# CONTINUOUS TIME APPLICATION OF THE ANDERSON-MOORE(AIM) ALGORITHM FOR IMPOSING THE SADDLE POINT PROPERTY IN DYNAMIC MODELS

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Abstract: (Anderson and Moore, 1983; Anderson and Moore, 1985) describe a powerful method for solving discrete time linear saddle point models. This paper shows how one can apply the technique to continuous time models.

## 1. INTRODUCTION AND SUMMARY

## 2. THE ALGORITHM

Perfect foresight models with solutions determined by saddle point property.

$$\begin{bmatrix} H_{-\tau} & \dots & H_{\theta} \end{bmatrix} \begin{bmatrix} \frac{d^0 x}{dt^0}(0) \\ \vdots \\ \frac{d^{\theta-1} x}{dt^{\theta}}(0) \end{bmatrix} = 0$$

With initial conditions:

$$Z \begin{bmatrix} \frac{d^0 x}{dt^0}(0) \\ \vdots \\ \frac{d^{\theta-1} x}{dt^{\theta-1}}(0) \end{bmatrix} = \xi$$

[1] I wish to thank Christiopher Sims for helpful comments. I am responsible for any remaining errors. The views expressed herein are mine and do not necessarily represent the views of the Board of Governors of the Federal Reserve System.

To deal with inhomogeneous system one can characterize the system in terms of deviations from steady state.

We will endeavor to write the system in the form:(Bellman, 1970)

$$\frac{dx}{dt}(t) = Ax(t)$$

If A has distinct eigenvalues then

$$x(t) = \exp^{At} x(0)$$

One can investigate the stability properties of the system by analyzing $A$ even when $A$ has repeated roots.

The AIM algorithm transition matrix computation produces A and, when necessary, generates auxiliary conditions which are important for establishing a full set of initial conditions for the solutions.(Anderson, 1997)

To apply the technique, one need only compute the dominant invariant space vectors spanning space assoicated with all positive roots.

$$VA = MV$$

But to motivate the solution, recall that one can always write

$$\begin{bmatrix} V \\ W \end{bmatrix} A = \begin{bmatrix} M & \\ & B \end{bmatrix} \begin{bmatrix} V \\ W \end{bmatrix}$$

with all the eigenvalues of $M$ positive and all the eigenvalues of $B$ zero or negative so that

$$A = \begin{bmatrix} V \\ W \end{bmatrix}^{-1} \begin{bmatrix} M & \\ & B \end{bmatrix} \begin{bmatrix} V \\ W \end{bmatrix}$$

so that

$$\begin{bmatrix} V \\ W \end{bmatrix} \frac{dx}{dt} = \begin{bmatrix} M & \\ & B \end{bmatrix} \begin{bmatrix} V \\ W \end{bmatrix} x$$

so that by choosing $x$ so that

$$Vx = 0$$

one has

$$\frac{d \begin{bmatrix} Vx \\ Wx \end{bmatrix}}{dt} = \begin{bmatrix} 0 \\ BWx \end{bmatrix}$$

and one can rest assured $Wx$ converges

$$\frac{d(Wx)}{dt} = B(Wx)$$

Combine results

$$\begin{bmatrix} V \\ W \end{bmatrix} x(t) = \begin{bmatrix} 0 & \\ & \exp^{Bt} \end{bmatrix} \begin{bmatrix} V \\ W \end{bmatrix} x(0)$$

so that

$$x(t) = \begin{bmatrix} V \\ W \end{bmatrix}^{-1} \begin{bmatrix} 0 & \\ & \exp^{Bt} \end{bmatrix} \begin{bmatrix} V \\ W \end{bmatrix} x(0)$$

One need only choose initial conditions guaranteeing that the initial part of the trajectory is orthogonal to the left invarinat space associated with positive roots, that the initial part of the trajectory not violate any constraints uncovered in computing the transition matrix, and the other original initial conditions.

$$Q = \begin{bmatrix} Z \\ Z^{\#} \\ V \end{bmatrix}$$

For most economic models, one will want an adequate number of constraints to identify a single trajectory.

