Pulse Sequences II: Spatial Encoding and Image Reconstruction

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Educational Syllabus, MR Physics for Physicists, International Society for Magnetic Resonance in Medicine, 2004.

Introduction

The topic of signal and image processing for MRI is vast, and it is necessary to restrict our attention to just a few central concepts. These will be Fourier transformation, sampling, filtering, convolution and the convolution theorem, the point spread function, and aliasing

Fourier Transformation

A broad class of functions, including those that we treat in MRI, may be regarded as a continuous superposition of complex exponentials. The Fourier transform (FT) of a time-domain function g(t) is a frequency-domain function G(ν), or spectrum, defined by:

$$G(\nu) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi\nu t} dt \qquad [1]$$

where j is the imaginary unit quantity. Physically, Eq. [1] shows how to extract from g(t) the amplitude of the frequency component at frequency ν. Conversely, the inverse FT (IFT) describes the synthesis of a time domain signal from sinusoidal components:

$$g(t) = \int_{-\infty}^{\infty} G(\nu) e^{j2\pi\nu t} d\nu \qquad [2]$$

From the above, we say that time, t, and frequency, ν, form a FT pair. In applications to image processing, the two-dimensional Fourier transform (2D-FT) frequently arises, with spatial position vectors, **x** = (x, y), and spatial frequency vectors, **k** = (k_x, k_y), forming the relevant FT pair. The 2-D FT and its inverse are then defined by:

$$G(k_x, k_y) = \iint g(x, y) e^{-j2\pi(k_x x + k_y y)} dxdy \qquad [3]$$

and

$$g(x, y) = \iint G(k_x, k_y) e^{j2\pi(k_x x + k_y y)} dk_x dk_y \qquad [4]$$

Another important result which can be shown to be equivalent to the above is the Fourier series expansion: *The Fourier series* of a periodic function $g_{per}(t)$, with period T, is:

$$g_{per}(t) = \sum_{-\infty}^{+\infty} \alpha_n e^{j2\pi nt/T} \qquad [5]$$

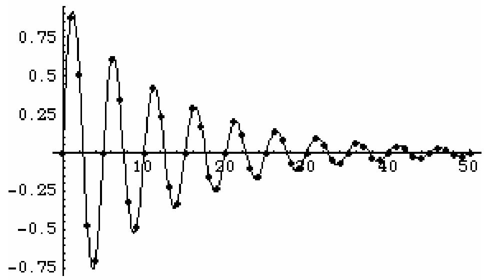
Using the properties of trigonometric integrals, one finds

$$\alpha_n = \frac{1}{T} \int_{-T}^{T} g_{per}(t) e^{-j2\pi nt/T} dt \qquad [6]$$

Eq. [5] shows that a periodic function can be regarded as a denumerable superposition of complex exponentials defined over its period.

Sampling of Continuous Data

MRI data acquisition and analysis deals with sampled, rather than continuous, data. The induced signal g(t) in the receiver coil as a function of time is an analog signal, which, after mixing down to low frequency, looks like the solid line in the illustration below. The analog-to-digital converter samples this at a certain rate, the sampling frequency ν_s. This gives the sampled data points, $g_{samp}(t)$, as illustrated. The reciprocal of ν_s is the time between the sampling events, and is called the sampling interval or dwell time, denoted ΔT. Clearly, sampling the continuous signal at different rates will lead to different sets of sampled data.



Sampling can be described formally as the product of g(t) with a sampling function, Samp(t), which equals unity at each of the sampling times and is otherwise zero. In terms of the Dirac delta function, δ, it can be written as:

$$Samp(t;\Delta T) = \sum_{n=-\infty}^{+\infty} \delta(t - n\Delta T) \qquad [7]$$

so that

$$g_{samp}(t) = g(t) \bullet \sum_{n=-\infty}^{+\infty} \delta(t - n\Delta T) = \sum_{n=-\infty}^{+\infty} g(n\Delta T)\delta(t - n\Delta T) \qquad [8]$$

Note that in reality only a finite number of samples can be obtained, as further discussed below.

