s) \\ &\quad - 6\left( \mu_I(x, s) \right)^2 \mu_I^{(2)}(x, s) - \left( \mu_I(x, s) \right)^4 \end{aligned}$$

As specified before, the selection of the Gaussian distribution as the sampling kernel was driven by a desire for the sampling filter to be invariant with respect to particular transformations of  $x$ . It may be desirable to analyze the sampled measurements of the array of  $I(x)$  values in dimensionless units (i.e. invariant with respect to certain transformations of  $I$ ). Note that the third and fourth central moments shown above have dimension equal to the degree of the moment and the units of measurement. (i.e. if the pixel value reflects lumens as a unit of measure, the third central moment is measured in (lumens)<sup>3</sup>) Dimensionless measures may be obtained by normalizing the central moments with multiples/exponents of the standard deviation. The resulting measures are described as *skewness* and *kurtosis*. Their local manifestations, given a sampling aperture  $s$ , may be defined as

$$\text{Local Skewness: } \gamma_I^{(3)}(x, s) = \frac{\mu_I^{(3)}(x, s)}{\left( \sigma_I(x, s) \right)^3}$$

$$\text{Local Kurtosis: } \gamma_I^{(4)}(x, s) = \frac{\mu_I^{(4)}(x, s)}{\left( \sigma_I(x, s) \right)^4}$$

Although these higher moments are of considerable interest, the remainder of this discussion will address the nature of scale, noise and boundary measures as they pertain to VCD.

## Boundaries and Scale

Boundaries of objects of varying size exhibit different responses when measured with differing Gaussian filters (a range of derivative of Gaussian kernels) through changing scale. The Laplacian of the Gaussian, in particular, elicits a medialness response from objects contained within the image with varying scale [Fritsch, 1993].

Figure 1 shows a computer generated phantom image of a teardrop shape. The measured signal to noise ratio (SNR) within that image has been constrained to 4 to 1. The images of Figure 2 are three local statistical measurements made of the teardrop using an aperture whose spatial

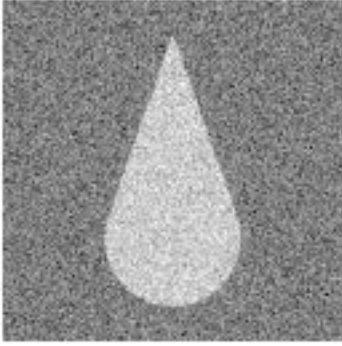


Figure 1 - A Teardrop with Signal to Noise of 4:1

standard deviation is 3 pixels wide. Figure 2 left shows the local mean values. Figure 2 center reflects the measured variances of the original teardrop. Figure 2 right shows local skewness.

A Laplacian of a Gaussian with measurement aperture  $s$  is applied to the variance values to measure the derivative of the variance with respect to changing scale  $s$ . Local areas whose measured variances do not change rapidly with increasing scale reflect coherent regions of either background or of signal (objects). Using this type of analysis, an optimal scale for object or local measurement may be determined.

### A modified VCD algorithm

Using the local statistical analysis presented above, a modification of the Whitaker VCD method has been developed. Whitaker uses a diffusion equation

$$\frac{\partial I}{\partial t} = \nabla \cdot \left( g(\nabla I(x,t) \otimes G(x,s(t))) \nabla I \right)$$

$$g(\nabla I(x,t) \otimes G(x,s(t))) = e^{-\frac{\|\nabla I(x,t) \otimes G(x,s(t))\|^2}{k^2}}$$

The control parameter  $k$  is the conductance parameter and it remains constant. It controls the rate of the variable conductance given a perceived intensity gradient at aperture  $s(t)$ . The aperture  $G(x,s(t))$  is a Gaussian sampling kernel

whose scale or aperture,  $s(t)$  is monotonically decreasing with time  $t$ .

By analyzing the images with the local statistical operators of variance and means, a local measure of the SNR may be made in a spatially adaptive fashion. The sample aperture may be controlled by seeking the largest scale at which local variance may be minimized. This in effect allows the best estimation of both the mean intensity as well as the variance of the noise. Therefore we replace  $s(t)$  with  $s(I_t(x))$  where

$$s(I_t(x)) \Rightarrow \frac{\partial \mu_{I_t}^{(2)}(x,s)}{\partial s} = 0$$

This constraint selects the largest scale where noise may be accurately estimated without the interference of intensity gradients introduced by boundaries. In practice, a zero value is never achieved, so a small threshold is used to select values of  $s$  that represent local minima of noise.