$$Q \begin{bmatrix} \frac{d^0 x}{dt^0}(0) \\ \vdots \\ \frac{d^{\theta-1} x}{dt^{\theta-1}}(0) \end{bmatrix} = \begin{bmatrix} \xi \\ 0 \\ 0 \end{bmatrix}$$

## 3. AN EXAMPLE

A recent paper by Sims(Sims, 1996) presents a stochastic version of the following example model.

$$w(t) = \rho \left( \int_{s=0}^{\infty} \exp^{-\rho s} W(t+s) ds \right) - \alpha (u(t) - u_n)$$
$$W(t) = \rho \left( \int_{s=0}^{\infty} \exp^{-\rho s} w(t-s) ds \right)$$
$$\dot{u}(t) = -\theta u(t) + \gamma W(t) + \mu$$

With initial conditions

$$W(0) = W_0$$
$$u(0) = u_0$$

One can rewrite the system as:

$$\dot{u}(t) = -\theta u(t) + \gamma W(t) + \mu$$
$$\dot{w}(t) = \rho(w(t) - W(t)) - \alpha \dot{u} + \rho\alpha(u(t) - u_n)$$
$$\dot{W}(t) = \rho(w(t) - W(t))$$

Or:

$$\begin{bmatrix} -\rho & \rho & -\alpha\rho & 1 & 0 & \alpha \\ -\rho & \rho & 0 & 0 & 1 & 0 \\ 0 & -\gamma & \theta & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{d^0 w}{dt^0}(0) \\ \frac{d^0 W}{dt^0}(0) \\ \frac{d^0 u}{dt^0}(0) \\ \frac{dw}{dt}(0) \\ \frac{dW}{dt}(0) \\ \frac{du}{dt}(0) \end{bmatrix}$$

With

$$Z \begin{bmatrix} \frac{d^0 x}{dt^0}(0) \\ \vdots \\ \frac{d^{\theta-1} x}{dt^{\theta-1}}(0) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w(0) \\ W(0) \\ u(0) \end{bmatrix} = \begin{bmatrix} W_0 \\ u_0 \end{bmatrix}$$

For the example problem, AIM generates:

$$A = \begin{bmatrix} \rho & -(\alpha\gamma) - \rho & \alpha(\rho + \theta) \\ \rho & -\rho & 0 \\ 0 & \gamma & -\theta \end{bmatrix}$$

There are no auxiliary conditions for this model. One can compute the eigenvalues for the system.

With $\{\alpha \to 0.1, \theta \to 0.1, \gamma \to 0.1, \rho \to 0.3\}$

$$\begin{bmatrix} 0.3 & -0.31 & 0.04 \\ 0.3 & -0.3 & 0 \\ 0 & 0.1 & -0.1 \end{bmatrix}$$

For the example model we will require one positive root.

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ e_w(\lambda^+) & e_W(\lambda^+) & e_u(\lambda^+) \end{bmatrix}$$

where the $e_{w,W,u}$ come from the components of the left eigenvector. So that one must have

$$\begin{bmatrix} w(0) \\ W(0) \\ u(0) \end{bmatrix} = \begin{bmatrix} -\frac{e_w(\lambda^+)}{e_u(\lambda^+)} & -\frac{e_W(\lambda^+)}{e_u(\lambda^+)} & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} W_0 \\ u_0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{e_w(\lambda^+)w_0 + e_W(\lambda^+)W_0}{e_u(\lambda^+)} \\ W_0 \\ u_0 \end{bmatrix}$$

With $\{\alpha \to \frac{1}{10}, \theta \to \frac{1}{10}, \gamma \to \frac{1}{10}, \rho \to \frac{3}{10}\}$

$$\begin{bmatrix} w(0) \\ W(0) \\ u(0) \end{bmatrix} = \begin{bmatrix} -0.242023u_0 + 0.78242W_0 \\ W_0 \\ u_0 \end{bmatrix}$$