Discrete Fourier Transformation

While the continuous FT and IFT are very convenient for developing physical insight and deriving basic theorems, sampled data requires use of the discrete FT (DFT). With time and frequency functions again denoted by g and G, the N-point DFT and inverse DFT are given by:

$$G(\tfrac{n}{N\Delta T}) = \sum_{m=0}^{N-1} g\left(m\Delta T\right) e^{-j2\pi nm/N} \qquad [9]$$

and

$$g(m\Delta T) = \frac{1}{N}\sum_{n=0}^{N-1} G\left(\frac{n}{N\Delta T}\right) e^{j2\pi nm/N} \qquad [10]$$

where g and G are now forced to be periodic with period N. For p, any integer,

$$G(\tfrac{n}{N\Delta T}) = G(\tfrac{pN+n}{N\Delta T}),\ g(m\Delta T) = g[(pN+m)\Delta T] \qquad [11]$$

The success of signal processing based upon the DFT is due largely to the fast FT (FFT) algorithm introduced by Cooley and Tukey in 1965, greatly accelerating these calculations.

Digital Filtering and Convolution

A digital filter T transforms an input sequence, x[n], into an output sequence, y[n]: T{x[n]} = y[n]. Filters which are *linear and time invariant* (LTI) possess especially favorable properties. A filter is linear if, given two input sequences, with $T\{x_1[n]\} = y_1[n]$ and $T\{x_2[n]\} = y_2[n]$, then $T\{\alpha\, x_1[n] + \beta\, x_2[n]\} = \alpha\, y_1[n] + \beta\, y_2[n]$. Time invariance means that a time shifted input leads to an equivalently time shifted output: T{x[n-D]} = y[n-D], where x[n-D] is a replicate of x[n] delayed by D steps, and similarly for y[n-D].

The *impulse response* of a filter T is the output, $h_T[n]$, resulting from application of T to a data stream represented by a unit impulse at the origin, that is, to x[n] ≡δ[n] ≡ {1, 0, 0, …}:

$T\{\delta[n]\} \equiv h_T[n]$ [12]

where $\delta[n]$ is the discrete delta function. An LTI filter T is fully characterized by its impulse response; knowing $h_T[n]$ leads to an explicit formula for the effect of T on any x[n]. This formula involves a convolution sum, as will be demonstrated below.

Convolution and the Convolution Theorem

Digital convolution of two data streams of length N, $x_1[n]$ and $x_2[n]$, is defined by

$$c[n] = x_1[n] * x_2[n] \equiv \sum_{k=0}^{N-1} x_1[k]\, x_2[n-k] \qquad [13]$$

where * is the symbol used to denote convolution. The summation describes a sum of products of the terms in x_1 marching forward, and the terms in x_2 marching backward. While confusing, Eq. [13] arises naturally from consideration of the impulse response of an LTI filter. For any k,

$\delta[n\text{–}k] = \{0, 0, 0, 1, 0, 0, \ldots\}$

with the lone "1" in the k-th position. A data stream x[n] can accordingly be written

$x[n] = x[0]\delta[n] + x[1]\delta[n–1] + x[2]\delta[n–2] + x[3]\delta[n–3] + x[4]\delta[n–4] + \ldots$

Application of a time-invariant filter T with an impulse response $h_T[n]$ to an individual term in this sequence is:

$T\{x[k]\delta[n–k]\} = x[k]\; T\{\delta[n–k]\} = x[k]\; h_T[n\text{-}k]$

so that operating with T on the entire sequence x[n] and using linearity, one finds

$$T\{x[n]\} = \sum_k x[k]\, h_T[n-k] \equiv x[n] * h_T[n] \qquad [14]$$

This demonstrates that knowing the impulse response $h_T[n]$ of an LTI filter T allows one to calculate the effect of T on an arbitrary data stream x[n] in terms of a convolution sum.