Conductance should be relative to the probability that the perceived intensity gradient is either noise or an object boundary. Given  $s(I_t(x))$ , we can compute  $\mu_{I_t}^{(2)}(x, s(I_t(x)))$ , the local variance at the optimal measurement scale. Gradient measurements are normalized with the expected noise distribution. That is, the conductance parameter  $k$  is replaced with the measured variance, making the conductance function

$$g(\nabla I(x,t) \otimes G(x, s(I_t(x)))) = e^{-\frac{\|\nabla I(x,t) \otimes G(x, s(I_t(x)))\|^2}{\mu_{I_t}^{(2)}(x, s(I_t(x)))}}$$

### Implementation

Each of the statistical measurements may be made using the convolution operation. Each of the higher moments are polynomial expressions of either the local means, the observed intensities, or combinations of the lower order moments.

At each iteration, statistical measurements at a range of scales were generated, and the values of the variance and local means were evaluated across increasing apertures size.

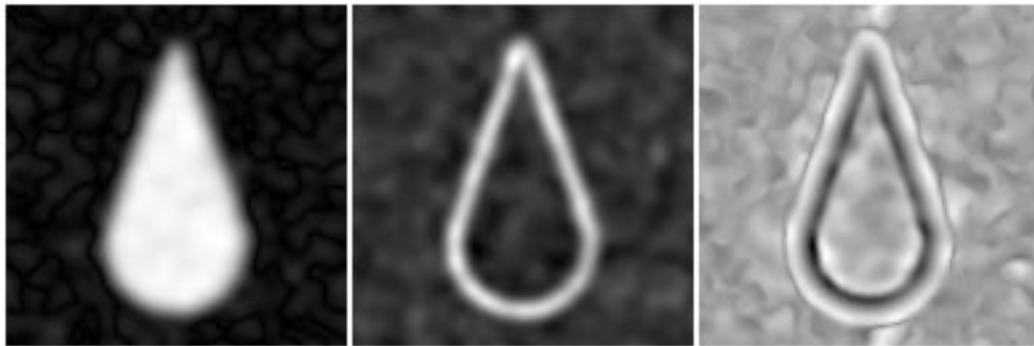


Figure 2: Local Statistical measures of the teardrop from fig. 1. (left: local means, center: local variance, right: local skewness)

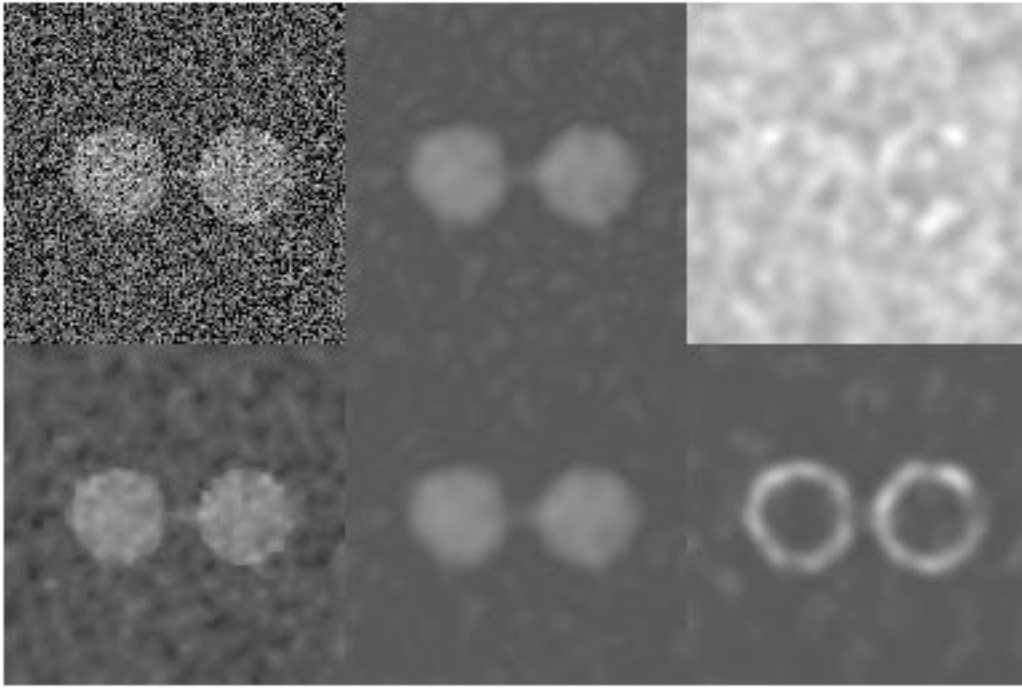


Figure 3: Results from the modified VCD (Top - before, Bottom - after; from left to right,  $I(x)$ ,  $\mu_1(x,s)$ ,  $\mu_1^{(2)}(x,s)$ )