With $\{\alpha \to \frac{1}{10}, \theta \to \frac{1}{10}, \gamma \to \frac{1}{10}, \rho \to \frac{3}{10}\}$

$$\exp^{Bt} = \begin{bmatrix} \frac{\cos(0.0834218t)}{e^{0.0826369t}} + \frac{i\sin(0.0834218t)}{e^{0.0826369t}} & 0 \\ 0 & \frac{\cos(0.0834218t)}{e^{0.0826369t}} - \frac{i\sin(0.0834218t)}{e^{0.0826369t}} \end{bmatrix}$$

With $\{\alpha \to 0.1, \theta \to 0.1, \gamma \to 0.1, \rho \to 0.3\}$

$$\begin{bmatrix} V \\ W \end{bmatrix} = \begin{bmatrix} 4.13185 & -3.23284 & 1 \\ 0.434077 + 2.08555i & -1.13358 - 2.53932i & 1 \\ 0.434077 - 2.08555i & -1.13358 + 2.53932i & 1 \end{bmatrix}$$

$$w(t) = \frac{-0.242023\cos(0.0834218t)\mathrm{u}_0}{e^{0.0826369t}} + \frac{-0.630612\sin(0.0834218t)\mathrm{u}_0}{e^{0.0826369t}} + \left(\frac{0.78242\cos(0.0834218t)}{e^{0.0826369t}} - \frac{0.127269\sin(0.0834218t)}{e^{0.0826369t}}\right)W_0$$

$$W(t) = \frac{-0.870357\sin(0.0834218t)\mathrm{u}_0}{e^{0.0826369t}} + \left(\frac{1.\cos(0.0834218t)}{e^{0.0826369t}} + \frac{0.208136\sin(0.0834218t)}{e^{0.0826369t}}\right)W_0$$

$$u(t) = \frac{1.\cos(0.0834218t)\mathrm{u}_0}{e^{0.0826369t}} + \frac{-0.208136\sin(0.0834218t)\mathrm{u}_0}{e^{0.0826369t}} + \frac{1.19873\sin(0.0834218t)\mathrm{W}_0}{e^{0.0826369t}}$$

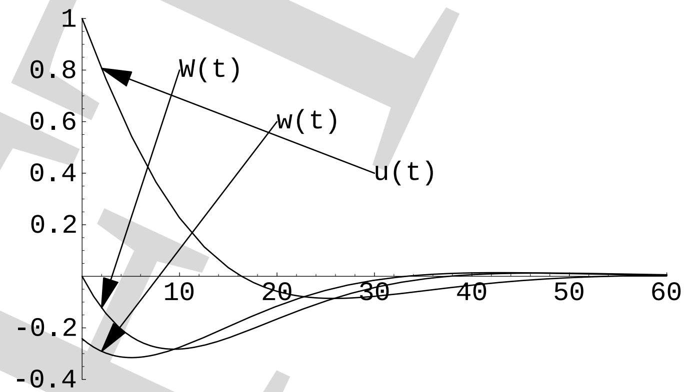


Fig. 1. Impulse Response to Unit Change in $u(0)$

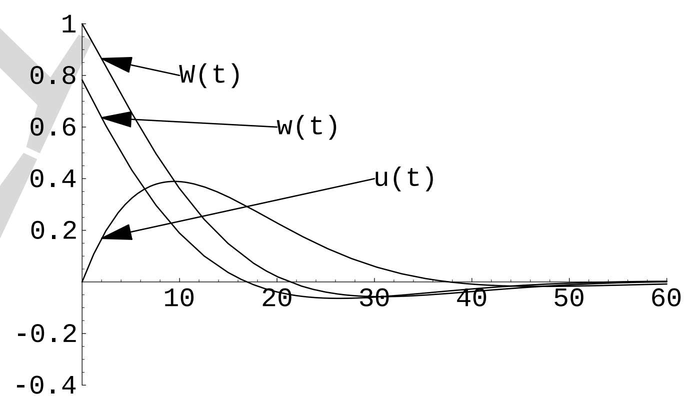


Fig. 2. Impulse Response to Unit Change in $W(0)$

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