Ref. (1) states that "[the convolution theorem] is possibly the most important and powerful tool in modern scientific analysis". That must mean it's pretty important! Let the DFT of the periodic sampled function $x_1[n]$ be denoted $X_1[k]$: $DFT\{x_1[n]\} = X_1[k]$. Similarly, let $DFT\{x_2[n]\} = X_2[k]$. Then the time-domain convolution theorem states that:

$DFT\{x_1[n] * x_2[n]\}=X_1[k]\; X_2[k]$ [15]

where the right hand side is the term-by-term multiplicative product of the indicated sequences. The frequency-domain convolution theorem states that:

$DFT\{x_1[n]\; x_2[n]\}=X_1[k]*\; X_2[k]$ [16]

Eqs. [15] and [16] together indicate that multiplication in one domain, e.g. time, is equivalent to convolution in the other domain, e.g. frequency.

These results are identical to those for the continuous FT, with convolution defined by

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(s)y(t-s)ds \qquad [17]$$

Similarly to Eq. [13], this follows naturally from considerations of analog filters. The convolution theorems are written, for two functions f(t) and g(t) with FT's $F(\nu)$ and $G(\nu)$,

$FT\{f(t)*g(t)\} = F(\nu)\bullet G(\nu)$ [18]

and

$FT\{f(t)\bullet g(t)\} = F(\nu)*G(\nu)$ [19]

For later use, we note that because $f(t) = IFT\{F(\nu)\}$ and $g(t)=IFT\{G(\nu)\}$, Eq. [18] becomes

$FT\{IFT\{F(\nu)\}*IFT\{G(\nu)\}\} = F(\nu)\bullet G(\nu)$. Taking the IFT of both sides, we have equivalently:

$$IFT\{F(\nu)\}*IFT\{G(\nu)\}=IFT\{F(\nu)\bullet G(\nu)\} \qquad [20]$$

Note that these relations, written here in terms of time and frequency functions, actually apply to any pair of domains related by Fourier transformation. This includes the x and k spaces of MRI.

Applications of the Convolution Theorem--Data Truncation and Sampling

In an idealized treatment of two-dimensional k-space MRI data acquisition, one obtains a continuous function $s(k_x, k_y)$ which is related to the image, $\rho(x, y)$, by the imaging equation:

$$\rho(x, y) = IFT\{s(k_x, k_y)\}. \qquad [21]$$

However, the actual data matrix is discrete and non-infinite in extent. The convolution theorem helps us understand the effect of this. We consider the one-dimensional case, with obvious extensions to two dimensions. The finite k-space matrix indicates that the ideal infinite continuous data function s(k) is effectively set to zero outside of some values k_{min} and k_{max}, typically with $-k_{min} = k_{max} \equiv k_m$. This is equivalent to multiplication of s(k) by a function:

$$\mathrm{Rect}(k) = \begin{cases} 0, & k < -k_m \\ 1, & -k_m < k < k_m \\ 0, & k > k_m \end{cases} \qquad [22]$$

Thus, the derived density is:

$$\rho_{trunc}(x) = IFT\{s(k) \bullet Rect(k)\}=IFT\{s(k)\}*IFT\{Rect(k)\}$$

$$=\rho(x)*2k_m\sin(2\pi k_m x)/(2\pi k_m x) \qquad [23]$$

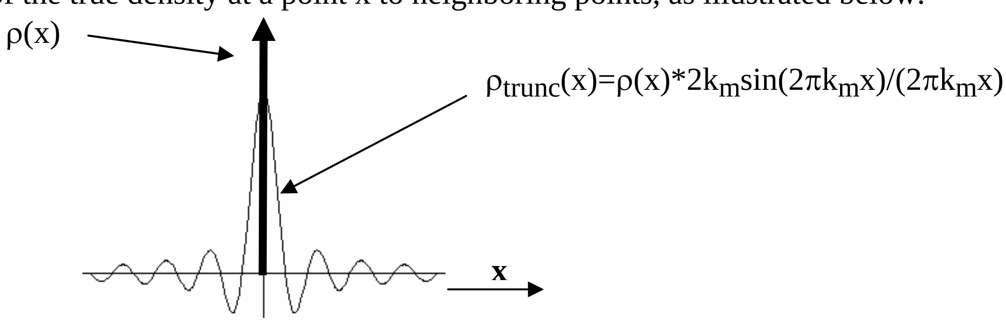
using Eq. [21] written for one dimension with{x, k} as Fourier variables, and a straightforward calculation of IFT{Rect(k)}. Physically, consider a point object located at the origin: $\rho(x)=\delta(x)$. Then from Eq. [23] one has for the image density resulting from the actual sample density:

$$\delta(x) \rightarrow \delta(x)*2k_m\sin(2\pi k_m x)/(2\pi k_m x)=\delta(x)*PSF_{trunc} \qquad [24]$$

where

$$PSF_{trunc} = 2k_m\sin(2\pi k_m x)/(2\pi k_m x)$$

denotes the *point spread function* resulting from data truncation. Thus, truncation leads to a smearing out of the true density at a point x to neighboring points, as illustrated below.



More generally, we see that the effect of a PSF can be written:

$$\rho(x)\rightarrow\rho(x)*PSF \qquad [25]$$

Sampling in k-space can be thought of as multiplication of the truncated data by a sampling function:

$$Samp(k;\Delta k) = \sum_{m=-\infty}^{\infty} \delta(k - m\Delta k) \qquad [26]$$

indicating sampling in k space at intervals of Δk. One finds that IFT(Samp(k; Δk)) = (1/Δk)Samp(x, 1/Δk). Therefore, the derived density from truncated sampled data is seen to be:

$$\rho_{trunc,samp}(x) = \text{IFT}\{s(k) \bullet \text{Rect}(k) \bullet \text{Samp}(k; \Delta k)\}$$
$$= \text{IFT}\{s(k)\} * \text{IFT}\{\text{Rect}(k)\} * \text{IFT}\{\text{Samp}(k; \Delta k)\} \qquad [27]$$
$$= \rho(x) * 2k_m \sin(2\pi k_m x)/(2\pi k_m x) * (1/\Delta k)\text{Samp}(x, 1/\Delta k)$$

The first and last terms, $\rho_{samp}(x) = \rho(x)*(1/\Delta k)\text{Samp}(x, 1/\Delta k)$, indicate the important result that sampling has led to replication of the original density at intervals of 1/Δk in the spatial domain.

The read dimension in 2D-FT MRI is filled by sampling at intervals ΔT, with $\Delta k = \gamma G_r \Delta T$, where G_r is the read gradient and γ is the gyromagnetic ratio divided by 2π. The other dimension of k-space is filled via the more complex process of phase encoding. Nevertheless, after the k-space matrix is filled, these two dimensions are equivalent and the discussion above applies equally to both read and phase encode directions.

Application of the Convolution Theorem--The Point Spread Function Due to T_2* Decay

We have seen above that multiplication of the k-space data by a function H(k) leads to a point spread function given by PSF = IFT{H(k)}. As another example, we can examine the blurring of gradient echo images by the T_2* decay occurring during acquisition of each k-space line. For convenience, we will incorporate truncation, but not sampling, in this discussion. Data for each k-space line which, without decay, would be s(k)•rect(k), becomes