Laplacian of Gaussian measurements were made of the variance, reflecting the change in variance with respect to increasing aperture size  $s$ , and a  $s(I_1(x))$  was selected that represented a local minimum in the noise. Each conductance value therefore is spatially varying depending on the local nature of the image. Although slow, the operation can be simplified to polynomial combinations of the primary local mean and the observed intensity values, making the process tractable.

### Results

Figure 3 shows the results of the modified VCD system on a simple object with SNR of 1:1. The object reflects features of differing scale, with the connecting bar between the two circles often dropping out during isotropic analysis. Using the modified VCD, no individual parameters were set for the image. Instead, measurement aperture was selected at each iteration for optimal estimation of mean and variance values. Notice the large amount of noise in the original image reflected by the relatively high measured variances (right top corner). After VCD, the noise is reduced somewhat, and the improvement in SNR may be characterized by the lower local variances (lower right corner). It is also important that the mean values of object and background remain constant. Perturbations of intensity as a filtering side effect will often result in poor classification if the object recognition tools depend on absolute intensity. The steady means are reflected in the center before and after images, where measured local means remain essentially unchanged.

Figure 4 shows results for the VCD system applied to a medical MRI image. The original image (shown in the

upper right corner) is not easily classified using intensity windows. Although there are clear separations in intensity among the various tissue values, noise that contributes to the misclassification of certain pixels (lower left corner). The VCD filtered images are depicted on the right.

### Future Work

In earlier work, I have shown how VCD can be applied to multiparameter images, in particular those produced by MRI, by basing spatial dissimilarities on distance measures in feature space that are defined according to SPR, i.e. according to probability distributions of these feature vectors that are estimated from sets of training pixels. Using the covariant probability distributions among multiple parameters as a basis for the dissimilarity measures that affect conductance in VCD generates a more robust means for reducing noise while preserving boundary information [Yoo, 1993].

While the resulting user-supervised SPR-constrained VCD filtering mechanism provides a principled means of measuring dissimilarity or gradients of possibly incommensurable within-pixel data parameters, there remains the problems associated with the variations of local intensity and the non-stationary nature of intensity and contrast in MR images. The development of local multiparameter statistical systems will address these issues and will require the generation of local measures of covariance, implying a matrix in the place of the current scalar measures of variance.

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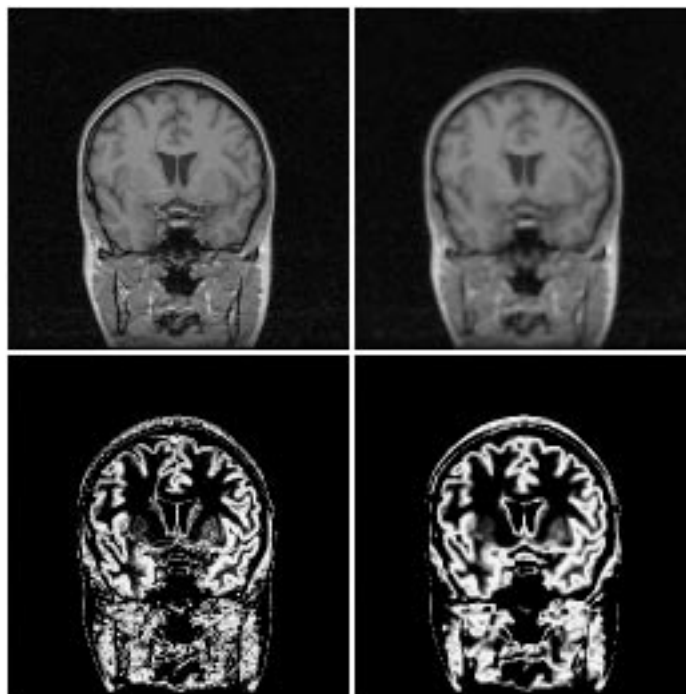


Figure 4: Results from the modified VCD system on an MRI image. (left- original, right - after VCD, top -  $I(x)$ , bottom, classification of grey matter using an intensity window classifier)