$$s(k) \bullet rect(k) \rightarrow s(k) \bullet rect(k) \bullet e^{-t/T_2^*}$$

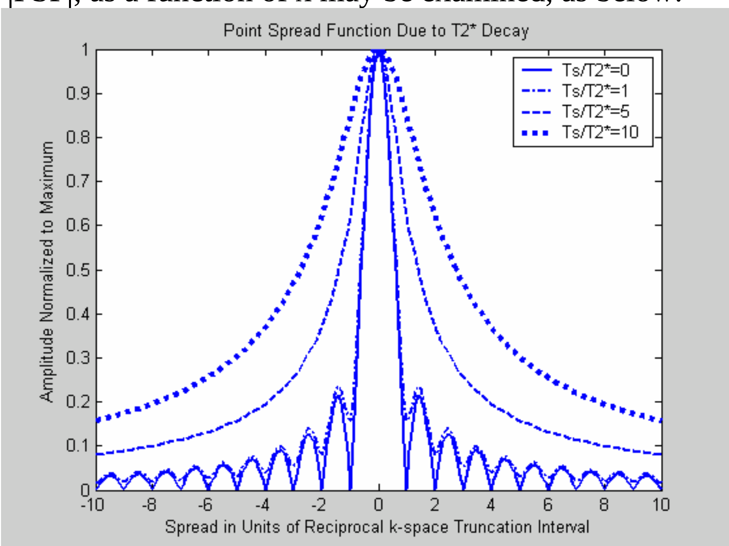
The gradient echo occurs at the center of the read gradient window, so we write

$$k = \gamma G_r t',$$

with t' = t – TE. Then e^{-t/T_2^*} may be written $e^{-TE/T_2^*} e^{-k/\gamma G T_2^*}$, and the point spread function is

$$\text{PSF}(x) = \text{IFT}(rect(k) e^{-TE/T_2^*} e^{-k/\gamma G T_2^*}).$$

A straightforward calculation then yields a complex function PSF(x) parameterized by $k_m/\gamma G T_2$*. We evaluate the effect of this by using the fact that the total sampling time for each line of k-space is given by $T_s = N \bullet \Delta T = 2k_m/\gamma G$, remembering that k-space sampling extends from -km to +km. The PSF may then be expressed in terms of T_s/T2*, and plots of the magnitude of the PSF, |PSF|, as a function of x may be examined, as below:

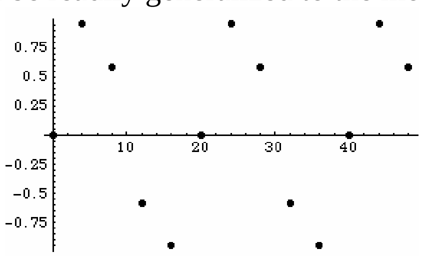


When T_2* decay is absent, that is, T_s/T_2*=0, the results are identical to those obtained from truncation alone. When T_s/T_2* = 1, there is virtually no further broadening beyond that due to windowing. As the sampling time becomes substantially longer than T_2*, we see progressive spreading of the |PSF|, indicating progressive blurring of the image. The PSF due to T_2 decay during data acquisition in a spin echo experiment can be calculated in a similar fashion, as can all of these effects for long echo-train length acquisitions in fast imaging experiments.

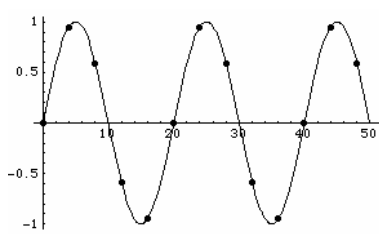
Aliasing

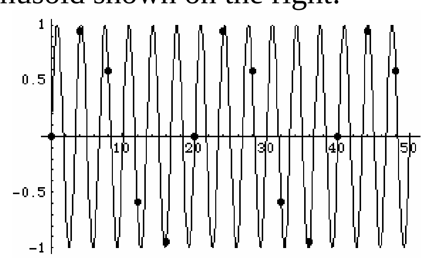
An aliased frequency is a high frequency temporal or spatial signal component that is represented at an erroneously low frequency. This results from sampling at too low a rate to faithfully capture high frequency components.

Recall that any signal we are likely to care about can be represented by the sum of sine waves. Therefore, we can restrict our attention to sinusoids, with the knowledge that the results can be readily generalized to the more general case. A digitized sine wave can be visualized as:



Is it obvious what continuous sine wave this was derived from? A natural choice is shown below (left), but the data points fit equally well onto the sinusoid shown on the right:





Thus, while the two analog signals illustrated above are very different, the sampled data streams derived from them will be identical if they are digitized at the rate illustrated. While there are an infinite number of higher frequency sinusoids that can fit the sampled data, there is a unique lowest-frequency sinusoid which fits the data. All of the high frequency sinusoids, when sampled, will be assigned to this lower frequency, that is, they will be aliased. The extension to a general, non-sinusoidal, sampled signal is clear; each Fourier component is assigned to the lowest possible frequency that fits the sampled data. Thus, some frequency components of a signal may be aliased, while others may not be.

The Nyquist criterion provides an explicit recipe for avoiding aliasing. This states that for a bandlimited time-domain signal with highest frequency component ν_{max}, if the sampling rate, ν_s, satisfies

$$\nu_s > 2\nu_{max} \qquad [28]$$

then aliasing will not occur. More formally, we wish to characterize the spectrum, $G_{samp}(\nu)$, of data stream obtained by sampling g(t):

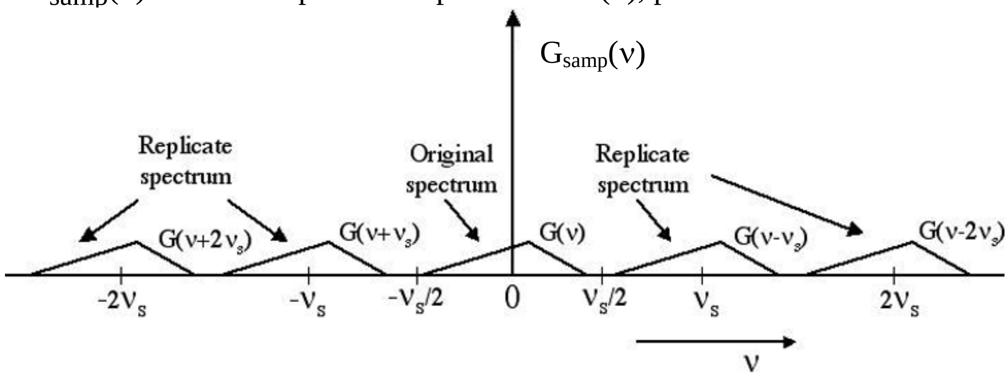
$$g_{samp}(t) = g(t) \bullet \sum_{n=-\infty}^{+\infty} \delta(t - n\Delta T) = \frac{1}{\Delta T} \sum_{m=-\infty}^{m=+\infty} g(t) e^{j2\pi m\nu\Delta T} \qquad [29]$$

where the second equality follows from a Fourier series expansion of the sum of Dirac delta functions. Fourier transforming both sides, we obtain

$$G_{samp}(\nu) = \frac{1}{\Delta T} \sum_{m=-\infty}^{m=+\infty} G(\nu - m/\Delta T) = \frac{1}{\Delta T} \sum_{m=-\infty}^{m=+\infty} G(\nu - m\nu_s)$$

[30]

so that $G_{samp}(\nu)$ consists of periodic replicates of $G(\nu)$, placed at intervals of $1/\Delta T = \nu_s$:



Thus, if the original spectrum $G(\nu)$ is zero for $\nu < -\nu_s/2$ and for $\nu > \nu_s/2$, then the spectral replicates will not intrude on the interval where the original spectrum resides, that is, on the interval $\nu \in (-\nu_s/2, \nu_s/2)$. Therefore, the original spectrum is preserved without distortion as long as the frequency spectrum $G(\nu)$ is bandlimited to frequencies $|\nu| < \nu_s/2$. This is the Nyquist criterion.

How does this apply to image data? Analogous to the above discussion, sampling of k-space must be of high enough frequency to properly represent high-frequency spatial components. We noted in our discussion of Eq. [23] that k-space sampling leads to replication of image density at intervals of $1/\Delta k$ in the spatial domain. To avoid having these replicates (actually, they are blurred replicates, due to the middle term, $2k_m \sin(2\pi k_m x)/(2\pi k_m x)$, of Eq. [27]) overlie each other, the spatial extent W of the object must be smaller than $1/\Delta k$:

$$\Delta k < 1/W \qquad [31]$$

This can be expressed in terms of k-space sampling most simply by considering the read (subscript "r") direction. Recall that sampling bandwidth is related to the field of view by:

$$\gamma G_r FOV_r = \text{Sampling Bandwidth} = 1/\Delta T. \qquad [32]$$

Now, from the definition of k-space,

$$\Delta k_r = \gamma G_r \Delta T \qquad [33]$$

Thus, using Eq. [31], aliasing can be avoided by requiring that the sampling interval satisfy:

$$\Delta T < 1/(\gamma G_r W_r). \qquad [34]$$

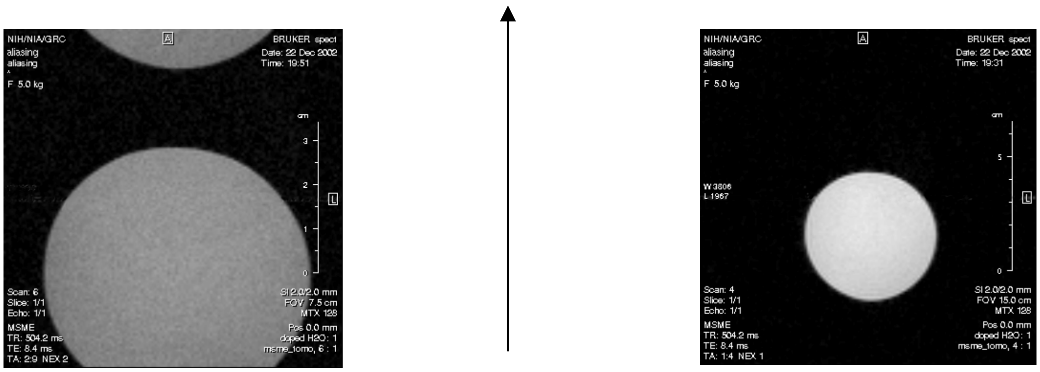
With $1/\Delta T = \gamma G_r / \Delta k_r$, Eq. [32] becomes the well-known result that

$$FOV_r = 1/\Delta k_r.$$

Therefore, the condition Eq. [31] to avoid aliasing can be recast into the familiar form

$$W_r < FOV_r. \qquad [35]$$

Because of the correspondence of sampling density in k-space to the rate of sampling in the time domain given by Eq. [33], with constant G_r, a short dwell time, resulting in a high receiver bandwidth, can be used to avoid aliasing. This is often not a problem–it means collecting more data, but doesn't add to the acquisition time of the image. This is illustrated below for a spherical phantom. The sampling rate is too low in the left hand image, leading to the apparent "wrap around" in the read (vertical) direction, where high-frequency regions are interpreted as low frequency due to insufficiently rapid sampling. The problem is solved in the right hand image, in which the read sampling rate was increased by a factor of two with constant G_r, resulting in a doubling of the FOV.



We note that Eq. [31] applies to the phase-encode (subscript "PE") dimension as well, where, instead of Eq. [33], one has:

$$\Delta k_{PE} = \gamma \Delta G_{PE} \tau \qquad [36]$$

Here, ΔG_{PE} is the phase encode gradient step and τ is its duration. Therefore we require

$$\gamma \Delta G_{PE} \tau < 1/W_{PE}$$

so that aliasing can be avoided by taking, for example,

$$\Delta G_{PE} < 1/(\gamma \tau W_{PE}) \qquad [37]$$

With $FOV_{PE} = 1/\Delta k_{PE}$, we generally write the non-aliasing condition as $W_{PE} < FOV_{PE}$, as before. Avoiding aliasing in the phase encode direction is considerably more problematic than in the read direction. In the phase encode direction, higher frequency sampling, that is, more closely spaced phase encode steps, requires addition of more shots if resolution is to be maintained. Because the duration of the image acquisition is given by Nshots * TR, where TR is the interval between excitation pulses and Nshots is the total number of shots, this results in longer imaging time.

References

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6. Spencer Richard G.S. (Editor). Advanced Signal Processing in Magnetic Resonance Imaging, Part 2. International J. Imaging Systems and Technology 10(3), 1999.
The two above collections of papers provide some insight into modern developments in MR signal processing.